Example 5.5 (Essential Variety). This is an example taken from [13]. Let $\mathcal{E}$ be the essential variety defined as:

$$\mathcal{E} = \left\{ M \in \mathbb{R}^{3\times 3} \mid \det(M) = 0,\ 2(MM^T)M - \mathrm{tr}(MM^T)M = 0 \right\},$$

where $\det(M)$ is the determinant of M and $\mathrm{tr}(MM^T)$ is the trace of MM^T.

If we write the matrix M as

$$\begin{pmatrix} a & b & c \\ u & v & w \\ x & y & z \end{pmatrix},$$

then the 10 cubics defining $\mathcal{E}$ are:

$$\begin{aligned}
&avz - awy - buz + bwx + cuy - cvx,\\
&(2a^2 + 2b^2 + 2c^2)a + (2au + 2bv + 2cw)u + (2ax + 2by + 2cz)x - ga,\\
&(2a^2 + 2b^2 + 2c^2)b + (2au + 2bv + 2cw)v + (2ax + 2by + 2cz)y - gb,\\
&(2a^2 + 2b^2 + 2c^2)c + (2au + 2bv + 2cw)w + (2ax + 2by + 2cz)z - gc,\\
&(2au + 2bv + 2cw)a + (2u^2 + 2v^2 + 2w^2)u + (2ux + 2vy + 2wz)x - gu,\\
&(2au + 2bv + 2cw)b + (2u^2 + 2v^2 + 2w^2)v + (2ux + 2vy + 2wz)y - gv,\\
&(2au + 2bv + 2cw)c + (2u^2 + 2v^2 + 2w^2)w + (2ux + 2vy + 2wz)z - gw,\\
&(2ax + 2by + 2cz)a + (2ux + 2vy + 2wz)u + (2x^2 + 2y^2 + 2z^2)x - gx,\\
&(2ax + 2by + 2cz)b + (2ux + 2vy + 2wz)v + (2x^2 + 2y^2 + 2z^2)y - gy,\\
&(2ax + 2by + 2cz)c + (2ux + 2vy + 2wz)w + (2x^2 + 2y^2 + 2z^2)z - gz,
\end{aligned}$$

where $g = (a^2 + b^2 + c^2 + u^2 + v^2 + w^2 + x^2 + y^2 + z^2)$. Let I denote the ideal generated by these 10 cubics. Take these 10 cubics as input and we obtain in 800 sec. only one minimal prime of $\sqrt[re]{I}$, which is the ideal I itself. Thus I is a real ideal.

ACKNOWLEDGMENT

We thank Yue Ren for his help with the implementation of our algorithms. We also thank Erich L. Kaltofen for his valuable suggestions on the complexity of factorizing multivariate polynomials.

REFERENCES

[1] Basu, S., Pollack, R., Roy, M.-F., 2006. Algorithms in Real Algebraic Geometry. Springer Berlin Heidelberg, Berlin, Heidelberg.

[2] Becker, E., Neuhaus, R., 1993. Computation of real radicals of polynomial ideals. In: Eyssette, F., Galligo, A. (Eds.), Computational Algebraic Geometry. Vol. 109. Birkhäuser, Boston, MA, pp. 1–20.

[3] Blanco, C., Jeronimo, G., Solernó, P., 2004. Computing generators of the ideal of a smooth affine algebraic variety. Journal of Symbolic Computation 38 (1), 843–872.

[4] Bochnak, J., Coste, M., Roy, M.-F., 1998. Real algebraic geometry. Vol. 36. Springer, Berlin, Heidelberg.

[5] Brake, D. A., Hauenstein, J. D., Liddell, Jr., A. C., 2016. Validating the completeness of the real solution set of a system of polynomial equations. In: Proceedings of the 2016 International Symposium on Symbolic and Algebraic Computation. ISSAC'16. ACM, New York, NY, USA, pp. 143–150.

[6] Chen, C., Davenport, J. H., May, J. P., Maza, M. M., Xia, B., Xiao, R., 2010. Triangular decomposition of semi-algebraic systems. In: Proceedings of the 2010 International Symposium on Symbolic and Algebraic Computation. ISSAC'10. ACM, New York, NY, USA, pp. 187–194.

[7] Chen, C., Davenport, J. H., May, J. P., Maza, M. M., Xia, B., Xiao, R., 2013. Triangular decomposition of semi-algebraic systems. Journal of Symbolic Computation 49, 3–26.

[8] Chen, C., Davenport, J. H., Moreno Maza, M., Xia, B., Xiao, R., 2011. Computing with semi-algebraic sets represented by triangular decomposition. In: Proceedings of the 36th International Symposium on Symbolic and Algebraic Computation. ISSAC'11. ACM, New York, NY, USA, pp. 75–82.

[9] Cox, D., Little, J., O'shea, D., 1992. Ideals, varieties, and algorithms. Vol. 3. Springer.

[10] Durvye, C., Lecerf, G., 2008. A concise proof of the Kronecker polynomial system solver from scratch. Expositiones Mathematicae 26 (2), 101–139.

[11] Eisenbud, D., 1995. Commutative Algebra: with a View Toward Algebraic Geometry. Vol. 150. Springer New York, New York, NY.

[12] Everett, H., Lazard, D., Lazard, S., Safey El Din, M., 2009. The Voronoi diagram of three lines. Discrete & Computational Geometry 42 (1), 94–130.

[13] Fløystad, G., Kileel, J., Ottaviani, G., 2017. The Chow form of the essential variety in computer vision. Journal of Symbolic Computation.

[14] Gelfand, I. M., Kapranov, M. M., Zelevinsky, A. V., 1994. Discriminants, Resultants, and Multidimensional Determinants. Birkhäuser Boston, Boston, MA.

[15] Giusti, M., Lecerf, G., Salvy, B., 2001. A Gröbner free alternative for polynomial system solving. Journal of complexity 17 (1), 154–211.

[16] Jeronimo, G., Krick, T., Sabia, J., Sombra, M., 2004. The computational complexity of the Chow form. Foundations of Computational Mathematics 4 (1), 41–117.

[17] Kalkbrener, M., 1991. Three contributions to elimination theory. Ph.D. thesis, Johannes Kepler University, Linz.

[18] Kaltofen, E., 1989. Factorization of polynomials given by straight-line programs. Randomness and Computation 5, 375–412.

[19] Kaltofen, E., Li, B., Yang, Z., Zhi, L., 2008. Exact certification of global optimality of approximate factorizations via rationalizing sums-of-squares with floating point scalars. In: Proceedings of the Twenty-first International Symposium on Symbolic and Algebraic Computation. ISSAC'08. ACM, New York, NY, USA, pp. 155–164.

[20] Kaltofen, E., Trager, B. M., 1990. Computing with polynomials given by black boxes for their evaluations: greatest common divisors, factorization, separation of numerators and denominators. Journal of Symbolic Computation 9 (3), 301–320.

[21] Krick, T., 2002. Straight-line programs in polynomial equation solving. Foundations of computational mathematics: Minneapolis 312, 96–136.

[22] Lasserre, J.-B., Laurent, M., Mourrain, B., Rostalski, P., Trébuchet, P., 2013. Moment matrices, border bases and real radical computation. Journal of Symbolic Computation 51, 63–85.

[23] Lasserre, J. B., Laurent, M., Rostalski, P., 2008. Semidefinite characterization and computation of zero-dimensional real radical ideals. Foundations of Computational Mathematics 8 (5), 607–647.

[24] Lax, P., 2005. On the discriminant of real symmetric matrices. Selected Papers Volume II, 577–586.

[25] Lazard, D., 1991. A new method for solving algebraic systems of positive dimension. Discrete Applied Mathematics 33 (1-3), 147–160.

[26] Lecerf, G., 2000. Computing an equidimensional decomposition of an algebraic variety by means of geometric resolutions. In: Proceedings of the 2000 International Symposium on Symbolic and Algebraic Computation. ISSAC'00. ACM, New York, NY, USA, pp. 209–216.

[27] Lecerf, G., 2003. Computing the equidimensional decomposition of an algebraic closed set by means of lifting fibers. Journal of Complexity 19 (4), 564–596.

[28] Ma, Y., Wang, C., Zhi, L., 2016. A certificate for semidefinite relaxations in computing positive-dimensional real radical ideals. Journal of Symbolic Computation 72, 1–20.

[29] Marshall, M., 2008. Positive polynomials and sums of squares. No. 146. American Mathematical Soc.

[30] Moreno Maza, M., 1997. Calculs de pgcd au-dessus des tours d'extensions simples et résolution des systèmes d'équations algébriques. Ph.D. thesis, Université Paris 6.

[31] Neuhaus, R., 1998. Computation of real radicals of polynomial ideals – II. Journal of Pure and Applied Algebra 124 (1), 261–280.

[32] Poteaux, A., Schost, É., 2013. On the complexity of computing with zero-dimensional triangular sets. Journal of Symbolic Computation 50 (Supplement C), 110 – 138.

[33] Safey El Din, M., 2005. Finding sampling points on real hypersurfaces is easier in singular situations. MEGA (Effective Methods in Algebraic Geometry) Electronic proceedings.

[34] Safey El Din, M., 2007. RAGLib (Real Algebraic Geometry Library), Maple package.

[35] Safey El Din, M., 2007. Testing sign conditions on a multivariate polynomial and applications. Mathematics in Computer Science 1 (1), 177–207.

[36] Safey El Din, M., Schost, É., 2003. Polar varieties and computation of one point in each connected component of a smooth real algebraic set. In: Proceedings of the 2003 International Symposium on Symbolic and Algebraic Computation. ISSAC'03. ACM, New York, NY, USA, pp. 224–231.

[37] Safey El Din, M., Schost, É., 2004. Properness defects of projections and computation of at least one point in each connected component of a real algebraic set. Discrete & Computational Geometry 32 (3), 417–430.

[38] Safey El Din, M., Schost, É., 2017. A nearly optimal algorithm for deciding connectivity queries in smooth and bounded real algebraic sets. Journal of the ACM 63 (6), 48:1–48:37.
URL http://doi.acm.org/10.1145/2996450

[39] Schost, É., 2003. Computing parametric geometric resolutions. Applicable Algebra in Engineering, Communication and Computing 13 (5), 349–393.

[40] Spang, S. J., 2007. On the computation of the real radical. Ph.D. thesis, Technische Universität Kaiserslautern.

[41] Spang, S. J., 2008. A zero-dimensional approach to compute real radicals. The Computer Science Journal of Moldova 16 (1), 64–92.

[42] Wang, D., 1998. Decomposing polynomial systems into simple systems. Journal of Symbolic Computation 25 (3), 295–314.

[43] Wu, W.-T., 1984. Basic principles of mechanical theorem proving in elementary geometries. Journal of Systems Science and Mathematical Sciences 4, 207–235.