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Global Optimization of Polynomials over Real Algebraic Sets*

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In memory of Wen-Tsün Wu (1919–2017)

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Abstract Let $f, g_1, \cdots, g_s$ be polynomials in $\mathbb{R}[X_1, \cdots, X_n]$. Based on topological properties of generalized critical values, the authors propose a method to compute the global infimum f^* of f over an arbitrary given real algebraic set $V = \{x \in \mathbb{R}^n \mid g_1(x) = 0, \cdots, g_s(x) = 0\}$, where V is not required to be compact or smooth. The authors also generalize this method to solve the problem of optimizing f over a basic closed semi-algebraic set $S = \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \cdots, g_s(x) \geq 0\}$.

Keywords Polynomial optimization, real algebraic set, generalized critical value.

1 Introduction

We consider the problem of optimizing a polynomial function f over an arbitrary given real algebraic set:

$$f^* = \inf_{x \in V} f(x), \tag{1}$$

where $V = \{x \in \mathbb{R}^n \mid g_1(x) = 0, \cdots, g_s(x) = 0\}$.

Polynomial optimization has wide applications in various fields, such as control theory[1], operational research[2], signal processing[3], computer vision[4], and so on. Nevertheless, polynomial optimization is NP-hard (see, e.g., [5]). Various numerical and symbolic methods have been developed to solve polynomial optimization problems efficiently. In [6], Wu Wen-Tsun introduced a new method based on Ritt-Wu zero-decomposition theorem[7–9] to prove geometry

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theorems involving inequalities. Furthermore, in [10, 11], Wu presented a finiteness theorem which is a variant of Sard's lemma[12–14] for optimizing a polynomial in a closed and bounded domain defined by polynomial equality, inequality constrains and a non-zero condition. Wu's method has been extended and used by Wu Tianjiao and other people to solve nontrivial nonlinear programming problems[15–21].

There are other kinds of symbolic methods for solving the optimization problem (1). We can rewrite (1) as a quantifier elimination problem which can be solved by cylindrical algebraic decomposition algorithm[22]. This algorithm can deal with general cases and it has been improved in many ways, see e.g. [23–32]. Its complexity is doubly exponential in the number of variables. In [33, Chapter 14, Section 14.2], an algorithm based on block elimination is presented for solving polynomial optimization problems and its complexity is singly exponential in the number of variables. However, this algorithm uses techniques such as infinitesimal deformations which do not provide practical results. In [34, 35], practical algorithms are given for a variant of the real quantifier elimination problem which requires that the input satisfy certain conditions. On the other hand, applying topological properties of generalized critical values and polar varieties, Safey El Din[36] proposed a probabilistic algorithm to compute $\inf_{x\in\mathbb{R}^n} f(x)$. The complexity of the algorithm is $O(n^7D^{4n})$ where D is the degree of f. Together with Greuet, they generalized this approach to the constrained case with some regularity assumptions: $\langle g_1,\cdots,g_s\rangle$ is a radical and equidimensional ideal and the complex variety $\{x\in\mathbb{C}^n \mid g_1(x)=0,\cdots,g_s(x)=0\}$ has finitely many singular points[37].

There are other numerical methods for solving the optimization problem (1). For example, when the real algebraic set V is compact, Lasserre[38] introduced a hierarchy of semidefinite programming (SDP) relaxations for computing the global infimum f^*. Suppose V is smooth and f^* is reached on V, Nie, et al.[39–41] proposed Jacobian SDP relaxation to solve (1). If one does not know whether f^* is reached on V, Schweighofer[42] introduced gradient tentacle to deal with the unconstrained case, that is, $V=\mathbb{R}^n$. Hà and Phạm[43, 44] introduced truncated tangency variety to deal with the constrained case with the assumption that V is smooth. Guo, et al.[45, 46] improved the work of [42–44] by replacing gradient tentacle and truncated tangency variety with polar variety.

We aim at solving the global optimization of polynomials over arbitrary given real algebraic sets, i.e., we wish to drop all assumptions on V. Our work is mainly based on previous works by Rabier, Jelonek and Kurdyka. Let $f: M\to N$ be a C^2 mapping with M,N being C^2 Finsler manifolds. Assume that M is complete and N is connected. Rabier[47] proved that under a certain condition (see (13)), the mapping $f: M\to N$ is a locally trivial fibration outside the set of generalized critical values of f (see (5)). Jelonek and Kurdyka[48] proved that the set of generalized critical values of a polynomial mapping over some smooth variety has Lebesgue measure zero. In the following, we first recall some basic definitions and notations in [47, 48], then we describe briefly main results of this paper.

Let $\mathcal{L}(\mathbb{R}^n,\mathbb{R}^m)$ denote the set of linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$. For $A\in\mathcal{L}(\mathbb{R}^n,\mathbb{R}^m)$, set $v(A)=\inf_{\|y^*\|=1}\|A^*(y^*)\|$ where A^* is the adjoint of A. Let H be a linear subspace of $\mathbb{R}^n$, we denote by $v(A,H)=v(A|_H)$, where $A|_H$ is the restriction of A to H.

Let $V \subset \mathbb{R}^n$ be an equidimensional real algebraic set of dimension d and let $\mathbf{I}(V) = \langle g_1, \cdots, g_s \rangle$ be the vanishing ideal of V in $\mathbb{R}[X_1, \cdots, X_n]$. Let $J(x)$ be the Jacobian matrix of $g_1, \cdots, g_s$ at $x \in V$. Let $\mathcal{R}_{V,x}$ be the localization of $\mathbb{R}[X]/\mathbf{I}(V)$ at $\mathfrak{m}_x$ where $\mathfrak{m}_x$ is the maximal ideal of polynomial functions vanishing at x. The point x is said to be nonsingular in dimension d if $\mathcal{R}_{V,x}$ is a regular local ring of dimension d (see [49, Definition 3.3.9]). By the Jacobian criterion (see, e.g. [50, Corollary 5.6.14] or [51, Corollary 16.20]), $\mathcal{R}_{V,x}$ is regular if and only if the matrix $J(x)$ has rank $n-d$. Let $\mathrm{Reg}(V)$ denote the set of nonsingular points in dimension d of V and $\mathrm{Sing}(V)$ denote the set $V \setminus \mathrm{Reg}(V)$. We call $\mathrm{Reg}(V)$ the smooth part of V.

Let $f : V \to \mathbb{R}^m$ be a polynomial mapping and $M = \mathrm{Reg}(V)$. We denote by $f|_M$ the restriction of f to M. The set of critical values of $f|_M$ is denoted as

$$K_0(f, M) = \{y \in \mathbb{R}^m \mid \exists x \in M \text{ s.t. } f(x) = y \text{ and } v(df(x), T_x M) = 0\}, \tag{2}$$

where $df = (\frac{\partial f_i}{\partial X_j})$ $(i = 1, 2, \cdots, m,\ j = 1, 2, \cdots, n)$ and $T_x M$ is the tangent space of M at x. Similarly, the set of asymptotic critical values of $f|_M$ at infinity is denoted by

$$K_\infty(f, M) = \{y \in \mathbb{R}^m \mid \exists x_l \in M, \|x_l\| \to \infty \text{ s.t. } f(x_l) \to y \text{ and } \|x_l\| v(df(x_l), T_{x_l} M) \to 0\}. \tag{3}$$

Define

$$K_1(f, M) = \{y \in \mathbb{R}^m \mid \exists x_l \in M, x \in \mathrm{Sing}(V), x_l \to x,\ f(x_l) \to y \text{ and } v(df(x_l), T_{x_l} M) \to 0\}. \tag{4}$$

The set of generalized critical values of $f|_M$ defined as:

$$K(f, M) = K_0(f, M) \cup K_1(f, M) \cup K_\infty(f, M). \tag{5}$$

Here are some geometric explanations of generalized critical values. Given a critical point $x \in M$ of f, then $v(df(x), T_x M) = 0$ and the tangent map $df(x) : T_x M \to \mathbb{R}^m$ is not surjective[47]. When f is a polynomial function, the gradient $\nabla f(x)$ is contained in the normal space of M on x. For $K_\infty(f, M)$ and $K_1(f, M)$, there exists a sequence (x_l) in M such that the tangent maps tend to be non-surjective. When f is a polynomial function, the distance between the gradient $\nabla f(x_l)$ and the normal space of M on x_l converges to zero.

Remark 1.1 In [52], we set $K(f, M) = K_0(f, M) \cup K_\infty(f, M)$. The results in [52] have been restated according to the new definition of $K(f, M)$ given in (5). These changes are motivated by results in [53].

Let $B(f, M)$ denote the bifurcation set of $f|_M$[†], which is the smallest set such that $f|_M$ is a locally trivial fibration (see Definition 2.5) over its complement. Let $V = \mathbb{R}^n$ and f be a polynomial function. If f^* in (1) exists, it is contained in $B(f, \mathbb{R}^n)$, and thus contained in $K(f, \mathbb{R}^n)$ [36, Theorem 5]. This result has been generalized to the case where V is equidimensional and smooth[37]. Therefore, the optimization problem (1) is reduced to the problem of identifying

[†]Following [48, 54–56], $B(f, M)$ is called "bifurcation set" in this paper. In algebraic geometry, "ramification set" is used instead of "bifurcation set".

f^* from $K(f,V)$, where $K(f,V)$ is a finite set. Note that in this case, $K(f,V)$ is equal to $K_0(f,V) \cup K_\infty(f,V)$.

However, as it is well known that if $\mathrm{Sing}(V)$ is not empty, then V may not be a smooth manifold. Assume that V is equidimensional, the nonsingular part M of V is a smooth manifold. Now, we show that if M is not complete, then $B(f,M)$ is not necessarily contained in $K_0(f,M) \cup K_\infty(f,M)$ or $K(f,M)$ by the following examples.

Example 1.2 Consider the real algebraic variety

$$V = \left\{(x,y) \in \mathbb{R}^2 \mid x^4(y^2+1) - y^2(1+y) = 0\right\}.$$

Let f be a polynomial function with $f(x,y) = y$. In this case, $f^* = \inf_{x \in V} f(x) = -1$.

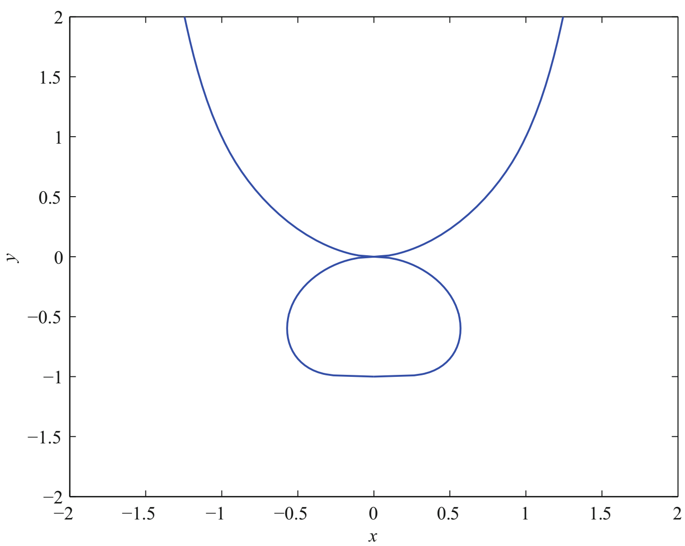

Figure 1 Example 1.2

We have $M = \mathrm{Reg}(V) = V \setminus \{(0,0)\}$ and $K_0(f,M) \cup K_\infty(f,M) = \{-1\}$, while $B(f,M) = \{-1,0\}$. Therefore we have $B(f,M) \not\subset K_0(f,M) \cup K_\infty(f,M)$.

The following example shows that $K(f,M)$ is still not sufficient to describe $B(f,M)$.

Example 1.3 Consider the real variety:

$$V = \left\{(x,y) \in \mathbb{R}^2 \mid x^3 + x^2 - y^2 = 0\right\}.$$

Let $f : V \to \mathbb{R}$ be a polynomial function with $f(x,y) = y$.

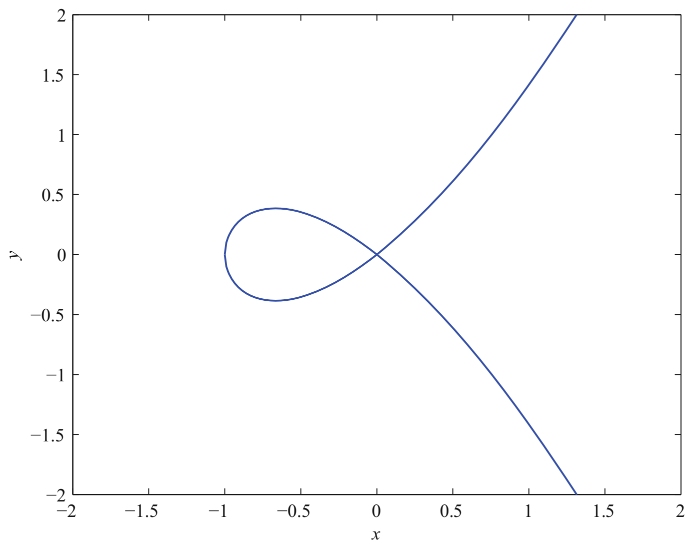

Figure 2 Example 1.3

The singular locus of V is $\mathrm{Sing}\,(V) = \{(0, 0)\}$. Let $M = \mathrm{Reg}\,(V)$, then

$$K\,(f, M) = \left\{a \in \mathbb{R} \mid 27a^2 - 4 = 0\right\} = \left\{\pm\frac{2\sqrt{3}}{9}\right\}.$$

However, we have $B(f, M) = \left\{\pm\frac{2\sqrt{3}}{9}, 0\right\}$, which is not contained in $K\,(f, M)$.

In order to characterize the bifurcation set $B(f, M)$ completely, we need to compute $\mathrm{Sing}\,(V) = \{x \in \mathbb{R}^n \mid h_1(x) = 0, \cdots, h_p(x) = 0\}$, and identify M with the smooth variety

$$\widetilde{V} = \left\{x \in \mathbb{R}^n, t \in \mathbb{R} \mid g_1(x) = 0, \cdots, g_s(x) = 0, (h_1^2 + \cdots + h_p^2)t - 1 = 0\right\}$$

in a higher dimensional space. On the other hand, if we are only interested in computing the infimum of f over V, then it is not necessary to compute $B(f, M)$.

Main contributions. We summarize main results of the paper below.

- Let V be an equidimensional real algebraic variety in $\mathbb{R}^n$, $f : V \to \mathbb{R}^m$ be a polynomial mapping, $M = \mathrm{Reg}\,(V)$ be the smooth part of V. We prove that $K\,(f, M)$ is a closed semi-algebraic set of dimension less than m and satisfies

  $$K\,(f, M) = \bigcup_{(k,j)} (\Gamma(k, j) \cap \mathbb{R}^m),$$

  where $\Gamma(k, j)$ (see (9)) has been defined in [48]. Moreover, $\Gamma(k, j)$ can be computed by using Gröbner bases if $m = 1$. The conclusion that $K\,(f, M)$ has dimension less than m has also been proved in [53], which is based on a notion called v-thin set. Our work is independent of the one given in [53]. Indeed, we prove this conclusion using lifting techniques.

- We prove that if f is a polynomial function, then the infimum $f^* = \inf_{x\in V} f(x)$ exists if and only if it is contained in $K_0\,(f, M)$, or $K_\infty\,(f, M)$, or the closure of $f(\mathrm{Sing}\,(V))$ (in the Euclidean topology). By performing recursive calls on equidimensional components of $\mathrm{Sing}\,(V)$, we obtain a finite set which contains f^*.
- We consider the problem of computing the infimum of a polynomial function over a basic closed semi-algebraic set S. We characterize the boundary of S by all combinations of the polynomials defining S and reduce the problem to solve finitely many polynomial optimization problems over real algebraic sets.

Outline of the paper. In Section 2, we introduce some basic definitions which are related to our work. In Section 3, we show some properties of $K_1\,(f, M)$ and $K\,(f, M)$. In Section 4, we give an algorithm to solve the problem of optimizing a polynomial function over an arbitrary given real algebraic variety. We also estimate the degrees of the output polynomials. In Section 5, we generalize the method to solve polynomial optimization problems over semi-algebraic sets.

2 Preliminaries

Let $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ denote the set of linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Proposition 2.1 [54, Proposition 2.2] *Let $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ and $\Sigma \subset \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ be the set of non-surjective mappings. Then $v(A) = dist(A, \Sigma)$.*

Definition 2.2 [48, Definition 2.3] Let $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ and $H = \{x \in X \mid B_j = \sum_{i=1}^n b_{ji}x_i = 0, j = 1, \cdots, r\}$ be a linear subspace of X with $\dim H = n - r$. Denote by B the coefficients matrix of $B_1, \cdots, B_r$ and identify A with the matrix of A. Assume that $n \geq m + r$. Let C be the $((m + r) \times n)$ matrix given by rows of A and B. For index set $I = (i_1, \cdots, i_{m+r}) \subset \{1, 2, \cdots, n\}$, let M_I denote the $((m + r) \times (m + r))$ minor of C given by columns indexed by I. For $J \subset I$ with $|J| = m + r - 1$, let $M_J(j)$ denote the $((m+r-1) \times (m+r-1)$ minor of C given by columns indexed by J and by deleting the j-th row, where $j \in \{1, 2, \cdots, m\}$. If there exist J and j such that $M_J(j) \neq 0$, we set

$$g'(A, H) = \max_I \left\{ \min_{J\subset I, 1\leq j\leq m} \frac{|M_I|}{|M_J(j)|} \right\}, \tag{6}$$

otherwise, we set $g'(A, H) = 0$.

To simplify notation, we will write $v(A, H)$, $g'(A, H)$ as $v(A)$, $g'(A)$ respectively when there is no confusion about the domain of A. It is shown in [56, Propostions 2.4 and 2.5] that one can replace the function v by g' in the definition of $K\,(f, M)$.

Example 1.2 (continued) Let $g = x^4(y^2 + 1) - y^2(1 + y)$. Recall that $f = y$. The Jacobian matrix of f and g with respect to x, y is

$$\mathrm{Jac}(f, g) = \begin{pmatrix} 0 & 1 \\ 4x^3(y^2 + 1) & 2x^4y - 3y^2 - 2y \end{pmatrix}.$$

Let $A=(0,1), B=(4x^3(y^2+1), 2x^4y-3y^2-2y)$, and $C=\mathrm{Jac}(f,g)$. The corresponding index sets are: $I=(1,2), J=(1)$ or $(2), j=1$. Then

$$g'\left(df(x,y),T_{(x,y)}M\right)=\max_I\left\{\min_{J\subset I}\frac{|M_I|}{|M_J(1)|}\right\}=\min\left\{1,\frac{|4x^3(y^2+1)|}{|2x^4y-3y^2-2y|}\right\}.$$

Therefore, we have $K_0\left(f,M\right)=\{-1\}, K_\infty\left(f,M\right)=\emptyset$ and $K_1\left(f,M\right)=\{0\}$.

Example 1.3 (continued) Let $g=x^3+x^2-y^2$. Recall that $f=y$. The Jacobian matrix of f and g with respect to x,y is

$$\mathrm{Jac}(f,g)=\begin{pmatrix} 0 & 1 \\ 3x^2+2x & -2y\end{pmatrix}.$$

Let $A=(0,1), B=(3x^2+2x,-2y)$, and $C=\mathrm{Jac}(f,g)$. The corresponding index sets are: $I=(1,2), J=(1)$ or $(2), j=1$. Then we have

$$g'\left(df(x,y),T_{(x,y)}M\right)=\max_I\left\{\min_{J\subset I}\frac{|M_I|}{|M_J(1)|}\right\}=\min\left\{1,\frac{|3x^2+2x|}{|2y|}\right\}=\min\left\{1,\frac{|3x^2+2x|}{2\sqrt{x^3+x^2}}\right\}.$$

We derive that

- $K_0\left(f,M\right)=\{y\in\mathbb{R}\mid 3x^2+2x=0,(x,y)\in M\}=\left\{\pm\frac{2\sqrt{3}}{9}\right\}$;
- $K_\infty\left(f,M\right)=\emptyset$ since $\lim_{l\to\infty}g'\left(df(x,y),T_{(x,y)}M\right)=1$ for any $(x_l,y_l)\in M$ such that $\|(x_l,y_l)\|\to\infty$.
- $K_1\left(f,M\right)=\emptyset$ since $\lim_{l\to\infty}g'\left(df(x,y),T_{(x,y)}M\right)=1$ for any $(x_l,y_l)\in M$ such that $(x_l,y_l)\to(0,0)$ (recall that $(0,0)$ is the only point in $\mathrm{Sing}\,(V)$).

Let $V\subset\mathbb{R}^n$ be an equidimensional real algebraic set of dimension d and $\mathbf{I}\,(V)=\langle g_1,\cdots,g_s\rangle$ be the vanishing ideal of V in $\mathbb{R}[X_1,\cdots,X_n]$. The Jacobian matrix of $g_1,\cdots,g_s$ at $x\in V$ is

$$J(x)=\begin{pmatrix}\frac{\partial g_1}{\partial X_1}(x) & \cdots & \frac{\partial g_1}{\partial X_n}(x)\\ \vdots & & \vdots\\ \frac{\partial g_s}{\partial X_1}(x) & \cdots & \frac{\partial g_s}{\partial X_n}(x)\end{pmatrix}.$$

For every $x\in M=\mathrm{Reg}\,(V)$, the matrix $J(x)$ has rank $r=n-d$. Let B be the $(r\times n)$ submatrix given by the first r rows of $J(x)$. Without loss of generality, we may assume that B has full row rank, i.e., $\mathrm{rank}(B)=r$.

Let $f:V\to\mathbb{R}^m$ be a dominant polynomial mapping and

$$C=\begin{pmatrix}df(x)\\ B\end{pmatrix}\in\mathbb{R}^{(m+r)\times n}.$$

Since f is dominant, we have $d\geq m$ and $n\geq m+r$. Given an index set $I=(i_1,\cdots,i_{m+r})\subset\{1,2,\cdots,n\}$, let $M_I(x)$ denote the $(m+r)\times(m+r)$ minor of C given by columns indexed by

I. For every $j \in I$ and $k \in \{1, 2, \cdots, m\}$, we denote by $M_{I(k,j)}(x)$ the $(m+r-1) \times (m+r-1)$ minor obtained by deleting the j-th column and the k-th row of $M_I(x)$. Clearly, $M_I(x)$ and $M_{I(k,j)}(x)$ are polynomial functions on V. Set

$$W_{I(k,j)}(x) = \frac{M_I(x)}{M_{I(k,j)}(x)}, \tag{7}$$

where for $M_{I(k,j)} \equiv 0$ we put $W_{I(k,j)} \equiv 0$.

Let $q = \binom{n}{m+r}$ and $M_{I_1}, \cdots, M_{I_q}$ be all possible $(m+r) \times (m+r)$ minors of C. For every $l \in \{1, 2, \cdots, q\}$, let $k_l \in \{1, 2, \cdots, m\}$ and $j_l \in I_l$, then the pair (k_l, j_l) determines a $(m+r-1) \times (m+r-1)$ minor $M_{I_l(k_l,j_l)}$ of M_{I_l}. We denote the sequence $((k_1, j_1), \cdots, (k_q, j_q))$ by $(k, j) \in \mathbb{N}^q \times \mathbb{N}^q$. There exist l and (k_l, j_l) such that $W_{I_l(k_l,j_l)} \equiv 0$ (since otherwise $g' \equiv 0$ on M, which is impossible). Let (k, j) be a sequence such that $W_{I_l(k_l,j_l)} \equiv 0$ for some (k_l, j_l), we define a rational mapping:

$$\begin{aligned} \Phi_{(k,j)} : V &\dashrightarrow \mathbb{R}^m \times \mathbb{R}^{(n+1)\times q} \\ x &\mapsto (f(x), W_{I_1(k_1,j_1)}(x), x_1 W_{I_1(k_1,j_1)}(x), \cdots, x_n W_{I_1(k_1,j_1)}, \cdots, \\ &\quad W_{I_q(k_q,j_q)}(x), x_1 W_{I_q(k_q,j_q)}(x), \cdots, x_n W_{I_q(k_q,j_q)}(x)). \end{aligned} \tag{8}$$

If the sequence $(k, j) = ((k_1, j_1), \cdots, (k_q, j_q))$ such that all $W_{I_l(k_l,j_l)}(x) \equiv 0$ $(l = 1, 2, \cdots, q)$, then we put $\Phi_{(k,j)}(x) = (0, 0, \cdots, 0)$. Let

$$\Gamma(k, j) = \mathrm{cl}\left(\Phi_{(k,j)}(V)\right), \tag{9}$$

where $\mathrm{cl}(\cdot)$ stands for the closure of a set under the Euclidean topology. If V is smooth, that is, $V = M$, then it is a complete manifold in $\mathbb{R}^n$ and the following equality is an immediate corollary of [48, Lemma 4.3]:

$$K(f, V) = K_0(f, V) \cup K_\infty(f, V) = \bigcup_{(k,j)} (\Gamma(k, j) \cap \mathbb{R}^m), \tag{10}$$

where we identify $\mathbb{R}^m$ with $\mathbb{R}^m \times (0, 0, \cdots, 0)$. According to (10), one can compute $K(f, V)$ by considering each $\Gamma(k, j) \cap \mathbb{R}^m$. But this implementation leads to high computation complexity. The following proposition shows that $K(f, V)$ can be characterized by a simpler implementation.

Proposition 2.3 *Let $V \subset \mathbb{R}^n$ be a smooth equidimensional real algebraic set and $f : V \to \mathbb{R}^m$ be a polynomial mapping. Let M_I, $M_{I(k,j)}$ be defined as above and $\alpha_{I(k,j)}$ be a real number corresponding to $M_{I(k,j)}$. $W_I(x)$ is defined as follows:*

$$W_I(x) = \frac{M_I}{\sum_{k,j} \alpha_{I(k,j)} M_{I(k,j)}}.$$

Define rational mapping Φ as follows:

$$\begin{aligned} \Phi : V &\dashrightarrow \mathbb{R}^m \times \mathbb{R}^{(n+1)\times q} \\ x &\mapsto \left(f(x), W_{I_1}(x), x_1 W_{I_1}(x), \cdots, x_n W_{I_1}(x), \cdots, W_{I_q}(x), x_1 W_{I_q}(x), \cdots, x_n W_{I_q}(x)\right). \end{aligned} \tag{11}$$

Let $\Gamma = \mathrm{cl}\,(\Phi(V))$, then for sufficiently general random numbers $\alpha_{I(k,j)}$, the following equality holds:

$$K\,(f, V) = \Gamma \cap \mathbb{R}^m, \tag{12}$$

where we identify $\mathbb{R}^m$ with $\mathbb{R}^m \times (0, \cdots, 0)$.

Here, "sufficiently general" means that $\alpha_{I(k,j)}$ can be chosen outside a zero-measure set. We refer to the proof of Lemma 4.4 in [48]. Thus in practice, each $\alpha_{I(k,j)}$ is selected randomly.

For an equidimensional real algebraic set V, the definition of Γ and $\Gamma(k, j)$ depend on $\Phi_{(k,j)}$ and Φ. Let $M = \mathrm{Reg}\,(V)$, Γ and $\Gamma(k, j)$ on M are the same as that for V. Combining (10) and (12), $\bigcup_{(k,j)}(\Gamma(k, j) \cap \mathbb{R}^m)$ is equal to $\Gamma \cap \mathbb{R}^m$. Since $\Gamma(k, j)$ and Γ depend only on $\Phi(k, j)$ and Φ, the equality remains true if V is an arbitrary equidimensional real algebraic set. In Section 3, we will prove that for each equidimensional real algebraic set V and $M = Reg(V)$, $K(f, M) = \bigcup_{(k,j)}(\Gamma(k, j) \cap \mathbb{R}^m)$. The following example is to illustrate this conclusion.

Example 2.4 Consider the real algebraic variety $V = \{(x, y) \in \mathbb{R}^2 \mid g(x, y) = 0\}$, where $g = x^2 - y^3$. Let f be a polynomial function with $f(x, y) = x$.

We have $M = \mathrm{Reg}\,(V) = V \setminus \{(0, 0)\}$, $K_0\,(f, M) \cup K_\infty\,(f, M) = \emptyset$, and $K_1\,(f, M) = \{0\}$. The Jacobian matrix of f and g with respect to x, y is

$$\mathrm{Jac}(f, g) = \begin{pmatrix} 1 & 0 \\ 2x & -3y^2 \end{pmatrix}.$$

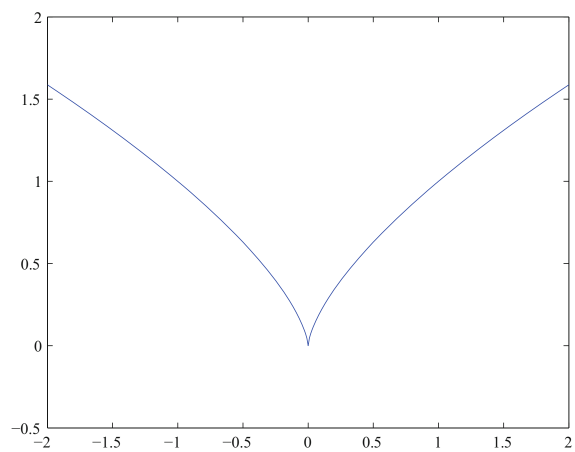

Figure 3 Example 2.4

There is only one index set $I_1 = (1, 2)$ and we can select (k_1, j_1) to be $((1, 1))$ or $((1, 2))$. With simple computation, we have that

$$M_{I_1} = -3y^2,\ \ \Gamma((1, 1)) \cap \mathbb{R} = \emptyset,\ \ \Gamma((1, 2)) \cap \mathbb{R} = \{0\}.$$

Thus in this example, $\bigcup_{(k,j)}(\Gamma(k, j) \cap \mathbb{R}) = K\,(f, M)$.

Definition 2.5 (Locally trivial fibration[57]) Let E, B be two topological spaces and $b \in B$. A continuous mapping $f : E \to B$ is called a locally trivial fibration, or a fiber bundle, with fiber $F = f^{-1}(b)$ if it satisfies the following properties:

(i) $f : E \to B$ is surjective;

(ii) For every point $x \in B$ there is a connected open neighborhood U of x in B and a homeomorphism ϕ such that the following diagram commutes:

$$\begin{tikzcd} F \times U \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & f^{-1}(U) \arrow[dl, "f"] \\ & U & \end{tikzcd}$$

where $\pi : F \times U \to U$ is the canonical projection to U.

The following is a special case of Theorem 6.1 in [47] which builds the connection between generalized critical values and local triviality of f.

Theorem 2.6 [47, Theorem 6.1] *Let $S \subset \mathbb{R}^n$ be a complete manifold, S' be an open subset of S and $f : S \to \mathbb{R}^m$ be a dominant polynomial mapping. Let $W \subset \mathbb{R}^m$ be a connected component of $\mathbb{R}^m \setminus K(f, S')$. Assume that the following condition holds:*

$$\text{There is no sequence } (x_l) \text{ from } S' \text{ such that } \lim_{n\to\infty} x_l \in \partial S' \text{ and } \lim_{n\to\infty} f(x_l) \in f(S'). \tag{13}$$

Then either $f^{-1}(W) = \emptyset$ or $f : f^{-1}(W) \to W$ is a locally trivial fibration.

3 Main Results

Let V be an equidimensional real algebraic set in $\mathbb{R}^n$ of dimension $d > 0$ and $f : V \to \mathbb{R}^m$ be a polynomial mapping. Recall that $\mathrm{Reg}\,(V)$ is the set of nonsingular points in dimension d of V, denote $M = \mathrm{Reg}\,(V)$ and $\mathrm{Sing}\,(V) = V \setminus M$. We first prove that $K_1(f, M)$ is a semi-algebraic set of dimension less than m. Then we show that $K(f, M)$ is exactly equal to $\bigcup_{(k,j)} (\varGamma(k, j) \cap \mathbb{R}^m)$. Finally, we prove that if f is a polynomial function and infimum $f^* = \inf_{x\in V} f(x)$ exists, then it is contained in $K_0(f, M)$ or $K_\infty(f, M)$ or the closure of $f(\mathrm{Sing}\,(V))$ (in the Euclidean topology).

3.1 Properties of $K_1(f, M)$

Let $\mathbf{I}(V) = \langle g_1, \cdots, g_s\rangle$ be the vanishing ideal of V in $\mathbb{R}[X_1, \cdots, X_n]$. Assume that

$$\mathrm{Sing}\,(V) = \{x \in \mathbb{R}^n \mid h_1(x) = 0, \cdots, h_p(x) = 0\},$$

where $h_1, \cdots, h_p \in \mathbb{R}[X_1, \cdots, X_n]$. Thus,

$$M = \{x \in \mathbb{R}^n \mid x \in V, \exists h_i(x) \neq 0, 1 \le i \le p\}. \tag{14}$$

For $i = 1, 2, \cdots, p$ and $k \in \mathbb{Z}_+$, let

$$V_i^k = \left\{(x, t) \in \mathbb{R}^{n+1} \mid g_1(x) = 0, \cdots, g_s(x) = 0, h_i(x)\ t^k = 1\right\}. \tag{15}$$

Then for every odd $k \in \mathbb{N}$, it follows from (14) that

$$M = \bigcup_{i=1}^{p} \pi(V_i^k), \tag{16}$$

where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ denotes the projection to the first n coordinates.

Lemma 3.1 *Let V_i^k be defined as in* (15). *If V_i^k is not empty, then it is an equidimensional smooth real algebraic set of dimension d in $\mathbb{R}^{n+1}$.*

Proof It is sufficient to prove the conclusion holds for V_1^1 and V_1^2. Denote $W = V_1^1$ and assume that $V_1^1 \neq \emptyset$. The semi-algebraic set $S = V \cap \{x \in \mathbb{R}^n \mid h_1(x) \neq 0\}$ is a non-empty open subset of V. Since every $x \in S$ is a nonsingular point in dimension d, we have $\dim S = d$ [49, Proposition 3.3.11]. Considering the mapping:

$$\begin{aligned} \phi : S &\longrightarrow W \\ x &\longmapsto (x, h_1(x)^{-1}), \end{aligned}$$

it is clear that ϕ is a bijective semi-algebraic mapping from S to W, hence $\dim W = \dim S = d$ by [49, Theorem 2.8.8].

We now show that W is equidimensional. Suppose on the contrary that W has an irreducible component W_1 with $\dim W_1 < \dim W = d$. Let W_2 be the Zariski closure of $W \setminus W_1$, then $W = W_1 \cup W_2$ and $W_1 \subset W_2$. Hence, $W_1 \setminus W_2$ is a non-empty open subset of W_1. Thus, for any $z = (x,t) \in W_1 \setminus W_2$, there exists an open neighborhood U_z of z in W_1 such that $U_z \subset W_1 \setminus W_2$. On the other hand, continuity of the mapping ϕ implies that there exists an open neighborhood U_x of x in S such that $\phi(U_x) \subset U_z$. Moreover, $\dim U_x$ is equal to $\dim S$ which is d. This leads to a contradiction:

$$d = \dim U_x = \dim \phi(U_x) \leq \dim U_z \leq \dim W_1 < d.$$

It remains to prove that W is smooth. Let (x,t) be an arbitrary point of W and T be a new variable. The Jacobian matrix of $g_1, \cdots, g_s, h_1T - 1$ at (x,t) is

$$J(x,t) = \begin{pmatrix} \frac{\partial g_1}{\partial X_1}(x) & \cdots & \frac{\partial g_1}{\partial X_n}(x) & 0 \\ \vdots & & \vdots & 0 \\ \frac{\partial g_s}{\partial X_1}(x) & \cdots & \frac{\partial g_s}{\partial X_n}(x) & 0 \\ \frac{\partial h_1}{\partial X_1}(x)t & \cdots & \frac{\partial h_1}{\partial X_n}(x)t & h_1(x) \end{pmatrix}.$$

The rank of the first s rows of $J(x,t)$ is $n-d$ since x is a nonsingular point of V. Then $\operatorname{rank}(J(x,t)) = n - d + 1$ as $h_1(x) \neq 0$. Therefore, (x,t) is a nonsingular point of $W \subset \mathbb{R}^{n+1}$.

For V_1^2, assume that $V_1^2 \neq \emptyset$ and take $S' = V \cap \{x \in \mathbb{R}^n \mid h_1(x) > 0\}$. Consider the following two injective mappings:

$$\begin{aligned} \phi_1 : S' &\longrightarrow V_1^2 \\ x &\longmapsto \left(x, h_1(x)^{-\frac{1}{2}}\right), \end{aligned} \qquad \begin{aligned} \phi_2 : S' &\longrightarrow V_1^2 \\ x &\longmapsto \left(x, -h_1(x)^{-\frac{1}{2}}\right). \end{aligned}$$

Then $V_1^2 = \phi_1(S') \cup \phi_2(S')$ and $\dim V_1^2 = \max\{\dim \phi_1(S'), \dim \phi_2(S')\} = \dim S'$. By similar arguments used for V_1^1, we can prove that V_1^2 is equidimensional and smooth. ∎

Lemma 3.2 *Let $f : V \to \mathbb{R}^m$ be a polynomial mapping. If the norms on $\mathbb{R}^n$ and $\mathbb{R}^m$ are semi-algebraic, then the function $x \mapsto v(df(x))$ is a continuous semi-algebraic function on M.*

Proof It proceeds similarly as the proof of [54, Proposition 2.4]. ∎

Remark 3.3 If V is not a smooth manifold, then the function $x \mapsto v(df(x))$ may not be continuous on V (see Example 1.3).

Lemma 3.4 *Let $f : V \to \mathbb{R}^m$ be a polynomial mapping and $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection to the first n coordinates. Let $\widetilde{f}$ denote the composition $f \circ \pi : V_i^k \to \mathbb{R}^m$. Then for each $y \in K_1(f, M)$, there exists an $i \in \{1, 2, \cdots, p\}$ and $L \in \mathbb{N}$ such that $y \in K_\infty(\widetilde{f}, V_i^L)$.*

Proof Let $y \in K_1(f, M)$, then there exists a sequence (x_l) in M and $\overline{x} \in \mathrm{Sing}(V)$ with $x_l \to \overline{x}$ such that $f(x_l) \to y$ and $v(df(x_l)) \to 0$. From (16), without loss of generality, we assume that (x_l) has an infinite subsequence (x_j) contained in $\pi(V_1^k)$ for any odd $k \in \mathbb{N}$ and a sequence $(x_j, t_j) = \left(x_j, h_1(x_j)^{-\frac{1}{k}}\right) \in V_1^k$ such that

$$\|(x_j, t_j)\| \to \infty, \widetilde{f}(x_j, t_j) \to y \text{ as } j \to \infty. \tag{17}$$

Let $M_1^k = \pi(V_1^k)$, then $\overline{x} \in \mathrm{cl}\left(M_1^k\right)$. By the Curve Selection Lemma [49, Theorem 2.5.5], there exists a continuous semi-algebraic mapping $\gamma : [0, 1] \to \mathbb{R}^n$ such that $\gamma(0) = \overline{x}$ and $\gamma((0, 1]) \subset M_1^k$. Then the function $t \mapsto v(df(\gamma(t)))$ is a continuous semi-algebraic function on $[0, 1]$ due to Lemma 3.2 and the fact that the composition of two semi-algebraic mappings is semi-algebraic (cf. [49, Proposition 2.2.6]). Thus,

$$v(df(\gamma(0))) = v(df(\overline{x})) = 0.$$

That is, $t = 0$ is a zero of the function $v(df(\gamma(t)))$. On the other hand, the function $t \mapsto h_1(\gamma(t))$ is also a continuous semi-algebraic function on $[0, 1]$. Hence,

$$h_1(\gamma(0)) = h_1(\overline{x}) = 0.$$

Indeed, $t = 0$ is the only zero of $h_1(\gamma(t))$ on $[0, 1]$ since $h_1(\gamma(0, 1])$ is contained in $h_1(M_1^k)$ and $0 \notin h_1(M_1^k)$. Hence, on the interval $[0, 1]$, the zero set of $h_1(\gamma(t))$ is contained in that of $v(df(\gamma(t)))$. By Łojasiewicz inequality [49, Corollary 2.6.7], there exist $L_1 \in \mathbb{N}$ and a constant $c > 0$ such that

$$|v(df(\gamma(t)))|^{L_1} \le c \cdot |h_1(\gamma(t))| \text{ on } [0, 1].$$

Let $L = L_1 + 1$, we have

$$|h_1(\gamma(t))|^{-\frac{1}{L}} v(df(\gamma(t))) \to 0 \text{ as } t \to 0,$$

which is equivalent to

$$|h_1(x_j)|^{-\frac{1}{L}} v(df(x_j)) \to 0 \text{ as } j \to \infty. \tag{18}$$

Let $z_j = (x_j, h_1(x_j)^{-\frac{1}{L}}) \in V_1^L$. Since (x_j) is bounded, (18) implies

$$\|z_j\| v(df(x_j)) \to 0 \text{ as } j \to \infty. \tag{19}$$

By (17), we have $\|z_j\| \to \infty, \widetilde{f}(z_j) \to y,$ as $j \to \infty$. We claim that $v(d\widetilde{f}(z_j)) \leq v(df(x_j))$.

Let us identify $\mathbb{R}^n$ as a subspace in $\mathbb{R}^{n+1}$. Since $g_1, \cdots, g_s \in \mathbf{I}\left(V_1^L\right)$, the tangent space of V_1^L at z_j is contained in the tangent space of M at x_j, that is, $T_{z_j}(V_1^L) \subset T_{x_j}(M)$. Let Σ_1 denote the set of non-surjective linear mappings from $T_{z_j}(V_1^L)$ to $\mathbb{R}^m$ and Σ_2 denote the set of non-surjective linear mappings from $T_{x_j}(M)$ to $\mathbb{R}^m$. Then Σ_2 is contained in Σ_1 and the claim $v(d\widetilde{f}(z_j)) \leq v(df(x_j))$ follows from Proposition 2.1. Combing (19) and the claim, we have

$$\|z_j\| v(d\widetilde{f}(z_j)) \to 0 \text{ as } j \to \infty.$$

Note that $y = \lim_{j\to\infty} \widetilde{f}(z_j)$, which concludes that $y \in K_\infty(\widetilde{f}, V_1^L)$. ∎

Theorem 3.5 *Let $f : V \to \mathbb{R}^m$ be a polynomial mapping. Then $K_1(f, M)$ is a semi-algebraic set of Lebesgue measure zero. In particular, it has dimension less than m.*

Proof By Lemma 3.1, every non-empty V_i^k is a smooth real algebraic set. Therefore, it follows from [48, Theorem 3.3] that $K(\widetilde{f}, V_i^k)$ is of Lebesgue measure zero. Finally, according to Lemma 3.4, $K_1(f, M)$ is contained in the union $\bigcup_{i,k} K_\infty(\widetilde{f}, V_i^k)$. Since the union of countable zero-measure sets is still of measure zero, $K_1(f, M)$ is also of Lebesgue measure zero. ∎

Remark 3.6 After we finished this paper, we found that the conclusion in Theorem 3.5 has also been proved in [53]. However, our proof is mainly in the scope of real algebraic geometry, which is different from the one in [53]. Therefore we retain this result here, and the same reason for Corollary 3.8 in the next subsection.

3.2 Properties of $K(f, M)$

It has been shown in [48, Theorem 3.3]) that if V is smooth and equidimensional, then $K(f, V)$ is a closed semi-algebraic set of dimension less than m. Moreover, it follows immediately from [48, Lemma 4.3] that $K(f, V)$ is equal to the set $\bigcup_{(k,j)}(\Gamma(k,j) \cap \mathbb{R}^m)$ (see (10)). In the following, we generalize these results to all equidimensional real algebraic sets.

Theorem 3.7 *Let $\Gamma(k,j)$ be defined as in* (9)*. Then it is true that*

$$K(f, M) = \bigcup_{(k,j)} (\Gamma(k,j) \cap \mathbb{R}^m), \tag{20}$$

where we identify $\mathbb{R}^m$ with $\mathbb{R}^m \times (0, \cdots, 0)$.

Proof By [56, Propostions 2.4 and 2.5], the function v can be replaced by g' in the definition of $K(f, M)$.

We first prove that for every (k, j), the set $\Gamma(k,j) \cap \mathbb{R}^m$ is contained in $K(f, M)$. Let $y \in \Gamma(k,j) \cap \mathbb{R}^m$, then there is a sequence (x_l) in M such that

$$\frac{M_{I_i}(x_l)}{M_{I_i(k_i,j_i)}(x_l)} \to 0, \quad \|x_l\| \frac{M_{I_i}(x_l)}{M_{I_i(k_i,j_i)}(x_l)} \to 0 \text{ as } l \to \infty$$

for all index sets I_i. Assume that (x_l) has a subsequence converging to some $\overline{x}$ in V, then $y = f(\overline{x})$ and $M_{I_i}(\overline{x}) = 0$ for any I_i, therefore, we have $y \in K_0\,(f, M) \cup K_1\,(f, M)$ (either $\overline{x} \in M$ leads to $y \in K_0\,(f, M)$, or $\overline{x} \in \mathrm{Sing}\,(V)$ leads to $y \in K_1\,(f, M)$). Otherwise, we have

$$\|x_l\| \to \infty,\ f(x_l) \to y,\ \|x_l\| g'(df(x_l)) \to 0,\ \text{as } l \to \infty,$$

which implies that $y \in K_\infty\,(f, M)$.

Conversely, if $y \in K\,(f, M)$, the proof breaks up into two cases:

Case I: Suppose that $y \in K_0\,(f, M) \cup K_1\,(f, M)$, then there exists a sequence (x_l) in M with $\lim_{l\to\infty} x_l \in V$ such that $\lim_{l\to\infty} f(x_l) = y$ and $\lim_{l\to\infty} g'(df(x_l)) = 0$. Thus, for all index sets I_i, there exists (k_i, j_i) such that

$$\frac{M_{I_i}(x_l)}{M_{I_i(k_i,j_i)}(x_l)} \to 0.$$

This means $y \in \mathrm{cl}\left(\Phi_{(k,j)}(V)\right) \cap \mathbb{R}^m = \Gamma(k, j) \cap \mathbb{R}^m$, where (k, j) is the sequence $(k_1, j_1), \cdots, (k_q, j_q)$.

Case II: Suppose that $y \in K_\infty\,(f, M)$, then there exists a sequence (x_l) in M such that

$$\|x_l\| \to \infty, \quad f(x_l) \to y, \quad \|x_l\| g'(df(x_l)) \to 0,\ \text{as } l \to \infty.$$

Therefore, $y \in \mathrm{cl}\left(\Phi_{(k,j)}(V)\right) \cap \mathbb{R}^m = \Gamma(k, j) \cap \mathbb{R}^m$. ▮

Corollary 3.8 *Let $f : V \to \mathbb{R}^m$ be a polynomial mapping. Then $K\,(f, M)$ is a closed semi-algebraic set of Lebesgue measure zero. In particular, it has dimension less than m.*

Proof It follows from Theorem 3.7 that $K\,(f, M)$ is a closed semi-algebraic set.

According to Theorem 3.5, $K_1\,(f, M)$ is a semi-algebraic set of Lebesgue measure zero. It remains to prove that $K_0\,(f, M) \cup K_\infty\,(f, M)$ is also of Lebesgue measure zero. Consider the following variety:

$$\widetilde{V} = \left\{x \in \mathbb{R}^n, t \in \mathbb{R} \mid g_1(x) = 0, \cdots, g_s(x) = 0, (h_1^2 + \cdots + h_p^2)t - 1 = 0\right\}.$$

Let $\widetilde{f}$ be the composition $f \circ \pi : \widetilde{V} \to \mathbb{R}^m$. For points $x \in M$ and $z = (x, t) \in \widetilde{V}$, by the same argument used in the final part of the proof of Lemma 3.4, we have $v(d\widetilde{f}(z)) \le v(df(x))$. Thus by definition, $K_0\,(f, M) \cup K_\infty\,(f, M)$ is contained in $K(\widetilde{f}, \widetilde{V})$. By [48, Theorem 3.3], we have that $K(\widetilde{f}, \widetilde{V})$ is of Lebesgue measure zero. This completes the proof. ▮

Remark 3.9 By [47, Lemma 8.1], Theorem 3.5 and Corollary 3.8 remain valid if the real field $\mathbb{R}$ is replaced by the complex field $\mathbb{C}$ and V is replaced by a complex variety.

3.3 Application to Polynomial Optimization Problems

Now let $m = 1$, we consider the problem of optimizing a polynomial function f over an arbitrary given real affine variety V.

Theorem 3.10 *Let $f \in \mathbb{R}[X_1, \cdots, X_n]$, $V \subset \mathbb{R}^n$ be an equidimensional real algebraic set of dimension $d > 0$, and $M = Reg(V)$. Then $\inf_{x\in V} f(x) > -\infty$ if and only if $\inf_{x\in V} f(x)$ is contained in the following set:*

$$C = K_0(f, M) \cup K_\infty(f, M) \cup \mathrm{cl}(f(Sing(V))),$$

Proof Assume that $f^* = \inf_{x\in V} f(x) > -\infty$. Suppose on the contrary that $f^* \notin C$. Let V be the topological space equipped with the subtopology of $\mathbb{R}^n$, then M is an open subset of V (since $\mathrm{Sing}(V)$ is closed in V). Thus, ∂M is contained in $V \setminus M$ which is $\mathrm{Sing}(V)$. Then by Theorem 2.6 [47, Theorem 6.1], there exists an open neighborhood $U = (f^* - \varepsilon, f^* + \varepsilon) \subset \mathbb{R}$ of f^* such that $f : f^{-1}(U) \to U$ is a locally trivial fibration. In another word, there exists a homeomorphism ϕ such that the following diagram

$$\begin{tikzcd} f^{-1}(f^*) \times U \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & f^{-1}(U) \arrow[dl, "f"] \\ & U & \end{tikzcd}$$

commutes. Furthermore, since $f^* \notin \mathrm{cl}(f(\mathrm{Sing}(V)))$, we can take ε small enough so that U does not intersect $\mathrm{cl}(f(\mathrm{Sing}(V)))$. Then we have $f^{-1}(U) \cap \mathrm{Sing}(V) = \emptyset$, which implies that $f^{-1}(U) \subset M$. Hence, for $y = f^* - \frac{\varepsilon}{2} \in U$, there exists $x \in f^{-1}(U) \subset M$ such that $f(x) = y$. This is contradictory to the optimality of f^*. ∎

From Theorem 3.7 and Corollary 3.8, we see that if f is a polynomial function then $K(f, M)$ is a finite set and it is equal to $\bigcup_{(k,j)}(\Gamma(k, j) \cap \mathbb{R})$. Moreover, as we have discussed below Proposition 2.3, the set $\bigcup_{(k,j)}(\Gamma(k, j) \cap \mathbb{R})$ is equal to $\Gamma \cap \mathbb{R}$. Recall that $\Gamma = \mathrm{cl}(\Phi(M))$, where Φ is the rational mapping defined as in (11). It holds that $\Gamma \cap \mathbb{R} = \mathrm{cl}(\Phi(M) \cap \mathbb{R})$. By the finiteness of $\Gamma \cap \mathbb{R}$, we deduce that it is equal to the Zariski closure of $\Phi(V) \cap \mathbb{R}$. Thus, it can be computed via Gröbner bases (see e.g. [58, §3.3 , Theorem 2]). Therefore, we can obtain $K(f, M)$ by computing $\Gamma \cap \mathbb{R}$. Furthermore, by recursive calls on all equidimensional components of $\mathrm{Sing}(V)$, we get a finite set which contains f^* (according to Theorem 3.10). The details of the algorithm for computing f^* will be given in Section 4.

4 The Algorithm

In this section, we give an algorithm to compute the global infimum of a polynomial function restricted to an arbitrary given real algebraic set. We assume that the input of the algorithm are polynomials with rational coefficients and the computations are performed in polynomial rings over the rational field $\mathbb{Q}$. We remark that one may drop this assumption if the subroutines recalled below can be performed in polynomial rings over the real field $\mathbb{R}$.

Let f be a polynomial in $\mathbb{Q}[X] = \mathbb{Q}[X_1, \cdots, X_n]$ and V be the real algebraic set defined by $G = \{g_1, \cdots, g_s\}$, i.e., $V = \{x \in \mathbb{R}^n \mid g_1(x) = 0, \cdots, g_s(x) = 0\}$. We give an algorithm to compute the global infimum $f^* = \inf_{x\in V} f(x)$. The following are several standard subroutines which will be used in our main algorithm.

Subroutine RealRadical. It takes as input a finite set of polynomials $G \subset \mathbb{Q}[X]$ and returns a set of generators of the ideal $\mathbf{I}(V) \subset \mathbb{Q}[X]$, where V is the set of common real zeros of G. For details of this subroutine, we refer to [59–63].

Subroutine EquidimensionalDecomposition. It takes as input a finite set of polynomials $G \subset \mathbb{Q}[X]$ and returns sets of generators of equidimensional components of $\langle G \rangle$ (see e.g. [50]). This subroutine computes complex equidimensional components. Subroutine RealRadical outputs a real radical ideal, so the input of EquidimensionalDecomposition is always a real radical ideal. In this case, the real part of complex equidimensional components coincide with real equidimensional components.

Subroutine Image. Its input consists of polynomials defining a rational mapping $\Phi : V \to \mathbb{R}^N$ and a set of generators of $\mathbf{I}(V)$. Its output is a set of polynomials defining the Zariski closure of $\Phi(V)$. This subroutine is based on the elimination theory and it can be performed by the computation of Gröbner bases (see e.g. [58, §3.3 , Theorem 2]).

Subroutine GenCritValues. It takes as input a polynomial function $f \in \mathbb{Q}[X]$ and a finite set $G \subset \mathbb{Q}[X]$ satisfying that $\langle G \rangle$ is real equidimensional. It returns a finite set $\mathscr{H}$ of the form

$$\{\Psi \mid \Psi \text{ is a finite set of univariate polynomials}\} \tag{21}$$

such that $\bigcup_{\Psi \in \mathscr{H}} \boldsymbol{V}_{\mathbb{R}}(\Psi)$ contains $K(f, M)$, where M is the smooth part of $\boldsymbol{V}_{\mathbb{R}}(G)$. This is done by computing univariate polynomials representing the corresponding set $\Gamma \cap \mathbb{R}$. This subroutine works similarly as the algorithm described in [48, Section 5.1], while here all computations are performed with polynomials with coefficients in $\mathbb{Q}$ rather than in $\mathbb{C}$. We refer to [58, §3.3, Theorem 2] for the correctness of this subroutine.

Subroutine FindInfimum. This subroutine is introduced in [37]. We slightly modify it to adapt our situation. It takes as input a polynomial function $f \in \mathbb{Q}[X]$, a set of polynomials $G \subset \mathbb{Q}[X]$ and a finite set $\mathscr{H}$ of the form as in (21). Assume that V is the real algebraic set defined by G and all local extrema of $f|_V$ are contained in the union $\bigcup_{\Psi \in \mathscr{H}} \boldsymbol{V}_{\mathbb{R}}(\Psi)$. It returns:

- $-\infty$ if f is not bounded below on $\boldsymbol{V}_{\mathbb{R}}(G)$;
- An interval U isolating f^* if f^* is finite;

The subroutine FindInfimum given in [37] assumes that $\langle G \rangle$ is radical and equidimensional and its complex variety has only finitely many singular points. We remark that this assumption does not spoil the adaption here because one can use algorithm in [64–67] to test whether an arbitrary given real algebraic set is empty.

Subroutine $\mathsf{LocalExtEquidim}$**.** It takes as input a polynomial function $f \in \mathbb{Q}[X]$ and a finite set $G \subset \mathbb{Q}[X]$ satisfying that $\langle G \rangle$ is real and equidimensional. Denote by V the real algebraic set of G. It returns a finite set res of the form as in (21) such that all local extrema of $f|_V$ are contained in the union $\bigcup_{\Psi \in \mathsf{res}} \boldsymbol{V}_{\mathbb{R}}(\Psi)$. Let $M = \mathrm{Reg}\,(V)$. As we have seen in the proof of Theorem 3.10, all local extrema of $f|_V$ are contained in the union of $K_0\,(f, M)$, $K_\infty\,(f, M)$, and the closure of $f(\mathrm{Sing}\,(V))$. Therefore, all local extrema of $f|_V$ are contained in the union of $K\,(f, M)$ and $\mathrm{cl}\,(f(\mathrm{Sing}\,(V)))$. Moreover, by Corollary 3.8, the set $K\,(f, M)$ is finite and it can be computed using the subroutine $\mathsf{GenCritValues}$. Therefore, by doing recursive calls on equidimensional components of $\mathrm{Sing}\,(V)$, we can obtain all local extrema of $f|_V$. For computing $\mathrm{Sing}\,(V)$, we denote the Jacobian matrix of G with respect to variables $[X_1, \cdots, X_n]$ by $\mathsf{Jac}(G)$, and the set of all $r \times r$ minors of the matrix $\mathsf{Jac}(G)$ by $\mathsf{Minors}(\mathsf{Jac}(G), r)$.

$\mathsf{LocalExtEquidim}(f, G)$

1. $d = \dim\langle G \rangle$;
2. if $d \leq 0$ then return $\mathsf{Image}(f, G)$;
3. $K_G = \mathsf{GenCritValues}(f, G)$;
4. $\mathsf{res} = \mathsf{res} \cup K_G$;
5. $S_G = \mathsf{RealRadical}(\mathsf{Minors}(\mathsf{Jac}(G), n - d))$;
6. if $\dim S_G \leq 0$, then $\mathsf{res} = \mathsf{res} \cup \mathsf{Image}(f, S_G)$;
7. else,
   - $\{G'_0, \cdots, G'_l\} = \mathsf{EquidimensionalDecomposition}(S_G)$;
   - for $0 \leq i \leq l$ do
     - $\mathsf{res} = \mathsf{res} \cup \mathsf{LocalExtEquidim}(f, G'_i)$;
8. return res.

We now describe the main algorithm. It takes as input a polynomial $f \in \mathbb{Q}[X]$, a finite set of polynomials $G \subset \mathbb{Q}[X]$ and returns the global infimum f^* of f over the real algebraic set defined by G.

$\mathsf{Optimize}(f, G)$

1. $\mathsf{res} = \{\}$;
2. $G = \mathsf{RealRadical}(G)$;
3. if $1 \in G$ then return $+\infty$;
4. if $\dim\langle G \rangle = 0$ then
   - $\mathsf{res} = \mathsf{Image}(f, G)$;

- return $\mathsf{FindInfimum}(f, G, \mathsf{res})$;

5. $\{G_0, \cdots, G_d\} = \mathsf{EquidimensionalDecomposition}(G)$;
6. for $0 \le i \le d$ do
   - $\mathsf{res} = \mathsf{res} \cup \mathsf{LocalExtEquidim}(f, G_i)$;
7. return $\mathsf{FindInfimum}(f, G, \mathsf{res})$.

We prove the correctness and termination of the algorithm Optimize and give an example to illustrate the main steps of the algorithm.

Theorem 4.1 *Let f be a polynomial in $\mathbb{Q}[X_1, \cdots, X_n]$ and V be the real algebraic set defined by polynomials in G. Denote by f^* the global infimum $\inf_{x \in V} f(x)$. The algorithm* Optimize *terminates after a finite number of steps and returns:*

- $+\infty$ *if V is empty;*
- $-\infty$ *if f is not bounded below on V;*
- *An interval U isolating f^* if f^* is finite.*

Proof We assume that V has an equidimensional decomposition: $V = \bigcup_{i=0}^{d} V_i$. Then f^* is contained in $\mathrm{cl}\,(f(V)) = \bigcup_{i=1}^{d} \mathrm{cl}\,(f(V_i))$. Thus f^* is a global infimum for some $f|_{V_i}$. The correctness of Optimize relies on the correctness of LocalExtEquidim and FindInfimum. Below, we focus on proving the correctness of LocalExtEquidim and for that of FindInfimum, we refer to [37].

Now let us assume that the ideal $\langle G \rangle$ is real and equidimensional. We show that the algorithm $\mathsf{LocalExtEquidim}(f, G)$ will terminate after a finite number of steps, and outputs a finite set res of the form (21) such that all local extrema of $f|_V$ are contained in the union $\bigcup_{\Psi \in \mathsf{res}} \boldsymbol{V}_{\mathbb{R}}(\Psi)$.

With the notations in the description of the algorithm $\mathsf{LocalExtEquidim}(f, G)$, $\langle S_G \rangle$ is the vanishing ideal of $\mathrm{Sing}\,(V)$, which implies that $\langle S_G \rangle$ has dimension strictly less than $\dim\langle G \rangle$. Thus for every G_i' appeared in the recursive call, $\langle G_i' \rangle$ has dimension strictly less than $\dim\langle G \rangle$. This concludes the termination of the algorithm.

Next we prove the correctness of this algorithm by induction on the dimension of the ideal $\langle G \rangle$. The case where $\langle G \rangle$ is zero-dimensional is immediate. Hence we assume below that the dimension d of $\langle G \rangle$ is positive and the algorithm is correct on inputs with its second argument defining real algebraic sets of dimension less than d.

Let $y \in \mathbb{R}$ be a local extremum of $f|_V$ and M be the set $\mathrm{Reg}\,(V)$. By the same argument as in the proof of Theorem 3.10, we can show that y is contained in the union of $K\,(f, M)$ and $\mathrm{cl}\,(f(\mathrm{Sing}\,(V)))$. If $y \in K\,(f, M)$ then we are done (see description of GenCritValues and Theorem 3.7). Otherwise, for $i = 0, 1, \cdots, l$, let V_i' be the real algebraic set of G_i', then $\mathrm{Sing}\,(V) = \bigcup_{i=0}^{l} V_i'$ and so y is contained in some $\mathrm{cl}\,(f(V_i'))$. This means y is a local extremum of $f|_{V_i'}$. Since $\langle G_i' \rangle$ has dimension less than d, the correctness of LocalExtEquidim follows by the induction hypothesis. ∎

Example 4.2 Consider the real algebraic set:

$$V = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + (x_2^2 + x_3)^3 = 0\right\}.$$

Let f be the polynomial function: $f(x_1, x_2, x_3) = -x_3$. We want to compute the infimum $\inf_{x \in V} f(x)$.

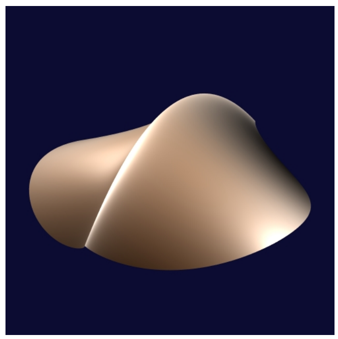

Figure 4 Example 4.2

Let $g = x_1^2 + (x_2^2 + x_3)^3$. Note that V is irreducible over $\mathbb{Q}$. Applying the algorithm $\mathsf{Optimize}(f, G)$ with $G = \{g\}$, we obtain:

- $V_0 = V,\ \mathrm{Sing}\,(V_0) = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = 0, x_2^2 + x_3 = 0\right\}, M_0 = V_0 \setminus \mathrm{Sing}\,(V_0)$,

$$\mathrm{Jac}(f, G) = \begin{pmatrix} 0 & 0 & -1 \\ 2x_1 & 6x_2(x_2^2 + x_3)^2 & 3(x_2^2 + x_3)^2 \end{pmatrix}.$$

Choose index sets $I_l \subset \{1, 2, 3\}$ with $|I_l| = 2$ (so $l = 1, 2, 3$), for $k = 1$ and $j \in I_l$, compute the corresponding rational functions $W_{I_i(k,j)}(x)$ (see (7)):

$$I_1 = (1, 2), \quad M_{I_1} = 0, \quad W_{I_1(1,1)} = W_{I_1(1,2)} = 0;$$
$$I_2 = (1, 3), \quad M_{I_2} = 2x_1, \quad W_{I_2(1,1)} = 2x_1/3(x_2^2 + x_3)^2, \quad W_{I_2(1,3)} = 1;$$
$$I_3 = (2, 3), \quad M_{I_3} = 6x_2(x_2^2 + x_3)^2, \quad W_{I_3(1,2)} = 2x_2, \quad W_{I_3(1,3)} = 1.$$

Since $\mathit{\Gamma} \cap \mathbb{R} = \emptyset$, $K\,(f, M_0) = \emptyset$.

- $V_1 = \mathrm{Sing}\,(V_0)\,, \mathrm{Sing}\,(V_1) = \emptyset, M_1 = V_1$,

$$\mathrm{Jac}(f, G) = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 2x_2 & 1 \end{pmatrix},$$

$$I_1 = (1, 2, 3), \quad M_{I_1} = -2x_2, \quad W_{I_1(1,1)} = 0, \quad W_{I_1(1,2)} = -2x_2, \quad W_{I_1(1,3)} = -1.$$

Since $\mathit{\Gamma} \cap \mathbb{R} = \{0\}$, we have $K\,(f, M_1) = \{0\}$.

Finally, we have $\{x \in \mathbb{R}^3 \mid f(x) = 0\} \cap V = \{(0,0,0)\}$ and $\inf_{x\in V} f(x) = 0$.

The following are two examples taken from the appendix of [37]. Our algorithm can not output the results for Examples 3 and 5 in [37] due to the high complexity.

Example 4.3 (Greuet 2014 Example 1) $f = (x_1x_2 - 1)^2 + x_2^2 + x_3^2 + 42$, $g = x_3$. We obtain $\mathsf{res} = \{\varPsi = \{(a-42)(a-43)\}\}$ and $\inf\{f(x) \mid g(x) = 0\} = 42$.

Example 4.4 (Greuet 2014 Example 2) $f = (x_1^2+x_2^2-2)(x_1^2+x_2^2)$, $g = (x_1^2+x_2^2-1)(x_1-3)$. We obtain $\mathsf{res} = \{\varPsi = \{(a+1)(a-63)\}\}$ and $\inf\{f(x) \mid g(x) = 0\} = -1$.

Our algorithm can compute the global optimum $\inf_{x\in V} f(x)$ by making a recursive call on $\mathrm{Sing}\,(V)$. In the following example, we show that our algorithm does not always return the bifurcation set $B(f, M)$.

Example 4.5 Consider the following surface in $\mathbb{R}^3$:

$$V = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 - x_2^2x_3^2 + x_3^3 = 0\right\}$$

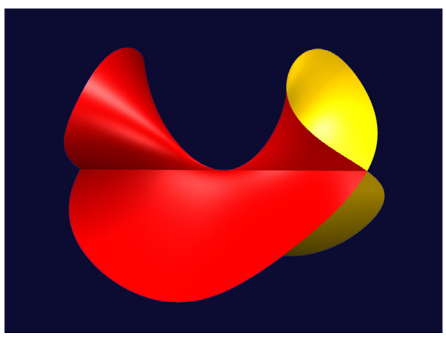

Figure 5 Example 4.5

Let f be the projection from V to the second coordinate, that is, $f(x_1, x_2, x_3) = x_2$. Note that V is irreducible over $\mathbb{Q}$ and $\mathrm{Sing}\,(V) = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = 0, x_3 = 0\right\}$. Taking $M = \mathrm{Reg}\,(V)$, we obtain that

$$K\,(f, M) = \emptyset.$$

Let $V_1 = \mathrm{Sing}\,(V)$ and $M_1 = \mathrm{Reg}\,(V_1)$. Then $M_1 = V_1$ and $f|_{M_1}$ is an identity mapping, thus,

$$K\,(f, M_1) = \emptyset.$$

But $B(f, M) = \{0\}$, since the local triviality of $f|_{M_1}$ cannot be extend to $f|_M$.

Theorem 4.6 *Let f be a polynomial in $\mathbb{Q}[X] = \mathbb{Q}[X_1, \cdots, X_n]$, G be a finite subset of $\mathbb{Q}[X]$. Let $D = \max\{\deg f, \deg g_1, \cdots, \deg g_s\}$ and $d = \dim\langle G\rangle$. Then the univariate polynomials obtained by the end of Step* 6 *in the algorithm* $\mathsf{Optimize}(f, G)$ *have degrees bounded by* $(nD)^{2^{O(dn^2)}}$.

Proof Let V be the real algebraic variety defined by G. If V is finite, then the image of $f(V)$ is finite and has cardinal bounded by D^n. Below we assume that $d > 0$ and retain the notations in the description of $\mathsf{Optimize}$.

The algorithm Optimize starts by calling the subroutine RealRadical. According to [59, Theorem 5.9], the generators of the real radical of $\langle G\rangle$ have degrees bounded by $D^{2^{O(n^2)}}$.

Next, the subroutine EquidimensionalDecomposition is called which results in polynomials with degrees bounded by $(D^{2^{O(n^2)}})^{2^{O(n)}} = D^{2^{O(n^2)}}$ (see e.g. [68]).

Then in the loop, one calls $d+1$ times the subroutine LocalExtEquidim with f, G_i as input for $0 \le i \le d$. Suppose that (f, G') is an input of LocalExtEquidim. Assume that $\langle G'\rangle$ is real and equidimensional of dimension $r > 0$. Let δ denote the maximal degree of polynomials in G'. The first step of LocalExtEquidim calls the subroutine GenCritValues which results in univariate polynomials of degrees bounded by $(D+(\delta-1)(n-r))^r\delta^{n-r}$ [48, Corollary 4.1]. Next one computes the singular locus of $\boldsymbol{V}_{\mathbb{R}}(G')$ using the Jacobian criterion and the subroutine RealRadical. This leads to a set of polynomials with degrees bounded by $((\delta-1)(n-r))^{2^{O(n^2)}}$. Assume that the singular locus obtained in the previous step is still of positive dimension, then one needs to call EquidimensionalDecomposition which will result in polynomials with degrees bounded by $((\delta-1)(n-r))^{2^{O(n^2)}}$ (see analysis for EquidimensionalDecomposition above). After that, one performs recursive calls on all equidimensional components of the singular locus. Let $T(D,\delta,r)$ denote the maximal degree of the output polynomials of LocalExtEquidim(f, G'). Then the following recurrence formula holds:

$$T(D,\delta,r) \le \max\left\{(D+(\delta-1)(n-r))^r\delta^{n-r}, T\left(D, ((\delta-1)(n-r))^{2^{O(n^2)}}, r-1\right)\right\}.$$

Solving this recurrence formula, we obtain that

$$T(D,\delta,d) \le (n\delta)^{2^{O(dn^2)}}.$$

Finally replacing δ by $D^{2^{O(n^2)}}$, which completes the proof. ∎

Remark 4.7 The complexity obtained above depends on the degree bound of generators of real radicals. In [63], it has been shown that if the complex variety of G is smooth, then its real radical has a set of generators with degrees bounded by D^n. Therefore, when the complex variety of G is equidimensional and smooth, then the complexity claimed in Theorem 4.6 can be reduced to

$$(D+(D^n-1)(n-d))^d\, D^{n(n-d)}.$$

For general cases, the complexity given in Theorem 4.6 is quite discouraging. In [63], an algorithm has been given to compute rational parametrizations for real radicals and the complexity of this algorithm is doubly exponential in the dimension of the input system. Thus, we may adapt this algorithm to polynomial optimization problems and reduce the complexity claimed in Theorem 4.6 to be only doubly exponential in the dimension of $\langle G\rangle$, which is left for future work.

5 Generalizations to Semi-Algebraic Cases

The algorithm Optimize can be used in a more general situation. Given a basic closed semi-algebraic set

$$S = \{x \in \mathbb{R}^n \mid g_1(x) \ge 0, \cdots, g_s(x) \ge 0\} \tag{22}$$

with $g_1, \cdots, g_s \in \mathbb{Q}[X_1, \cdots, X_n]$, let $f : S \to \mathbb{R}$ be a polynomial function on S. Consider the following problem:

$$\inf_{x \in S} f(x). \tag{23}$$

The next theorem shows that the optimal value of (23) can be computed effectively by repeatedly using the algorithm Optimize. We introduce some notations first:

- $\Lambda = \{\lambda \mid \lambda \subset \{1, 2, \cdots, s\}\}$;
- For each non-empty index set $\lambda \in \Lambda$, let $V_0^\lambda = \{x \in \mathbb{R}^n \mid g_i(x) = 0, i \in \lambda\}$. If $\lambda = \emptyset$, let $V_0^\lambda = \mathbb{R}^n$.
- For $i \geq 0$ and $\lambda \in \Lambda$, let $V_{i+1}^\lambda = \bigcup_l \mathrm{Sing}\left(V_{il}^\lambda\right)$, where V_{il}^λ is the l-th equidimensional component of V_i^λ;
- For each V_{il}^λ, let $M_{il}^\lambda = \mathrm{Reg}\left(V_{il}^\lambda\right)$ and $S_{il}^\lambda = M_{il}^\lambda \cap \{x \in \mathbb{R}^n \mid g_j(x) > 0, j \notin \lambda\}$ (if $\lambda = \{1, 2, \cdots, s\}$ then take $S_{il}^\lambda = M_{il}^\lambda$).

Theorem 5.1 *With the notations above, we have*

$$f^* = \inf_{x \in S} f(x) \in \bigcup_{\lambda, i, l} \left(K\left(f, M_{il}^\lambda\right) \cup f(V_{i0}^\lambda)\right). \tag{24}$$

Proof Since $S = \bigcup_{\lambda,i,l} S_{il}^\lambda$, we have $f^* = \inf\left\{f(x) \mid x \in S_{il}^\lambda\right\}$ for some fixed $i, l \geq 0$ and $\lambda \in \Lambda$. By arguments similar to the ones used in the proof of Theorem 3.10, we can show that

$$f^* \in K\left(f, S_{il}^\lambda\right) \cup \mathrm{cl}\left(f(\partial S_{il}^\lambda)\right). \tag{25}$$

Since $S_{il}^\lambda = M_{il}^\lambda \cap \{x \in \mathbb{R}^n \mid g_j(x) > 0, j \notin \lambda\}$ and $\{x \in \mathbb{R}^n \mid g_j(x) > 0, j \notin \lambda\}$ is an open subset of $\mathbb{R}^n$, it is straightforward to show that for $l > 0$,

$$K\left(f, S_{il}^\lambda\right) \subset K\left(f, M_{il}^\lambda\right). \tag{26}$$

Moreover, if $l = 0$, then $S_{i0}^\lambda = V_{i0}^\lambda$. Thus, if f^* is contained in $K\left(f, S_{il}^\lambda\right)$ or $f(V_{i0}^\lambda)$ then we are done.

Now we assume that $f^* \in \mathrm{cl}\left(f(\partial S_{il}^\lambda)\right)$, then $f^* \in \mathrm{cl}\left(f(\partial M_{il}^\lambda)\right)$ or $f^* \in \mathrm{cl}\left(f(S_{i_1 l_1}^{\lambda_1})\right)$ for some $\lambda_1 \supsetneq \lambda$ and $i_1, l_1 \geq 0$. If $f^* \in \mathrm{cl}\left(f(\partial M_{il}^\lambda)\right)$, then $f^* \in K\left(f, M_{i_2 l_2}^\lambda\right)$ for some $i_2 > i$ and $l_2 < l$ since $\partial M_{il}^\lambda \subset \mathrm{Sing}\left(V_{il}^\lambda\right)$. Otherwise, the conclusion follows by replacing S_{il}^λ with $S_{i_1 l_1}^{\lambda_1}$ and repeating this process until the index set λ equals to $\{1, 2, \cdots, s\}$. ∎

Example 5.2 Let f be the polynomial function: $f(x, y) = x$ and $S = \{(x, y) \in \mathbb{R}^2 \mid g_1(x, y) \leq 0,\ g_2(x, y) \leq 0\}$ be the semi-algebraic set defined by $g_1 = (x - 1)^2(x^2 + y^2) - 4x^2$ and $g_2 = (x - 1)y^2 - 1$. We show below how to compute the infimum $f^* = \inf_{x \in S} f(x)$.

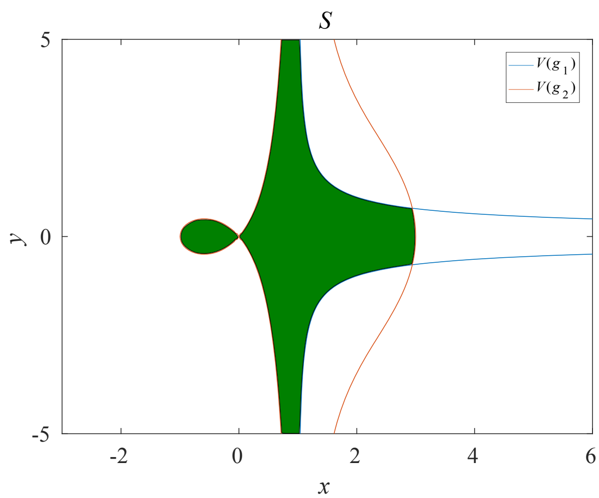

Figure 6 The semi-algebraic set S

Let $\lambda_1 = \emptyset, \lambda_2 = \{1\}, \lambda_3 = \{2\}, \lambda_4 = \{1, 2\}$.

- $\lambda_1 = \emptyset$, $V_0^{\lambda_1} = \mathbb{R}^2$, $M_{02}^{\lambda_1} = \mathbb{R}^2$, $S_{02}^{\lambda_1} = \{(x, y) \in \mathbb{R}^2 \mid g_1(x, y) < 0,\ g_2(x, y) < 0\}$. It is easy to check that
$$K\left(f, M_{02}^{\lambda_1}\right) = \emptyset.$$
- $\lambda_2 = \{1\}$, $V_0^{\lambda_2} = \{(x, y) \in \mathbb{R}^2 \mid g_1(x, y) = 0\}$, $\mathrm{Sing}\left(V_0^{\lambda_2}\right) = \{(0, 0)\}$, $M_{01}^{\lambda_2} = V_0^{\lambda_2} \setminus \{(0, 0)\}$, $S_{01}^{\lambda_2} = \{(x, y) \mid g_1 = 0,\ g_2 < 0\} \setminus \{(0, 0)\}$. We obtain
$$K\left(f, M_{01}^{\lambda_2}\right) = \{-1, 1, 3\},$$
$$V_1^{\lambda_2} = \mathrm{Sing}\left(V_0^{\lambda_2}\right) = \{(0, 0)\}, f\left(V_1^{\lambda_2}\right) = \{0\}.$$
- $\lambda_3 = \{2\}$, $V_0^{\lambda_3} = \{(x, y) \in \mathbb{R}^2 \mid g_2(x, y) = 0\}$, $\mathrm{Sing}\left(V_0^{\lambda_3}\right) = \emptyset$, $M_{01}^{\lambda_3} = V_0^{\lambda_3}$, $S_{01}^{\lambda_3} = \{(x, y) \mid g_1 < 0,\ g_2 = 0\}$. We obtain
$$K\left(f, M_{01}^{\lambda_3}\right) = \{1\}.$$
- $\lambda_4 = \{1, 2\}$, $V_0^{\lambda_4} = \{(x, y) \in \mathbb{R}^2 \mid g_1(x, y) = 0, g_2(x, y) = 0\}$, $\mathrm{Sing}\left(V_0^{\lambda_4}\right) = \emptyset$, $M_{00}^{\lambda_4} = V_0^{\lambda_4}$, $S_{00}^{\lambda_4} = V_0^{\lambda_4}$. Since $V_0^{\lambda_4}$ is finite, we only need to compute the image $f(V_0^{\lambda_4})$. We obtain the following univariate polynomial:
$$\psi = a^4 - 2a^3 - 3a^2 + a - 1.$$
The equation $\psi = 0$ has two real roots and one of which is less than -1. However by the algorithm FindInfimum we obtain that $f^* = -1$. This is because although the ideal $\langle g_1, g_2 \rangle$ is real in $\mathbb{Q}[x, y]$, it has complex points in $\mathbb{C}^2$. Thus, $\boldsymbol{V}_{\mathbb{R}}(\psi) \supsetneq f(V_0^{\lambda_4})$ and the smallest real root of the equation is not contained in $f(V_0^{\lambda_4})$.

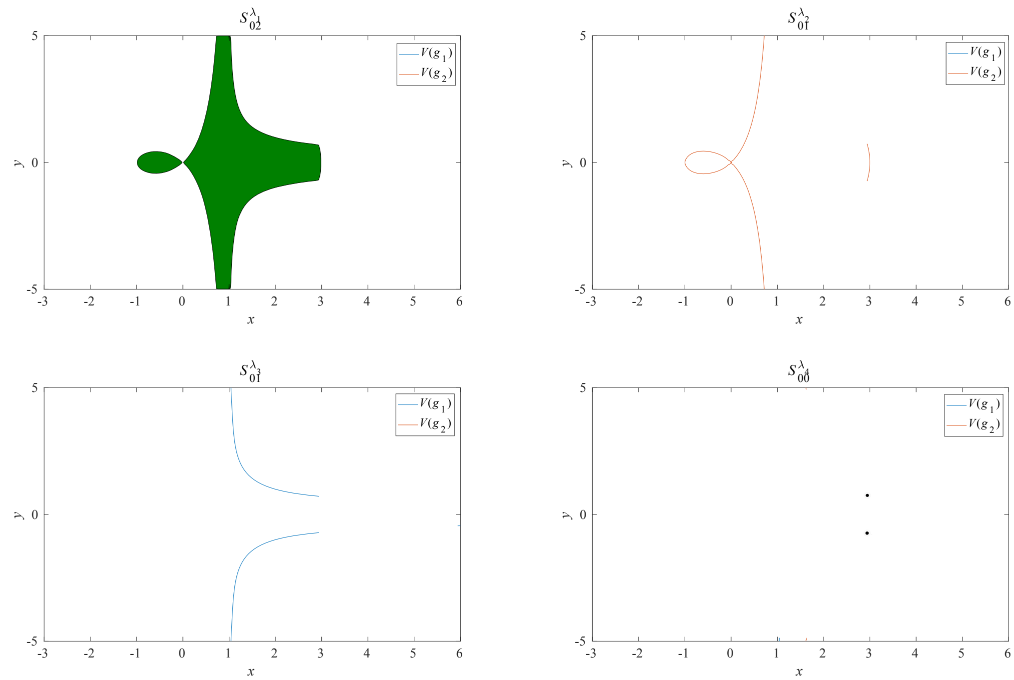

Figure 7 The semi-algebraic sets for λ_i

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