ISRAEL JOURNAL OF MATHEMATICS **00** (XXXX), 1–6
DOI: 10.1007/s000000000000000000000000

THE NON-ARCHIMEDEAN NIRGENDSNEGATIVSEMIDEFINITHEITSSTELLENSATZ IS NOT TRUE *

BY

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ABSTRACT

Klep and Schweighofer asked whether the Nirgendsnegativsemidefinitheitsstellensatz holds for a symmetric noncommutative polynomial whose evaluations at bounded self-adjoint operators on any nontrivial Hilbert space are not negative semidefinite. We provide an example to show the open problem has a negative answer.

Contents

1. Introduction

This paper considers polynomials generated by noncommutative variables $\underline{X} := \{X_1, X_2, \ldots, X_m\}$ with coefficients from $k \in \{\mathbb{C}, \mathbb{R}\}$, where $\mathbb{R}$, $\mathbb{C}$ are real numbers and complex numbers respectively. Let $\mathbb{N} := \{1, 2, \ldots\}$ be the set of natural

* This research is supported by the National Key Research Project of China 2018YFA0306702 and the National Natural Science Foundation of China 12071467.

numbers. Let

$$p = \sum_{\omega \in \mathcal{W}_m} p_\omega \omega$$

be a polynomial in $k\langle \underline{X} \rangle$ with finitely many nonzero coefficients p_ω, and let $\mathcal{W}_m$ be the set of words generated by $\{X_1, X_2, \ldots, X_m\}$. The length of the longest word appearing in a polynomial p is defined as the degree of p. We define the transpose of a polynomial p as

$$p^* = \sum_{\omega \in \mathcal{W}_m} p_\omega^* \omega^*,$$

where $\omega^* = X_{i_k} \cdots X_{i_2} X_{i_1}$ for the word $\omega = X_{i_1} X_{i_2} \cdots X_{i_k}$. If $p = p^*$, we say p is symmetric. The set of symmetric polynomials is denoted by $\mathrm{Sym}\, k\langle \underline{X} \rangle$.

Let $\mathcal{H}$ denote a separable $k-$Hilbert space, and let $B(\mathcal{H})$ denote the set of bounded operators on $\mathcal{H}$. We evaluate a polynomial p at $\underline{A} = (A_1, A_2, \ldots, A_m)$, where each A_i is a self-adjoint operator in $B(\mathcal{H})$ for $1 \le i \le m$. The evaluation of a tuple $\underline{A} = (A_1, A_2, \ldots, A_m)$ on the empty word is $\mathrm{Id}_{\mathcal{H}}$, which is the identity operator in $\mathcal{H}$.

Let S be a subset of symmetric polynomials $\mathrm{Sym}\, k\langle \underline{X} \rangle$. The noncommutative semialgebraic set K_S consists of tuples $\underline{A} = (A_1, \ldots, A_m)$ of bounded self-adjoint operators on a nontrivial k-Hilbert space $\mathcal{H}$ such that $s(\underline{A})$ is positive semidefinite for all $s \in S$, i.e.,

$$K_S = \{\underline{A} \in B(\mathcal{H})^m \mid A_i^* = A_i,\ 0 \le i \le m,\ s(\underline{A}) \succeq 0,\ \forall s \in S\}.$$

Let M_S be the quadratic module defined by the set of elements of the form

$$M_S = \left\{ \sum_{i=1}^{n} g_i^* s_i g_i \mid g_i \in k\langle \underline{X} \rangle,\ s_i \in S \cup \{1\},\ n \in \mathbb{N} \right\}.$$

When S is an empty set, the quadratic module $M_\emptyset$ is denoted by SoS.

If M_S is an Archimedean quadratic module, i.e., there exists an $N \in \mathbb{N}$, such that $N - X_1^2 - X_2^2 - \cdots - X_m^2 \in M_S$, then it has been proved in Theorem 1.4 [5] that $p(\underline{A})$ is not negative semidefinite for any $\underline{A} \in K_S$ if and only if there exist a positive integer $r \in \mathbb{N}$ and polynomials $g_1, \ldots g_r \in k\langle \underline{X} \rangle$ such that

$$\sum_{i=1}^{r} g_i^* p g_i \in 1 + M_S.$$

Different proofs have been given in Theorem 5 in [2], Theorem 4.2 in [3], and Proposition 17 in [9].

2. A negative answer to Klep and Schweighofer's open problem

Klep and Schweighofer posted an interesting open problem in [5].

OPEN PROBLEM: [5] *Given a symmetric polynomial* $p \in k\langle \underline{X} \rangle$, *are the following two conditions equivalent?*

(*a*) $p(A_1, \dots, A_m)$ *is not negative semidefinite for any nontrivial* k*-Hilbert space* $\mathcal{H}$ ($\mathcal{H} \neq 0$) *and bounded self-adjoint operators* $A_1, \dots A_m$ *on* $\mathcal{H}$;

(*b*) *There exist* $r \in \mathbb{N}$ *and polynomials* $g_1, \dots g_r \in k\langle \underline{X} \rangle$ *such that*

$$\sum_{i=1}^{r} g_i^* p g_i \in 1 + \mathrm{SoS}. \tag{1}$$

If f is a commutative polynomial in $\mathbb{R}[X_1, \dots, X_m]$, then we have

$$f(A_1, \dots, A_m) \not\preceq 0 \iff f(A_1, \dots, A_m) > 0, \text{ where } A_i \in \mathbb{R}, 1 \leq i \leq m.$$

According to the Positivstellensatz in [6, 10], we have $f > 0$ on $\mathbb{R}^m$ if and only if there exist $r \in \mathbb{N}$ and polynomials $g_1, \dots g_r \in \mathbb{R}[X_1, \dots, X_m]$ such that

$$\left(\sum_{i=1}^{r} g_i^2 \right) f \in 1 + \mathrm{SoS}.$$

Therefore, for commutative cases ($\dim \mathcal{H} = 1$), Klep and Schweighofer's question has a positive answer. However, their open problem has a negative answer for symmetric noncommutative polynomials whose evaluations at bounded operators on any nontrivial Hilbert space are not negative semidefinite.

Example 1: Let the polynomial $f(X_1, X_2)$ be given as

$$f(X_1, X_2) = X_1 X_2^2 X_1 - X_2 X_1^2 X_2 + 1. \tag{2}$$

THEOREM 2.1: *The polynomial* f *given in (2) satisfies condition* (*a*) *but does not satisfy condition* (*b*) *in the open problem. Hence, the non-Archimedean Nirgendsnegativsemidefinitheitsstellensatz is not true.*

Proof. Firstly, we show that f satisfies condition (a),

$$f(A, B) \not\preceq 0, \; \forall A = A^*, B = B^*, A, B \in B(\mathcal{H}).$$

For any Hilbert space $\mathcal{H}$ and bounded self-adjoint operator $A, B \in B(\mathcal{H})$, we have $AB = (BA)^*$. Therefore, the bounded operator AB^2A and BA^2B are both positive semidefinite. Moreover, the set of bounded operators $B(\mathcal{H})$ is an example of C^*-algebra [1]. Let the symbol $\|\cdot\|$ denote the operator norm. We have the following equations

$$a = \|AB^2A\| = \|(BA)^*BA\| = \|BA\|^2 = \|AB\|^2 = \|BA^2B\|.$$

If $a = 0$, then we have $AB^2A = BA^2B = 0$, and $f(A,B) = \mathrm{Id}_{\mathcal{H}}$ is not negative semidefinite.

If $a > 0$, let $\langle\cdot,\cdot\rangle$ denote inner product on $\mathcal{H}$. There exists a sequence of unit vectors $\{v_i\} \in \mathcal{H}$ such that

$$a = \lim_{i\to\infty} \langle AB^2Av_i, v_i\rangle.$$

Hence, there exists a number $j \in \mathbb{N}$ such that

$$\langle AB^2Av_j, v_j\rangle > a - \frac{1}{2}, \quad -\langle BA^2Bv_j, v_j\rangle \geq -a.$$

Adding two sides of the inequalities, we derive that $f(A,B)$ is not negative semidefinite as

$$\langle f(A,B)v_j, v_j\rangle > \frac{1}{2}.$$

Secondly, we show that f does not satisfy condition (b). In [4, Lemma 2.1], Helton proved that for any given symmetric noncommutative polynomial p, there exists a symmetric matrix $\mathcal{M}_p$, not dependent on $\underline{X}$, and a vector $V(\underline{X})$ of monomials in $\underline{X}$ such that

$$p(\underline{X}) = V(\underline{X})^T\mathcal{M}_pV(\underline{X}).$$

Furthermore, the vector $V(\underline{X})$ can always be chosen as $V^d(\underline{X})$ which contains the monomials whose degree is at most d, where $d = \lceil \deg(p)/2\rceil$. As shown in [8, Theorem 2.1], a symmetric polynomial $p \in$ SoS if and only if there exists a positive semidefinite matrix $\mathcal{M}_p$ such that

$$p(\underline{X}) = V(\underline{X})^T\mathcal{M}_pV(\underline{X}).$$

For a monomial ω in p, we can find its corresponding entries whose index (v,u) satisfies $\omega = v^*u$ in $\mathcal{M}_p$, and the coefficient of ω in p equals to $\sum_{\omega=v^*u}\mathcal{M}_p(v,u)$. The matrix $\mathcal{M}_p$ is not unique, but for a monomial ω in p whose degree is $2d$, it has only one corresponding entry, i.e., there is a unique choice such that $\omega = v^*u$ where degrees of u and v are equal to d.

For the polynomial f defined in (2), we have

$$f(X_1,X_2) = V^2(\underline{X})^*\mathcal{M}_fV^2(\underline{X})$$

where a matrix $\mathcal{M}_f$ can be written as

$$\mathcal{M}_f = \begin{array}{c} \\ 1 \\ X_1 \\ X_2 \\ X_1^2 \\ X_1X_2 \\ X_2X_1 \\ X_2^2 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & X_1 & X_2 & X_1^2 & X_1X_2 & X_2X_1 & X_2^2 \end{array} \\ \begin{pmatrix} 1 & & & & & & \\ & 0 & & & & & \\ & & 0 & & & & \\ & & & 0 & & & \\ & & & & -1 & & \\ & & & & & 1 & \\ & & & & & & 0 \end{pmatrix} \end{array}$$

and $V^2(\underline{X})$ is the monomial vector

$$V^2(\underline{X})^* = \begin{pmatrix} 1 & X_1 & X_2 & X_1^2 & X_2X_1 & X_1X_2 & X_2^2 \end{pmatrix}$$

For any $r \in \mathbb{N}$ and $g_1, g_2, \ldots, g_r \in k\langle \underline{X} \rangle$, we have the symmetric polynomial

$$F = \sum_{i=1}^{r} g_i^* f g_i = \sum_{i=1}^{r} g_i^* X_1 X_2^2 X_1 g_i - \sum_{i=1}^{r} g_i^* X_2 X_1^2 X_2 g_i + \sum_{i=1}^{r} g_i^* g_i.$$

Define $D = \max\{d_1, \ldots, d_r\}$, where d_i denotes the degree of g_i for $1 \leq i \leq r$. There always exists a monomial u with the maximal degree D in some g_i for $1 \leq i \leq r$.

For any matrix $\mathcal{M}_0$ which satisfies that

$$F = V^{D+2}(\underline{X}) \mathcal{M}_0 V^{D+2}(\underline{X}),$$

the diagonal entry in $\mathcal{M}_0$ indexed by (X_1X_2u, X_1X_2u) must be the coefficient of the monomial $u^* X_2 X_1^2 X_2 u$ which is negative in F since the degree of $u^* X_2 X_1^2 X_2 u$ is $2D + 4$. Therefore, $\mathcal{M}_0$ can not be positive semidefinite, i.e., we have

$$F = \sum_{i=1}^{r} g_i^* f g_i \notin \text{SoS}.$$

Since $1 + \text{SoS} \subseteq \text{SoS}$, we have shown that there are no $r \in \mathbb{N}$ and $g_1, g_2, \ldots, g_r \in k\langle \underline{X} \rangle$, such that condition (b) holds. ∎

Remark 2.1: Similar to the proof of Theorem 2.1, we can show that the following symmetric polynomial with complex coefficients

$$g(X_1, X_2) = 2\mathrm{i} X_2 X_1 - 2\mathrm{i} X_1 X_2 + 1 \tag{3}$$

satisfies condition (a) but does not satisfy condition (b) [7, Theorem 3.1]. However, if we allow evaluations at unbounded operators, then the polynomial g does not satisfy condition (a) [7].

Acknowledgments: We are grateful to the anonymous referee for the constructive comments which lead to the example given in Remark 2.1. We would like to thank Zhihong Yang, Tao Zheng and Tianshi Yu for helpful discussions.

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