

Lifts of Non-compact Convex Sets and Cone Factorizations

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Abstract

In this paper we generalize the factorization theorem of Gouveia, Parrilo and Thomas to a broader class of convex sets. Given a general convex set, we define a slack operator associated to the set and its polar according to whether the convex set is full dimensional, whether it is a translated cone and whether it contains lines. We strengthen the condition of a cone lift by requiring not only the convex set is the image of an affine slice of a given closed convex cone, but also its recession cone is the image of the linear slice of the closed convex cone. We show that the generalized lift of a convex set can also be characterized by the cone factorization of a properly defined slack operator.

Key words: lift; convex set; recession cone; polyhedron; cone factorization; nonnegative rank; positive semidefinite rank.

1 Introduction

Given a linear programming problem, how to reformulate it to a standard form with fewer constraints is an important problem. In [11], Yannakakis proved that the nonnegative rank of a slack matrix of a polytope P is the minimum number k such that P is the linear image of an affine slice of the nonnegative quadrant \mathbb{R}_+^k . In [3, 6], Yannakakis's result was generalized to decide whether a convex body C (a compact convex set containing the origin in its interior) is the linear image of an affine slice of a given convex cone K (K -lift) via cone factorizations of slack operators. Although it was claimed that results in [6] hold for all convex sets, we notice that it is more complicated to identify whether a non-compact convex set C containing no lines has a K -lift since C could be generated by not only extreme points but also extreme directions. Moreover, if a convex set contains lines, then it has no extreme points or extreme

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directions. Furthermore, if the convex set does not contain the origin in its interior, linear functions corresponding to its polar cannot characterize the convex set completely (see Example 3.1). These facts motivate us to study how to extend the definitions of K -lift and the slack operator to a general convex set and show the relationship between lifts of convex sets and cone factorizations of slack operators when the convex set is not a convex body.

Our contribution: Let C be a closed convex set in \mathbb{R}^n and K a closed convex cone in \mathbb{R}^m . We consider how to generalize the factorization theorem in [6] to a broader class of convex sets. Our main results are as follows.

- When C contains the origin, since $C = C^{oo}$, C can be described by all vectors in C^o . When C does not contain the origin, we show that a convex set C can be characterized completely by vectors in C^o , $0^+C^o = \{l \mid \langle l, x \rangle \leq 0, \forall x \in C\}$ and $C_3 = \{l \mid \langle l, x \rangle \leq -1, \forall x \in C\}$.
- We extend the question of when a given convex body C is the linear image of an affine slice of a convex cone to the case where C is not compact and may not contain the origin in its interior and may contain lines. We introduce two ways to characterize the existence of a K -lift of C . The first one is based on all points in C and the second one uses only extreme points, extreme directions and an orthogonal basis of the lineality space of C if it contains lines. Although the first method can be used to check the existence of a K -lift of any convex set, it is difficult to use, see Remark 3.10. Therefore, in the paper, we focus on the second method. We extend Definition 1,2 and Theorem 1 in [6] to a broader class of convex sets and show that the generalized lift of a convex set can also be characterized by the cone factorization of properly defined slack operator according to whether the convex set is full dimensional, whether it is a translated cone and whether it contains a line.
- We specialize the results of the cone lift of general convex sets to polyhedra and show that the conclusion can be strengthened when C and K are both polyhedra. When K is a positive semidefinite convex cone, we give a lower bound on the positive semidefinite rank of a polyhedron, which generalizes the result in [7]. We also extend results in [4, 5, 7] to identify whether a given nonnegative matrix is a slack matrix of a polyhedron and characterize the rank of a slack matrix in terms of the dimension of a polyhedron.

The paper is organized as follows. In Section 2, we provide some preliminaries about convex sets and cones. Some well-known results in convex analysis are recalled. In Section 3, we generalize the factorization theorem in [6, Theorem 1] to convex sets which are not convex bodies. In Section 4, we specialize results established in Section 3 to the case where the convex set is a polyhedron. Some results in [4, 5, 7] on the positive semidefinite rank of a slack matrix are extended to the case that the convex set is a polyhedron.

2 Preliminary

In this section, we recall some definitions and known results about convex sets in [9]. Let \mathbb{R}^n be a n -dimensional linear space, \mathcal{S}^m the space of real symmetric $m \times m$ matrices. A non-empty

subset $C \subset \mathbb{R}^n$ is said to be *convex* if $(1 - \lambda)x + \lambda y \in C$ whenever $x \in C$, $y \in C$ and $0 < \lambda < 1$. We denote $\text{cl}(C)$ and $\text{int}(C)$ as the *closure* and *interior* of C respectively. The *affine hull* of a convex set C , denoted by $\text{aff}(C)$, is the unique smallest affine set containing C . If a closed convex set C is compact and contains the origin in its interior, it is called a *convex body*.

A subset K of \mathbb{R}^n is called a *cone* if it is closed under nonnegative scalar multiplication, i.e. $\lambda x \in K$ when $x \in K$ and $\lambda \geq 0$. We denote the m -dimensional nonnegative quadrant by \mathbb{R}_+^m and the cone of $m \times m$ real symmetric positive semidefinite (psd) matrices by \mathcal{S}_+^m . A convex cone K is *pointed* if it is closed and $K \cap -K = \{0\}$. The *polar* of a non-empty convex cone K is defined as

$$K^\circ = \{x \in \mathbb{R}^n \mid \forall y \in K, \langle x, y \rangle \leq 0\}.$$

Given a set C , if there exists a cone C_0 and a vector $x \in \mathbb{R}^n$ such that $C = x + C_0$, then C is said to be a *translated cone*.

The *recession cone* 0^+C of a non-empty convex set C is the set including all vectors y satisfying $x + \lambda y \in C$ for every $\lambda \geq 0$ and $x \in C$. We say that C *recedes* in the direction of y . The set $0^+C \cap (-0^+C)$ is called the *lineality space* of C .

Let S_0 be a set of points in \mathbb{R}^n and S_1 a set of directions in \mathbb{R}^n . We define the *convex hull* $\text{co}(S)$ of $S = S_0 \cup S_1$ to be the smallest convex set C in \mathbb{R}^n such that $C \supseteq S_0$ and C recedes in all directions in S_1 .

Algebraically, a vector x belongs to $\text{co}(S)$ if and only if it can be expressed in the form

$$x = \lambda_1 x_1 + \cdots + \lambda_k x_k + \lambda_{k+1} x_{k+1} + \cdots + \lambda_m x_m, \quad \sum_{i=1}^k \lambda_i = 1,$$

where x_1, \dots, x_k are vectors in S_0 and x_{k+1}, \dots, x_m are vectors whose directions are in S_1 and $\lambda_i \geq 0$ for $1 \leq i \leq m$. If $S_0 = \{0\}$ and S_1 is not empty, then $C = \text{co}(S_0 \cup S_1)$ is a cone which is also denoted as $\text{cone}(S_1)$.

The *relative interior* $\text{ri}(C)$ of a convex set C in \mathbb{R}^n is defined as the interior when C is regarded as a subset of its affine hull $\text{aff}(C)$. A *face* of a convex set C is a convex subset C' of C such that every closed line segment in C with a relative interior in C' has both endpoints in C' . The zero-dimensional faces of C are called the *extreme points* of C . If C' is a half-line face of a convex set C , we shall call the direction of C' an *extreme direction* of C . If C is a convex cone, an *extreme ray* is a face which is a half-line emanating from the origin. Note that every extreme direction of a closed convex set C can also be regarded as an extreme ray of 0^+C . Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be two vectors in \mathbb{R}^n , the *inner product* of x, y in \mathbb{R}^n is expressed by $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$.

The *polar* of a non-empty convex set $C \subset \mathbb{R}^n$ is a closed convex set defined as

$$C^\circ = \{x \in \mathbb{R}^n \mid \forall y \in C, \langle x, y \rangle \leq 1\}.$$

We have $C^{\circ\circ} = \text{cl}(\text{co}(C \cup \{0\}))$.

The *indicator function* $\delta(\cdot, C)$ is defined by

$$\delta(x, C) = \begin{cases} 0 & \text{if } x \in C, \\ +\infty & \text{if } x \notin C. \end{cases}$$

The *support function* $\delta^*(x, C)$ of a convex set $C \in \mathbb{R}^n$ is defined by

$$\delta^*(x, C) = \sup\{\langle x, y \rangle \mid y \in C\}.$$

The *barrier cone* of C is defined as the effective domain of $\delta^*(x, C)$:

$$\text{dom}\delta^*(x, C) = \{x \mid \delta^*(x, C) < +\infty\}.$$

A convex function f is said to be *proper* if its epigraph is non-empty and contains no vertical lines, i.e. if $f(x) < +\infty$ for at least one x and $f(x) > -\infty$ for every x .

Theorem 2.1 [9, Theorem 8.3] *Let C be a non-empty closed convex set, and let $y \neq 0$. If there exists even one $x \in C$ such that the half-line $\{x + \lambda y \mid \lambda \geq 0\}$ is contained in C , then the same thing is true for every $x \in C$, i.e. one has $y \in 0^+C$.*

Theorem 2.2 [9, Theorem 18.5] *Let C be a closed convex set containing no lines, and let S be the set of all extreme points and extreme directions of C . Then $C = \text{co}(S)$.*

Theorem 2.3 [9, Theorem 8.7] *Let f be a closed proper convex function. Then all the non-empty level sets of the form $\{x \mid f(x) \leq \alpha\}$, $\alpha \in \mathbb{R}$, have the same recession cone and the same lineality space.*

Corollary 2.4 [9, Corollary 14.2.1] *The polar of the barrier cone of a non-empty closed convex set C is the recession cone of C .*

Theorem 2.5 [9, Theorem 13.1] *Let C be a convex set. Then $x \in \text{cl}(C)$ if and only if $\langle x, x^* \rangle \leq \delta^*(x, C)$ for every vector x^* .*

3 Cone lifts of non-compact convex sets

Let $\text{ext}_1(C)$ denote the set of extreme points of a closed convex set C and $\text{ext}_2(C)$ the set of extreme rays of a closed convex cone C . An extreme ray is also the common direction of vectors in this ray. In the following part of our paper, we represent each extreme ray by one vector and denote $\text{ext}_2(C)$ as the collection of such vectors.

If C is a compact convex set containing the origin in its interior, according to [6, Definition 1], a *K-lift* of $C \subset \mathbb{R}^n$ is a set $Q = K \cap L$ where $L \subset \mathbb{R}^m$ is an affine subspace and $\pi : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a linear map such that

$$C = \pi(K \cap L). \quad (3.1)$$

If L intersects the interior of K , we say that Q is a *proper K-lift* of C . Define the operator $S : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ to be $S(x, y) = 1 - \langle x, y \rangle$. The Slack operator S_C of C is the restriction of S to $\text{ext}_1(C) \times \text{ext}_1(C^\circ)$. The slack operator S_C is *K-factorizable* if there exist maps

$$A : \text{ext}_1(C) \rightarrow K, \quad B : \text{ext}_1(C^\circ) \rightarrow K^*$$

such that $S_C(x, y) = \langle A(x), B(y) \rangle$ for all $(x, y) \in \text{ext}_1(C) \times \text{ext}_1(C^\circ)$, see [6, Definition 2].

In this section, we explain how to generalize the argument in [6] to more general convex sets and show the relationship between cone lifts of convex sets and cone factorizations of slack operators when the closed convex set is not a convex body.

3.1 C is full dimensional

Assume that C is a full dimensional closed convex set in \mathbb{R}^n , we define

$$C^\circ = \{x \mid \delta^*(x, C) \leq 1\}, 0^+C^\circ = \{x \mid \delta^*(x, C) \leq 0\}, C_3 = \{x \mid \delta^*(x, C) \leq -1\}.$$

It is clear that C°, C_3 are closed convex sets containing no lines and 0^+C° is a closed pointed cone that contains C_3 . Let

$$D_1 = \text{ext}_1(C^\circ) \setminus \{0\}, D_2 = \text{ext}_2(0^+C^\circ) \cap \{x \mid \delta^*(x, C) = 0\}, D_3 = \text{ext}_1(C_3).$$

By Theorem 2.3, we have $0^+C^\circ = 0^+C_3$. Let

$$D_{32} = \text{ext}_2(0^+C^\circ) \cap \{x \mid \delta^*(x, C) = -1\}.$$

It is clear that $D_{32} \subseteq D_3$ but D_{32} is not always equal to D_3 .

Example 3.1 We consider a compact convex set

$$C = \{(x, y) \mid x + y \geq 1, x + y \leq 3, y - x \geq -1, y - x \leq 1\}.$$

Then, we have

$$\begin{aligned} C^\circ &= \{(x, y) \mid 2x + y \leq 1, x + 2y \leq 1, x \leq 1, y \leq 1\}, \\ 0^+C^\circ &= \{(x, y) \mid x \leq 0, y \leq 0\}, \\ C_3 &= \{(x, y) \mid x \leq -1, y \leq -1\}, \end{aligned}$$

see Figure 1. Furthermore, we have $D_{32} = \emptyset$ and

$$D_1 = \{(-1, 1), (\frac{1}{3}, \frac{1}{3}), (1, -1)\}, D_2 = \{(-1, 0), (0, -1)\}, D_3 = \{(-1, -1)\}.$$

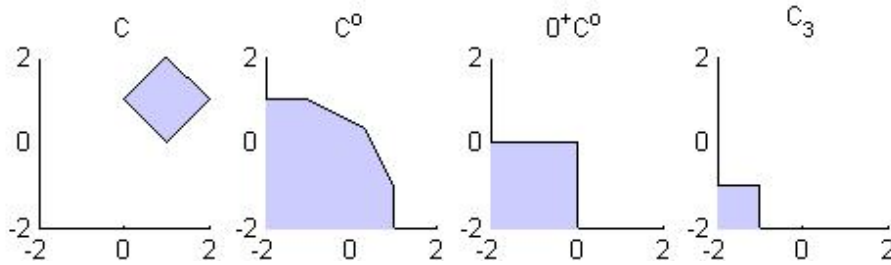


Figure 1: Example 3.1

Remark 3.2 According to Theorem 2.2 and 2.3, we can show:

1. The convex set C° can be expressed as convex combinations of all points in D_1 and all directions of vectors in D_2 and D_{32} .
2. The convex cone 0^+C° can be expressed as convex combinations of all directions of vectors in D_2 and D_{32} .
3. The convex set C_3 can be expressed as convex combinations of all points in D_3 and all directions of vectors in D_2 and D_{32} .

Theorem 3.3 *Given a full dimensional closed convex set $C \subset \mathbb{R}^n$, the following statements are true:*

1. *The set D_1 is empty if and only if C° is a closed cone. If D_1 is not empty, for every vector x in D_1 , we have $\delta^*(x, C) = 1$.*
2. *The set D_2 is empty if C contains the origin in its interior. When C is not compact and contains the origin on its boundary, D_2 is not empty and each extreme ray of 0^+C° is the direction of a vector in D_2 .*
3. *The set D_3 is empty if and only if C contains the origin. If D_3 is not empty, for every vector x in D_3 , $\delta^*(x, C) = -1$.*

Moreover, the convex cone generated by C° is $\text{dom}\delta^*(x, C)$.

Proof Since C° contains no lines, D_1 is empty if and only if the origin is the only extreme point of C° , i.e. C° is a closed cone. If there exists an extreme point $x \in D_1$ such that $\delta^*(x, C) < 1$, then there exists $\lambda > 0$ such that $\delta^*((1 + \lambda)x, C) \leq 1$ and $\delta^*((1 - \lambda)x, C) \leq 1$. So $(1 - \lambda)x$ and $(1 + \lambda)x$ are both in C° which contradicts to the fact that x is an extreme point of C° .

When C contains the origin in its interior, C° is compact and 0^+C° contains only zero vector. Hence, D_2 is empty. If C contains the origin, for every x in $\text{dom}\delta^*(x, C)$, we have $\delta^*(x, C) \geq 0$. If the origin is on its boundary, there exists a supporting hyperplane of C through the origin. So C° is not compact and 0^+C° contains a nonzero vector. Combined with the fact that $\delta^*(x, C) = 0$ for all x in 0^+C° , D_2 can represent all extreme rays of 0^+C° .

It is clear that C_3 is empty if and only if $\delta^*(y, C) \geq 0$ for all $y \in \mathbb{R}^n$. By Theorem 2.5, this is equal to say that C contains the origin. Therefore, D_3 is empty if and only if C contains the origin. Similar arguments can be used to show $\delta^*(x, C) = -1$ for every vector x in D_3 .

For every $x \in \text{cone}(C^\circ)$, there exists $\lambda \geq 0$ and $y \in C^\circ$ such that $x = \lambda y$. So $\delta^*(x, C) = \lambda \delta^*(y, C) < \infty$ and $x \in \text{dom}\delta^*(x, C)$. On the other hand, for each $x \in \text{dom}\delta^*(x, C)$, if $\delta^*(x, C) = M > 0$, then x/M is in C° and x is in $\text{cone}(C^\circ)$. Hence $\text{cone}(C^\circ) = \text{dom}\delta^*(x, C)$.

Remark 3.4 When C does not contain the origin, it is not easy to identify whether the set D_2 is empty. The convex set C in Example 3.1 does not contain the origin, $D_2 = \{(-1, 0), (0, -1)\}$. However, for the convex set C defined by $C = \{(x, y) \mid y \geq x+1, y \geq -x+1\}$, we have $C^\circ = 0^+C^\circ = \{(x, y) \mid x + y \leq 0, y - x \leq 0\}$. The extreme rays of 0^+C° are

$l_1 = (-1, -1)$ and $l_2 = (1, -1)$. We have $\delta^*(l_1, C) = \delta^*(l_2, C) = -1 < 0$. Hence, the set D_2 is empty.

When C contains the origin in its interior, by Theorem 3.3, D_2 and D_3 are empty and C can be characterized by D_1 alone. However, when C does not contain the origin in its interior, as shown by the following example, the linear functions with coefficients in D_1 or $D_1 \cup D_2$ cannot characterize C completely.

Example 3.1 (continued) In this example, every linear function $f(x) = \langle l_1, x \rangle$ where $l_1 \in D_1$ has maximal value 1 on C , therefore,

$$E_1 = \{(x, y) \mid c_1x + c_2y \leq 1, (c_1, c_2) \in D_1\} = \{(x, y) \mid -x + y \leq 1, x - y \leq 1, x + y \leq 3\}.$$

The linear function $f(x) = \langle l_2, x \rangle$ where $l_2 \in D_2$ has maximal value 0 on C and

$$E_2 = \{(x, y) \mid c_1x + c_2y \leq 0, (c_1, c_2) \in D_2\} = \{(x, y) \mid x \geq 0, y \geq 0\}.$$

The linear function $f(x) = \langle l_3, x \rangle$ where $l_3 \in D_3$ has maximal value -1 on C , hence,

$$E_3 = \{(x, y) \mid c_1x + c_2y \leq -1, (c_1, c_2) \in D_3\} = \{(x, y) \mid x + y \geq 1\}.$$

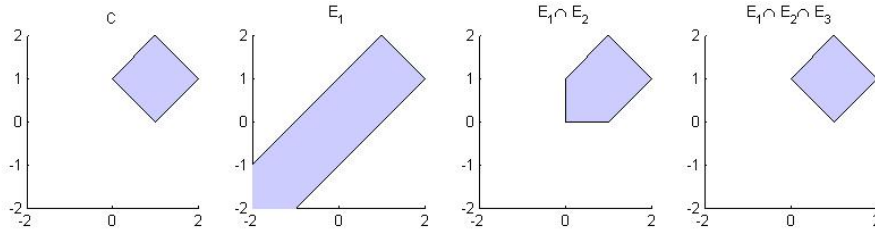


Figure 2: Example 3.1

We show below that a full dimensional closed convex set C can be characterized completely by elements in D_1, D_2 and D_3 .

Theorem 3.5 *Let $C \subset \mathbb{R}^n$ be a full dimensional closed convex set. Then we have*

$$x \in C \iff \begin{cases} \langle l_1, x \rangle \leq 1 & \forall l_1 \in D_1, \\ \langle l_2, x \rangle \leq 0 & \forall l_2 \in D_2, \\ \langle l_3, x \rangle \leq -1 & \forall l_3 \in D_3. \end{cases} \quad (3.2)$$

Proof Since C is full dimensional, C° contains no lines and D_1, D_2 and D_3 are well defined. The necessity is clear. Suppose on the other hand that x satisfies the conditions on the right hand side of (3.2), we shall show that $x \in C$. By Theorem 2.5, it is enough to show that $\langle l, x \rangle \leq \delta^*(l, C)$ for every $l \in \text{dom}\delta^*(x, C)$. Let $l_0 = \delta^*(l, C)$, we prove that $\langle l, x \rangle \leq l_0$ in all three cases below:

- If $l_0 > 0$, then we have $l/l_0 \in C^\circ$. By Remark 3.2, there exist $\lambda_i^1 \geq 0$, $\lambda_j^2 \geq 0$, $\lambda_k^3 \geq 0$ and $x_i \in D_1$, $y_j \in D_2$, $z_k \in D_{32}$ satisfying the following equality:

$$l/l_0 = \sum_i \lambda_i^1 x_i + \sum_j \lambda_j^2 y_j + \sum_k \lambda_k^3 z_k, \quad \sum_i \lambda_i^1 = 1.$$

According to the definitions of D_1, D_2, D_3 , we have

$$\langle l/l_0, x \rangle = \sum_i \lambda_i^1 \langle x_i, x \rangle + \sum_j \lambda_j^2 \langle y_j, x \rangle + \sum_k \lambda_k^3 \langle z_k, x \rangle \leq \sum_i \lambda_i^1 = 1.$$

- If $l_0 = 0$, then $l \in 0^+ C^\circ$. By Remark 3.2, there exist $\lambda_j^2 \geq 0$, $\lambda_k^3 \geq 0$ and $y_j \in D_2$, $z_k \in D_{32}$ satisfying the following equality:

$$l = \sum_j \lambda_j^2 y_j + \sum_k \lambda_k^3 z_k.$$

$$\text{So } \langle l, x \rangle = \sum_j \lambda_j^2 \langle y_j, x \rangle + \sum_k \lambda_k^3 \langle z_k, x \rangle \leq 0.$$

- If $l_0 < 0$, then $l/|l_0| \in C_3$. By Remark 3.2, there exist $\lambda_i^1 \geq 0$, $\lambda_j^2 \geq 0$, $\lambda_k^3 \geq 0$ and $x_i \in D_3$, $y_j \in D_2$, $z_k \in D_{32}$ satisfying the following equality:

$$l/|l_0| = \sum_i \lambda_i^1 x_i + \sum_j \lambda_j^2 y_j + \sum_k \lambda_k^3 z_k, \quad \sum_i \lambda_i^1 = 1.$$

$$\text{So } \langle l/|l_0|, x \rangle = \sum_i \lambda_i^1 \langle x_i, x \rangle + \sum_j \lambda_j^2 \langle y_j, x \rangle + \sum_k \lambda_k^3 \langle z_k, x \rangle \leq \sum_i -\lambda_i^1 = -1.$$

Theorem 3.6 *Suppose $C \subset \mathbb{R}^n$ is a full dimensional closed convex set. If there exists $x \in \mathbb{R}^n$ such that $1 - \langle l_1, x \rangle = 0$, $\forall l_1 \in D_1$, $-\langle l_2, x \rangle = 0$, $\forall l_2 \in D_2$, $-1 - \langle l_3, x \rangle = 0$, $\forall l_3 \in D_3$, then x is the only extreme point of C and C is a translated convex cone.*

Proof By Theorem 3.5, x is in C . By Theorem 3.3 and Remark 3.2, for every $l \in \text{dom} \delta^*(x, C)$, there exist $\lambda_i^1 \geq 0$, $i = 1, \dots, i_1$, $\lambda_j^2 \geq 0$, $j = 1, \dots, j_2$ and $\lambda_k^3 \geq 0$, $k = 1, \dots, k_3$ such that $l = \sum_{i=1}^{i_1} \lambda_i^1 x_i + \sum_{j=1}^{j_2} \lambda_j^2 y_j + \sum_{k=1}^{k_3} \lambda_k^3 z_k$ for $x_i \in D_1$, $y_j \in D_2$ and $z_k \in D_{32}$. Since for every $y_j \in D_2$, $\delta^*(y_j, C) = 0$, we have the following inequality:

$$\begin{aligned} \delta^*(l, C) &\leq \sum_{i=1}^{i_1} \lambda_i^1 \delta^*(x_i, C) + \sum_{k=1}^{k_3} \lambda_k^3 \delta^*(z_k, C) = \sum_{i=1}^{i_1} \lambda_i^1 - \sum_{k=1}^{k_3} \lambda_k^3 \\ &= \sum_{i=1}^{i_1} \lambda_i^1 \langle x_i, x \rangle + \sum_{k=1}^{k_3} \lambda_k^3 \langle z_k, x \rangle = \langle l, x \rangle. \end{aligned}$$

Furthermore, it is clear that $\delta^*(l, C) \leq \langle l, x \rangle + \delta(l, \text{cl}(\text{dom} \delta^*(x, C)))$ for every $l \in \mathbb{R}^n$. Take closure for both sides, we have:

$$\begin{aligned} \delta^*(l, C) &\leq \langle l, x \rangle + \delta(l, \text{cl}(\text{dom} \delta^*(x, C))) \\ &= \langle l, x \rangle + \delta(l, (0^+ C)^\circ) \quad (\text{by Corollary 2.4}) \\ &= \delta^*(l, x + 0^+ C). \end{aligned}$$

This implies that $C \subseteq x + 0^+C$. On the other hand, since $x \in C$, we have $x + 0^+C \subseteq C$. Therefore, $C = x + 0^+C$, i.e. C is a translated convex cone and contains x as the only extreme point.

3.1.1 C contains no lines

When $C \subset \mathbb{R}^n$ is a full dimensional closed convex set and contains no lines, we set the slack operator S_C to be

$$S_C = \begin{cases} S_C^1(x, y) = 1 - \langle x, y \rangle & \text{for } (x, y) \in C \times D_1, \\ S_C^2(x, y) = -\langle x, y \rangle & \text{for } (x, y) \in C \times D_2, \\ S_C^3(x, y) = -1 - \langle x, y \rangle & \text{for } (x, y) \in C \times D_3. \end{cases} \quad (3.3)$$

In this definition, D_1 , D_2 and D_3 are disjoint and may be empty for some convex set C . If one of them is empty, we just remove the corresponding slack operator from the definition.

Definition 3.7 Let $K \subset \mathbb{R}^m$ be a full dimensional closed convex cone and $C \subset \mathbb{R}^n$ a full dimensional convex set containing no lines. We say that the slack operator S_C defined by (3.3) is K -factorizable, if there exist maps

$$A : C \rightarrow K, \quad B_1 : D_1 \rightarrow K^*, \quad B_2 : D_2 \rightarrow K^*, \quad B_3 : D_3 \rightarrow K^*$$

such that

- $S_C^i(x, y) = \langle A(x), B_i(y) \rangle$ for all $(x, y) \in C \times D_i$ and $i = 1, 2, 3$.

Theorem 3.8 Let $K \subset \mathbb{R}^m$ be a full dimensional closed convex cone and $C \subset \mathbb{R}^n$ a full dimensional closed convex set containing no lines. Assume C is not a translated cone. If C has a proper K -lift defined by (3.1), then the slack operator S_C defined by (3.3) is K -factorizable. Conversely, if S_C defined by (3.3) is K -factorizable, then C has a K -lift defined by (3.1).

The proof of Theorem 3.13 can be modified slightly to show the correctness of Theorem 3.8.

Example 3.9 Consider $C = \{x \mid x \geq -1\}$. Let K be \mathcal{S}_+^3 and

$$L = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in \mathcal{S}^3 \mid a_{11} = 1, \quad a_{13} = 0, \quad a_{23} = 0, \quad a_{33} = a_{12} + 1 \right\}.$$

We construct a linear map π from \mathcal{S}^3 to \mathbb{R}^1 : $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \rightarrow a_{33}$. It is easy to check that

C has a K -lift, i.e. $C = \pi(K \cap L)$.

Now let us check whether the slack operator S_C defined by (3.3) is K -factorizable. Because C contains the origin in its interior, according to Theorem 3.3, D_2 and D_3 are empty. Since

$C^\circ = \{x \mid -1 \leq x \leq 0\}$, we have $D_1 = \text{ext}_1(C^\circ) \setminus 0 = \{(-1)\}$. Let us define the maps $A : C \rightarrow K$, $B_1 : D_1 \rightarrow K^*$ as

$$A(x) = \begin{pmatrix} 1 & x & 0 \\ x & x^2 & 0 \\ 0 & 0 & x+1 \end{pmatrix}, \quad B_1(y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -y \end{pmatrix}.$$

Since $1 - \langle x, y \rangle = \langle A(x), B_1(y) \rangle$ for all $(x, y) \in C \times D_1$, we claim that the slack operator S_C is K -factorizable.

Remark 3.10 Although Definition 3.7 and Theorem 3.8 have extended the argument in [6] to more general convex sets, it is not easy to use. The main reason is that we have to define the map A and check whether S_C is factorizable for all points in C . This is difficult since C usually contains infinite number of points even when it is a polyhedron.

By Theorem 2.2, if C contains no lines, every point in it can be expressed as the convex combination of extreme points $\text{ext}_1(C)$ and extreme directions $\text{ext}_2(0^+C)$. By introducing the cone lift of the recession cone 0^+C and defining its slack operator and cone factorization, we extend results in [6] to non-compact convex sets.

Definition 3.11 Let $K \subset \mathbb{R}^m$ be a full dimensional closed convex cone. A K -lift of a non-compact closed convex set $C \subset \mathbb{R}^n$ is a set $Q = K \cap L$ such that

$$C = \pi(K \cap L), \quad 0^+C = \pi(K \cap 0^+L) \quad (3.4)$$

where $L \subset \mathbb{R}^m$ is an affine subspace and $\pi : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a linear map. We say that Q is a proper K -lift of C , if $L \cap \text{int}(K) \neq \emptyset$.

We would like to emphasize that the condition $0^+C = \pi(K \cap 0^+L)$ is not redundant and cannot be deduced from the condition $C = \pi(K \cap L)$ in general, see the following example.

Example 3.9 (continued) Although we have $C = \pi(K \cap L)$, it is clear that

$$\mathbb{R}_+^1 = 0^+C \neq \pi(K \cap 0^+L),$$

since

$$0^+L = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \in \mathcal{S}^3 \mid a_{11} = 0, a_{13} = 0, a_{23} = 0, a_{33} = a_{12} \right\}$$

and $\pi(K \cap 0^+L) = \{0\}$.

We define the slack operator S_C of a full dimensional closed convex set C as

$$S_C = \begin{cases} S_C^1(x, y) = 1 - \langle x, y \rangle & \text{for } (x, y) \in \text{ext}_1(C) \times D_1, \\ S_C^2(x, y) = -\langle x, y \rangle & \text{for } (x, y) \in \text{ext}_1(C) \times D_2, \\ S_C^3(x, y) = -1 - \langle x, y \rangle & \text{for } (x, y) \in \text{ext}_1(C) \times D_3, \\ S_{0^+C}^i(x, y) = -\langle x, y \rangle & \text{for } (x, y) \in \text{ext}_2(0^+C) \times D_i, i = 1, 2, 3. \end{cases} \quad (3.5)$$

In this definition, D_1 , D_2 and D_3 are disjoint and may be empty for some convex set C . If one of them is empty, we just remove the corresponding slack operator from the definition.

Definition 3.12 Let $K \subset \mathbb{R}^m$ be a full dimensional closed convex cone and $C \subset \mathbb{R}^n$ a full dimensional closed convex set containing no lines. We say that the slack operator S_C defined by (3.5) is K -factorizable, if there exist maps

$$\begin{aligned} A_1 : \text{ext}_1(C) &\rightarrow K, \quad A_2 : \text{ext}_2(0^+C) \rightarrow K, \\ B_1 : D_1 &\rightarrow K^*, \quad B_2 : D_2 \rightarrow K^*, \quad B_3 : D_3 \rightarrow K^*, \end{aligned}$$

such that

- $S_c^i(x, y) = \langle A_1(x), B_i(y) \rangle$ for all $(x, y) \in \text{ext}_1(C) \times D_i$ and $i = 1, 2, 3$.
- $S_{0^+C}^i(x, y) = \langle A_2(x), B_i(y) \rangle$ for all $(x, y) \in \text{ext}_2(0^+C) \times D_i$ and $i = 1, 2, 3$.

Theorem 3.13 Let $K \subset \mathbb{R}^m$ be a full dimensional closed convex cone. Assume $C \subset \mathbb{R}^n$ is a full dimensional closed convex set containing no lines and C is not a translated cone. If C has a proper K -lift defined by (3.4), then the slack operator S_C defined by (3.5) is K -factorizable. Conversely, if S_C defined by (3.5) is K -factorizable, then C has a K -lift defined by (3.4).

Proof Suppose C has a proper K -lift, then we set $L = w_0 + L_0$ in \mathbb{R}^m where L_0 is a linear subspace, $w_0 \in \text{int}(K)$ and $\pi : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a linear map such that $C = \pi(K \cap L)$, $0^+C = \pi(K \cap 0^+L)$. Since $0^+L = L_0$, we have $0^+C = \pi(K \cap L_0)$. We need to construct maps A_1, A_2 and B_1, B_2, B_3 from the K -lift that factorize the slack operator S_C .

For every point $x_1 \in \text{ext}_1(C)$, there exists a point w_1 in the convex set $K \cap L$ such that $\pi(w_1) = x_1$. We define $A_1(x_1) := w_1$. Moreover, for every point $x_2 \in \text{ext}_2(0^+C)$, there exists a point w_2 in the convex set $K \cap L_0$ such that $\pi(w_2) = x_2$. We define $A_2(x_2) := w_2$.

The definitions of B_1, B_2 and B_3 are similar to those given in [6, Theorem 1], which use the properness condition to guarantee the strong duality holds. The only difference is that for $l_1 \in D_1$, $\max\{\langle l_1, x \rangle \mid x \in C\}$ is 1, for $l_2 \in D_2$, $\max\{\langle l_2, x \rangle \mid x \in C\}$ is 0 and for $l_3 \in D_3$, $\max\{\langle l_3, x \rangle \mid x \in C\}$ is -1 . Therefore, we only give the definitions and omit all proofs. For every $y_1 \in D_1$, we define $B_1(y_1) := z - \pi^*(y_1)$ where z is any point in $L_0^\perp \cap (K^* + \pi^*(y_1))$ that satisfies $\langle w_0, z \rangle = 1$. For every $y_2 \in D_2$, we define $B_2(y_2) := z - \pi^*(y_2)$ where z is any point in $L_0^\perp \cap (K^* + \pi^*(y_2))$ that satisfies $\langle w_0, z \rangle = 0$. For every $y_3 \in D_3$, we define $B_3(y_3) := z - \pi^*(y_3)$ where z is any point in $L_0^\perp \cap (K^* + \pi^*(y_3))$ that satisfies $\langle w_0, z \rangle = -1$. It remains to check that S_c^i and $S_{0^+C}^i$ have K -factorizations given in Definition 3.12. The K -factorization of S_c^i can be checked by the same method used in [6, Theorem 1]. For each $x_2 \in \text{ext}_2(0^+C)$ and $y_i \in D_i$, $i = 1, 2, 3$, we have

$$\begin{aligned} \langle x_2, y_i \rangle &= \langle \pi(w_2), y_i \rangle = \langle w_2, \pi^*(y_i) \rangle = \langle w_2, z - B_i(y_i) \rangle \\ &= -\langle w_2, B_i(y_i) \rangle = -\langle A_2(x_2), B_i(y_i) \rangle. \end{aligned}$$

Therefore, $S_{0^+C}^i$ is K -factorizable according to Definition 3.12.

Suppose on the other hand that S_C is K -factorizable, i.e. there exist maps A_1, A_2 and B_1, B_2, B_3 such that $S_c^i(x, y) = \langle A_1(x), B_i(y) \rangle$ for all $(x, y) \in \text{ext}_1(C) \times D_i$, $i = 1, 2, 3$ and $S_{0^+C}^i(x, y) = \langle A_2(x), B_i(y) \rangle$ for all $(x, y) \in \text{ext}_2(0^+C) \times D_i$, $i = 1, 2, 3$. We construct the affine space

$$L = \{(x, z) \in \mathbb{R}^n \times \mathbb{R}^m \mid 1 - \langle x, y_1 \rangle = \langle z, B_1(y_1) \rangle, \forall y_1 \in D_1, \\ - \langle x, y_2 \rangle = \langle z, B_2(y_2) \rangle, \forall y_2 \in D_2, -1 - \langle x, y_3 \rangle = \langle z, B_3(y_3) \rangle, \forall y_3 \in D_3\},$$

and let L_K be the projection of L onto the second component z .

We need to show firstly that $0 \notin L_K$. If $0 \in L_K$, there exists $x \in \mathbb{R}^n$ such that $1 - \langle x, y_1 \rangle = 0$, $\forall y_1 \in D_1$, $-\langle x, y_2 \rangle = 0$, $\forall y_2 \in D_2$, $-1 - \langle x, y_3 \rangle = 0$, $\forall y_3 \in D_3$. By Theorem 3.6, C is a translated cone and this contradicts to the assumption. For each $x \in \text{ext}_1(C)$, we have $A_1(x) \in K \cap L_K$, then $K \cap L_K \neq \emptyset$.

Since C contains no lines, we can show that for every $z \in K \cap L_K$, there exists unique $x_z \in \mathbb{R}^n$ such that $(x_z, z) \in L$. Hence, the map π from z to x_z is a well defined affine map. Since the origin is not in L_K , we can extend the map to a linear map: $\mathbb{R}^m \rightarrow \mathbb{R}^n$. We prove below that $C = \pi(K \cap L_K)$.

For every $x \in \mathbb{R}^n$, if there exists $z \in K$ such that $(x, z) \in L$, then $\langle x, y_1 \rangle = 1 - \langle z, B_1(y_1) \rangle \leq 1$, $\forall y_1 \in D_1$, $\langle x, y_2 \rangle = -\langle z, B_2(y_2) \rangle \leq 0$, $\forall y_2 \in D_2$, and $\langle x, y_3 \rangle = -1 - \langle z, B_3(y_3) \rangle \leq -1$, $\forall y_3 \in D_3$. By Theorem 3.5, we have $x \in C$. Hence, $\pi(K \cap L_K) \subseteq C$.

For every $x \in C$, there exist $\lambda_i^1 \geq 0$, $i = 1, \dots, i_1$, $\lambda_j^2 \geq 0$, $j = 1, \dots, j_2$ such that

$$x = \sum_{i=1}^{i_1} \lambda_i^1 x_i + \sum_{j=1}^{j_2} \lambda_j^2 y_j, \quad \sum_{i=1}^{i_1} \lambda_i^1 = 1,$$

where $x_i \in \text{ext}_1(C)$ and $y_j \in \text{ext}_2(0^+C)$. Let $z = \sum_{i=1}^{i_1} \lambda_i^1 A_1(x_i) + \sum_{j=1}^{j_2} \lambda_j^2 A_2(y_j)$. Since S_C is K -factorizable, it is easy to check that $z \in K \cap L_K$ and therefore, $x = \pi(z) \in \pi(K \cap L_K)$. We can deduce that $C = \pi(K \cap L_K)$.

Furthermore, we need to show that $0^+C = \pi(K \cap 0^+L_K)$. Since $C = \pi(K \cap L_K)$, we know that $0^+C \supseteq \pi(K \cap 0^+L_K)$. On the other hand, for every $x \in \text{ext}_2(0^+C)$, by the definition of L , we claim that $A_2(x)$ is in $K \cap 0^+L_K$. Hence, we have $0^+C = \pi(K \cap 0^+L_K)$.

The following example shows that the K -factorization of $S_c^i(x, y)$ for $(x, y) \in \text{ext}_1(C) \times D_i$ and $i = 1, 2, 3$ cannot guarantee that the convex set C has a K -lift defined by (3.4). It is necessary to consider the K -factorization of $S_{0^+C}^i(x, y)$ for all $(x, y) \in \text{ext}_2(0^+C) \times D_i$ and $i = 1, 2, 3$ too.

Example 3.14 Let C be a 3-dimensional polyhedron in \mathbb{R}^3 defined by the following

inequality:

$$C = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : \begin{pmatrix} 1 & \frac{\sqrt{3}}{3} & 0 \\ 0 & \frac{2\sqrt{3}}{3} & 0 \\ -1 & \frac{\sqrt{3}}{3} & 0 \\ -1 & -\frac{\sqrt{3}}{3} & 0 \\ 0 & -\frac{2\sqrt{3}}{3} & 0 \\ 1 & -\frac{\sqrt{3}}{3} & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

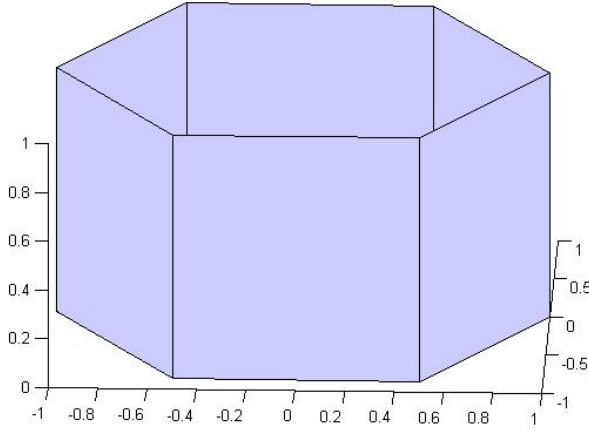


Figure 3: Example 3.14

The six vertices of C are $\{(\cos(i\pi/3), \sin(i\pi/3), 0), i = 0, \dots, 5\}$ and $0^+C = \text{cone}(\{(0, 0, 1)\})$. According to Definition 4.1, its slack matrix is

$$S := \begin{pmatrix} 0 & 0 & 1 & 2 & 2 & 1 & 0 \\ 1 & 0 & 0 & 1 & 2 & 2 & 0 \\ 2 & 1 & 0 & 0 & 1 & 2 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It has been shown in [6, Example 2] that the first 6×6 submatrix S_H of S has a \mathbb{R}_+^5 -factorization.

However, we claim that the matrix S does not have a \mathbb{R}_+^5 -factorization. If it does, we can assume it has the following nonnegative decomposition:

$$\begin{pmatrix} S_H & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}.$$

Since $A_{21}B_{11} + A_{22}B_{21} = 0$, we have $A_{22}B_{21} = 0$. We claim that $A_{22} = 0$. Otherwise, B_{21} will be zero and $S_H = A_{11}B_{11}$. This contradicts to the fact that S_H has no \mathbb{R}_+^4 -factorization. From $A_{21}B_{12} + A_{22}B_{22} = 1$ and $A_{22} = 0$, we have $B_{12} \neq 0$. However, since $A_{11}B_{12} + A_{12}B_{22} = 0$, there exists one column of A_{11} which is a zero column. Therefore, S_H has a nonnegative decomposition in \mathbb{R}_+^4 which is also a contradiction. Hence, according to Theorem 4.5, C has no \mathbb{R}_+^5 -lift.

A cone K is *nice* if $K^* + F^\perp$ is closed for all faces F of K [1]. In [8], Pataki points out that K is nice if and only if $F^* = K^* + F^\perp$ for all faces F of K . In [6, Corollary 1], they have shown that if K is a nice cone, then whenever a convex body C has a K -lift (not necessarily proper), the slack operator has a K -factorization. We extend their results to the case that C is not a convex body.

Corollary 3.15 *If K is nice cone and $C \subset \mathbb{R}^n$ is a full dimensional closed convex set containing no lines and C is not a translated cone, then whenever C has a K -lift, S_C has a K -factorization.*

Proof Let $Q := K \cap L$ in K and F be the minimal face of K containing Q , i.e. $Q \cap \text{ri}(F) \neq \emptyset$. If C has a K -lift defined by (3.4), then it is clear that $Q = F \cap L$. Furthermore, we have $0^+C = \pi(K \cap 0^+L) = \pi(0^+Q) = \pi(0^+(F \cap L)) = \pi(F \cap 0^+L)$. Therefore, Q also have a proper F -lift defined by (3.4). The F -factorization of slack operator S_C defined by (3.12) can be constructed by the same way as in the proof of Theorem 3.13. The maps A_1 and A_2 are also maps to K . Since K is a nice cone and F is a face of K , we have $F^* = K^* + F^\perp$ [8, Remark 1]. Similar to [6, Corollary 1], we can construct new maps $B'_i(y) = z \in K^*$ such that $B_i(y) - z \in F^\perp$ for $1 \leq i \leq 3$ and show that the slack operator S_C is K -factorizable.

In Theorem 3.13, we have assumed that the full dimensional closed convex set C is not a translated cone. If C is a translated cone, a K -lift of C can be defined as

$$C = b + \pi(K \cap L), \quad (3.6)$$

where $b \in \mathbb{R}^n$ is a constant vector, L is a linear space and $\pi : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a linear map. In this case, we only need to characterize $\text{ext}_2(0^+C)$. Without loss of generality, we can assume $b = 0$ and C is a cone. We define the slack operator S_C as

$$S_C(x, y) = -\langle x, y \rangle \text{ for } (x, y) \in \text{ext}_2(C) \times \text{ext}_2(C^\circ). \quad (3.7)$$

We say that the slack operator S_C is K -factorizable, if there exist maps

$$A : \text{ext}_2(C) \rightarrow K, \quad B : \text{ext}_2(C^\circ) \rightarrow K^*$$

such that

- $S_C(x, y) = \langle A(x), B(y) \rangle$ for all $(x, y) \in \text{ext}_2(C) \times \text{ext}_2(C^\circ)$.

Theorem 3.16 *Let K be a full dimensional closed convex cone in \mathbb{R}^m and C a full dimensional closed pointed convex cone in \mathbb{R}^n . If C has a proper K -lift defined by (3.6), then S_C defined by (3.7) is K -factorizable. Conversely, if S_C defined by (3.7) is K -factorizable, then C has a K -lift defined by (3.6).*

The proof that S_C is K -factorizable if C has a proper K -lift is similar to the one given for Theorem 3.13. We omit the proof here. Below we give a short proof to show that C has a K -lift if S_C is K -factorizable.

Proof Suppose S_C is K -factorizable, we construct the linear space

$$L = \{(x, z) \mid -\langle x, y \rangle = \langle z, B(y) \rangle, \forall y \in \text{ext}_2(C^\circ)\}.$$

Let L_K be the projection of L onto the second component z . Since C contains no lines, for every $z \in L_K$, there exists unique $x_z \in \mathbb{R}^n$ such that $(x_z, z) \in L$. Hence, we can define a linear map $\pi: L_K \rightarrow \mathbb{R}^n$. Since L is a linear space, we can extend π to a linear map: $\mathbb{R}^m \rightarrow \mathbb{R}^n$.

For every $x \in \mathbb{R}^n$, if there exists $z \in K$ such that (x, z) is in L , then $\langle x, y \rangle \leq 0$ for all $y \in \text{ext}_2(C^\circ)$. Hence $x \in C^{\circ\circ} = \text{cl}(C) = C$ since C is a closed convex set. We have $\pi(K \cap L) \subseteq C$. On the other hand, since S_C is K -factorizable, for every $x \in \text{ext}_2(C)$, $(x, A(x)) \in L$, hence $C \subseteq \pi(K \cap L)$. The proof is completed.

3.1.2 C contains lines

When C is a full dimensional closed convex set containing lines, Definition 3.7 and Theorem 3.8 can be generalized without any change. However, when C contains lines, it has no extreme points and 0^+C contains no extreme rays, so Definition 3.12 and Theorem 3.13 need to be adjusted properly.

Let L_1 denote the lineality space of C and $\{l_1, \dots, l_s\}$ be an orthogonal basis of L_1 . The convex set containing lines can be decomposed as

$$C = C_0 + L_1, \tag{3.8}$$

where $C_0 = C \cap L_1^\perp$ is a closed convex set containing no lines and L_1^\perp is the orthogonal complement of L_1 .

Lemma 3.17 *L_1^\perp is the affine hull of C° .*

Proof Since $C = C_0 + L_1$, we have $C^\circ = C_0^\circ \cap L_1^\perp$. The convex set C_0 contains no lines, then 0^+C_0 contains no lines. By Corollary 2.4 and Theorem 3.3, we claim $\text{cl}(\text{cone}(C_0^\circ)) = (0^+C_0)^\circ$. Since $(0^+C_0)^\circ$ contains an interior, it is clear that $\text{cone}(C_0^\circ)$ contains an interior. As C_0 is in L_1^\perp , $\text{cone}(C_0^\circ)$ contains L_1 . Hence $\text{cone}(C^\circ) = \text{cone}(C_0^\circ) \cap L_1^\perp$ has an interior in L_1^\perp . Furthermore, $C^\circ = C_0^\circ \cap L_1^\perp$ has an interior in L_1^\perp too. Hence L_1^\perp is the affine hull of C° .

We define the slack operator S_C of a full dimensional closed convex set C containing lines

as

$$S_C = \begin{cases} S_{C_0}^1(x, y) = 1 - \langle x, y \rangle & \text{for } (x, y) \in \text{ext}_1(C_0) \times D_1, \\ S_{C_0}^2(x, y) = -\langle x, y \rangle & \text{for } (x, y) \in \text{ext}_1(C_0) \times D_2, \\ S_{C_0}^3(x, y) = -1 - \langle x, y \rangle & \text{for } (x, y) \in \text{ext}_1(C_0) \times D_3, \\ S_{0^+C_0}^i(x, y) = -\langle x, y \rangle & \text{for } (x, y) \in \text{ext}_2(0^+C_0) \times D_i, i = 1, 2, 3, \\ S_{L_1}(x, y) = \langle x, y \rangle & \text{for } (x, y) \in \{l_1, \dots, l_s\} \times \{l_1, \dots, l_s\}. \end{cases} \quad (3.9)$$

Definition 3.18 Let $K \subset \mathbb{R}^m$ be a full dimensional closed convex cone and $C \subset \mathbb{R}^n$ a full dimensional non-compact closed convex set containing lines. We say that the slack operator S_C defined by (3.9) is K -factorizable, if there exist maps

$$\begin{aligned} A_1 : \text{ext}_1(C_0) &\rightarrow K, \quad A_2 : \text{ext}_2(0^+C_0) \rightarrow K, \quad A_3 : \{l_1, \dots, l_s\} \rightarrow K, \\ B_1 : D_1 &\rightarrow K^*, \quad B_2 : D_2 \rightarrow K^*, \quad B_3 : D_3 \rightarrow K^*, \quad F : \{l_1, \dots, l_s\} \rightarrow \mathbb{R}^m, \end{aligned}$$

such that

- $S_{C_0}^i(x, y) = \langle A_1(x), B_i(y) \rangle$ for all $(x, y) \in \text{ext}_1(C_0) \times D_i$ and $i = 1, 2, 3$,
- $S_{0^+C_0}^i(x, y) = \langle A_2(x), B_i(y) \rangle$ for all $(x, y) \in \text{ext}_2(0^+C_0) \times D_i$ and $i = 1, 2, 3$,
- $S_{L_1}(x, y) = \langle A_3(x), F(y) \rangle$ for all $(x, y) \in \{l_1, \dots, l_s\} \times \{l_1, \dots, l_s\}$,
- $\langle A_3(x), B_i(y) \rangle = 0$ for all $(x, y) \in \{l_1, \dots, l_s\} \times D_i$ and $i = 1, 2, 3$,
- $\langle A_1(x), F(y) \rangle = 0$ for all $(x, y) \in \text{ext}_1(C_0) \times \{l_1, \dots, l_s\}$,
- $\langle A_2(x), F(y) \rangle = 0$ for all $(x, y) \in \text{ext}_2(0^+C_0) \times \{l_1, \dots, l_s\}$.

Theorem 3.19 Let K be a full dimensional closed convex cone in \mathbb{R}^m . Assume C is a full dimensional closed convex set in \mathbb{R}^n which can be decomposed as (3.8) and C_0 is not a translated cone. If C has a proper K -lift defined by (3.4), then the slack operator S_C defined by (3.9) is K -factorizable. Conversely, if S_C defined by (3.9) is K -factorizable, then C has a K -lift defined by (3.4).

Proof Suppose C has a proper K -lift, then we set $L = w_0 + L_0$ in \mathbb{R}^m where L_0 is a linear subspace, $w_0 \in \text{int}(K)$ and $\pi : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a linear map such that $C = \pi(K \cap L)$, $0^+C = \pi(K \cap 0^+L)$. Since $0^+L = L_0$, we have $0^+C = \pi(K \cap L_0)$. We need to construct maps $A_1, A_2, A_3, B_1, B_2, B_3$ and F that factorize the slack operator S_C from the K -lift. We can define A_1, A_2, B_1, B_2, B_3 by the same way used in the proof of Theorem 3.13. For every l_i , $i = 1, \dots, s$, there exists a point $w_i \in K \cap L_0$ such that $\pi(w_i) = l_i$. So we define $A_3(l_i) := w_i$ for $i = 1, \dots, s$. Furthermore, we define $F(l_i) := \pi^*(l_i)$ for $i = 1, \dots, s$.

The equalities for $S_{C_0}^i, S_{0^+C_0}^i, i = 1, \dots, 3$ in Definition 3.18 can be checked by the same method used in the proof of Theorem 3.13. For each $x, y \in \{l_1, \dots, l_s\}$, we have

$$\langle x, y \rangle = \langle \pi(A_3(x)), y \rangle = \langle A_3(x), F(y) \rangle.$$

For each $x \in \{l_1, \dots, l_s\}$ and $y \in D_i$, we have

$$\langle A_3(x), B_i(y) \rangle = \langle A_3(x), z - \pi^*(y) \rangle = -\langle \pi(A_3(x)), y \rangle = -\langle x, y \rangle = 0.$$

For each $x \in \text{ext}_1(C_0)$ and $y \in \{l_1, \dots, l_s\}$, we have

$$\langle A_1(x), F(y) \rangle = \langle \pi(A_1(x)), y \rangle = \langle x, y \rangle = 0.$$

For each $x \in \text{ext}_2(0^+C_0)$ and $y \in \{l_1, \dots, l_s\}$, we have

$$\langle A_2(x), F(y) \rangle = \langle \pi(A_2(x)), y \rangle = \langle x, y \rangle = 0.$$

Therefore, S_C is K -factorizable.

Suppose S_C is K -factorizable. We need to construct an affine space L :

$$\begin{aligned} L = \{ & (x, z) \in \mathbb{R}^n \times \mathbb{R}^m \mid x = x_1 + x_2 \text{ such that } x_1 \in L_1^\perp \text{ and } x_2 \in L_1, \\ & 1 - \langle x_1, y_1 \rangle = \langle z, B_1(y_1) \rangle, \forall y_1 \in D_1, -\langle x_1, y_2 \rangle = \langle z, B_2(y_2) \rangle, \forall y_2 \in D_2, \\ & -1 - \langle x_1, y_3 \rangle = \langle z, B_3(y_3) \rangle, \forall y_3 \in D_3, \langle x_2, l_i \rangle = \langle z, F(l_i) \rangle, \forall i = 1, \dots, s\}. \end{aligned}$$

Let L_K be the projection of L onto the second component z .

We need to show that $0 \notin L_K$. If $0 \in L_K$, there exists $x = x_1 + x_2$ such that $1 - \langle x_1, y_1 \rangle = 0$, $\forall y_1 \in D_1$, $-\langle x_1, y_2 \rangle = 0$, $\forall y_2 \in D_2$, $-1 - \langle x_1, y_3 \rangle = 0$, $\forall y_3 \in D_3$. By Theorem 3.6, C_0 is a translated convex cone and this leads to a contradiction. Moreover, $K \cap L_K \neq \emptyset$ since for each $x \in \text{ext}_1(C_0)$, we have $A_1(x) \in K \cap L_K$.

Now we prove that for each $z \in K \cap L_K$, there exists unique $x \in \mathbb{R}^n$ such that $(x, z) \in L$. If for some $z \in K \cap L_K$, there exist two different points $x^1 = x_1^1 + x_2^1$ and $x^2 = x_1^2 + x_2^2$ such that both (x^1, z) and (x^2, z) are in L . Because $\langle x_1^1 - x_1^2, y \rangle = 0$ for $y \in D_1 \cup D_2 \cup D_3$, we have $\langle x_1^1 - x_1^2, y \rangle = 0$ for $y \in C^\circ$. According to Lemma 3.17, L_1^\perp is the affine hull of C° , hence $x_1^1 - x_1^2 \in L_1$. Since $x_1^1 - x_1^2$ is also in L_1^\perp , we have $x_1^1 - x_1^2 = 0$. Furthermore, because $\langle x_2^1 - x_2^2, l_i \rangle = 0$, $1 \leq i \leq s$ and $x_2^1 - x_2^2 \in L_1$, we also have $x_2^1 - x_2^2 = 0$. Hence, the map π from z to x_z is a well defined affine map. Since the origin is not in L_K , we can extend it to a linear map: $\mathbb{R}^m \rightarrow \mathbb{R}^n$. By the same method used in proving Theorem 3.13, we can show $C = \pi(K \cap L_K)$ and $0^+C = \pi(K \cap 0^+L_K)$.

Remark 3.20 If K is a nice cone, the proper condition can also be removed from Theorem 3.19. In this case, the K -lift of C and the K -factorization of the slack operator S_C is equivalent.

When C is a closed translated convex cone that contains lines, C can be decomposed as (3.8) and we have results similar to those given in Theorem 3.16. The slack operator S_C of a convex set C containing lines is defined as

$$S_C = \begin{cases} S_{C_0}(x, y) = -\langle x, y \rangle & \text{for } (x, y) \in \text{ext}_2(C_0) \times \text{ext}_2(C^\circ), \\ S_{L_1}(x, y) = -\langle x, y \rangle & \text{for } (x, y) \in \{l_1, \dots, l_s\} \times \{l_1, \dots, l_s\}. \end{cases} \quad (3.10)$$

Definition 3.21 We say that the slack operator S_C defined by (3.10) is K -factorizable, if there exist maps

$$\begin{aligned} A_2 : \text{ext}_2(C_0) &\rightarrow K, \quad A_3 : \{l_1, \dots, l_s\} \rightarrow K, \\ B : \text{ext}_2(C^o) &\rightarrow K^*, \quad F : \{l_1, \dots, l_s\} \rightarrow \mathbb{R}^m, \end{aligned}$$

such that

- $S_{C_0}(x, y) = \langle A_2(x), B(y) \rangle$ for all $(x, y) \in \text{ext}_2(C_0) \times \text{ext}_2(C^o)$,
- $S_{L_1}(x, y) = \langle A_3(x), F(y) \rangle$ for all $(x, y) \in \{l_1, \dots, l_s\} \times \{l_1, \dots, l_s\}$,
- $\langle A_2(x), F(y) \rangle = 0$ for all $(x, y) \in \text{ext}_2(C_0) \times \{l_1, \dots, l_s\}$,
- $\langle A_3(x), B(y) \rangle = 0$ for all $\{l_1, \dots, l_s\} \times \text{ext}_2(C^o)$.

Theorem 3.22 *Let K be a full dimensional convex cone in \mathbb{R}^m and C is a full dimensional closed translated convex cone in \mathbb{R}^n that contains lines and C can be decomposed as (3.8). If C has a proper K -lift defined by (3.6), then S_C defined by (3.10) is K -factorizable. Conversely, if S_C defined by (3.10) is K -factorizable, then C has a K -lift defined by (3.6).*

Theorem 3.22 can be proved using similar arguments for Theorem 3.16 and Theorem 3.19.

3.2 C is not full dimensional

Theorem 3.23 *Let C be a closed convex set in \mathbb{R}^n . The polar C^o contains lines if and only if C is contained in a non-trivial linear space. When C contains the origin, C is not full dimensional if and only if C^o contains lines.*

Proof C^o contains lines if and only if there exists $a \in \mathbb{R}^n$ such that $\delta^*(a, C) \leq 0$ and $\delta^*(-a, C) \leq 0$, i.e. C is contained in the set $\{x \mid a^T x = 0\}$.

When C contains the origin, C is not full dimensional if and only if there exists $a \in \mathbb{R}^n$ such that C is contained in $\{x \mid a^T x = 0\}$.

We assume that the convex set C is not full dimensional and contains no lines. Then C^o may or may not contain lines. If C^o contains no lines, we have the same results as the case that C is full dimensional. When C^o contains lines, there exists no extreme point or extreme direction in C^o and the sets D_1 , D_2 and D_3 are empty. Let L_2 denote the lineality space of C^o . Assume $C^o = C' + L_2$ such that $C' = C^o \cap L_2^\perp$. The closed convex set C' contains no lines. It is clear that $0^+ C' = 0^+ C^o \cap L_2^\perp$. Recall that $C_3 = \{x \mid \delta^*(x, C) \leq -1\}$, $C'_3 = C_3 \cap L_2^\perp$ contains no lines. Let

$$D'_1 = \text{ext}_1(C') \setminus \{0\}, \quad D'_2 = \text{ext}_2(0^+ C') \cap \{x \mid \delta^*(x, C') = 0\}, \quad D'_3 = \text{ext}_1(C'_3).$$

Let

$$D'_{32} = \text{ext}_2(0^+ C') \cap \{x \mid \delta^*(x, C') = -1\}.$$

Then $D'_{32} \subseteq D'_3$.

Theorem 3.24 *Assume a closed convex set $C \subset \mathbb{R}^n$ is not full dimensional and contains no lines. For a vector $x \in \mathbb{R}^n$, $x \in C$ if and only if for every $l \in D'_1$, $\langle l, x \rangle \leq 1$, for every $l \in D'_2$, $\langle l, x \rangle \leq 0$, for every $l \in D'_3$, $\langle l, x \rangle \leq -1$ and $x \in L_2^\perp$, where L_2 is the lineality space of C° .*

Proof Similar to the proof of Theorem 3.5.

Assume the closed convex set C is not full dimensional and contains no lines. By replacing D_i by D'_i for $i = 1, 2, 3$ in Theorem 3.5 and Definition 3.12, we can define the slack operator S_C and its K -factorization, then all results in Subsection 3.1.1 can be extended trivially to the case that C is not full dimensional. Although results in Subsection 3.1.2 can also be extended to the case that the closed convex set C is not full dimensional and contains lines, it becomes much more complicated and we omit the discussions here.

4 Cone lifts of polyhedra

Similar to [6, Section 3], we specialize results given in previous section to the case of cone lifts of polyhedra. Let $C \subset \mathbb{R}^n$ be a polyhedron defined by a set of linear inequalities:

$$C = \{x \in \mathbb{R}^n : f_1(x) \leq \alpha_1, \dots, f_{k_1}(x) \leq \alpha_{k_1}, g_1(x) \leq 0, \dots, g_{k_2}(x) \leq 0, \\ h_1(x) \leq -\beta_1, \dots, h_{k_3}(x) \leq -\beta_{k_3}\}, \quad (4.1)$$

where $\alpha_i > 0$ for $1 \leq i \leq k_1$ and $\beta_j > 0$ for $1 \leq j \leq k_3$. The recession cone of C has the following form:

$$0^+C = \{x \in \mathbb{R}^n : f_1(x) \leq 0, \dots, f_{k_1}(x) \leq 0, g_1(x) \leq 0, \dots, g_{k_2}(x) \leq 0, \\ h_1(x) \leq 0, \dots, h_{k_3}(x) \leq 0\}. \quad (4.2)$$

Let the convex set C be generated by a set of points c_1, \dots, c_t and directions r_1, \dots, r_s . We extend the definition of a slack matrix in [6, 11].

Definition 4.1 We define the *slack matrix* of C as $[S_1^T, S_2^T, S_3^T]^T$, where

1. $S_1 \in \mathbb{R}^{k_1 \times (t+s)}$ whose (i, j) -entry is $\alpha_i - f_i(c_j)$ for $i = 1, \dots, k_1$, $j = 1, \dots, t$ and $(i, t+j)$ -entry is $-f_i(r_j)$ for $i = 1, \dots, k_1$, $j = 1, \dots, s$.
2. $S_2 \in \mathbb{R}^{k_2 \times (t+s)}$ whose (i, j) -entry is $-g_i(c_j)$ for $i = 1, \dots, k_2$, $j = 1, \dots, t$ and $(i, t+j)$ -entry is $-g_i(r_j)$ for $i = 1, \dots, k_2$, $j = 1, \dots, s$.
3. $S_3 \in \mathbb{R}^{k_3 \times (t+s)}$ whose (i, j) -entry is $-\beta_i - h_i(c_j)$ for $i = 1, \dots, k_3$, $j = 1, \dots, t$ and $(i, t+j)$ -entry is $-h_i(r_j)$ for $i = 1, \dots, k_3$, $j = 1, \dots, s$.

Assume C is a full dimensional polyhedron containing no lines, the slack matrix S is called the *canonical slack matrix* of C if f_i, g_i, h_i represent the facets of C , $\alpha_i = 1, \beta_j = 1$ for $i = 1, \dots, k_1$, $j = 1, \dots, k_3$ and c_1, \dots, c_t and r_1, \dots, r_s are the vertices and the extreme directions of C respectively.

Definition 4.2 [6, Definition 7] Let $M = (M_{ij}) \in \mathbb{R}_+^{p \times q}$ be a nonnegative matrix and K a closed convex cone. Then, a K -factorization of M is a pair of ordered sets $a_1, \dots, a_p \in K$ and $b_1, \dots, b_q \in K^*$ such that $\langle a_i, b_j \rangle = M_{ij}$.

Definition 4.2 generalizes nonnegative factorizations of nonnegative matrices [11] to arbitrary closed convex cones. We generalize results [3, Theorem 13], [6, Theorem 3] and [11, Theorem 3] to show the equivalence between the K -lift of a polyhedron and the K -factorization of a slack matrix.

When C is a full dimensional polyhedron containing no lines and $K \subset \mathbb{R}^m$ a full dimensional polyhedral cone, the K -factorization of a slack operator is identical to the K -factorization of the canonical slack matrix of C . Theorem 4.3 can be deduced directly from Theorem 3.13 and 3.16 when C is full dimensional.

Theorem 4.3 *Let K be a full dimensional closed convex cone in \mathbb{R}^m . If a full dimensional polyhedron $C \subset \mathbb{R}^n$ containing no lines has a proper K -lift defined by 3.4, then every slack matrix of C admits a K -factorization. Conversely, if some slack matrix of C has a K -factorization, then C has a K -lift defined by 3.4.*

4.1 K is a polyhedral cone

Although we have pointed out in previous section the condition $0^+C = \pi(K \cap 0^+L)$ is not redundant and cannot be deduced from the condition $C = \pi(K \cap L)$ in general. When C and K are both polyhedra, (3.1) and (3.4) are equivalent.

Lemma 4.4 *Let $C \subset \mathbb{R}^n$ be a full dimensional polyhedron containing no lines and $K \subset \mathbb{R}^m$ a full dimensional polyhedral cone, then C has a K -lift defined by (3.1) if and only if it has a K -lift defined by (3.4).*

Proof It is sufficient to show that if there exists an affine space L and a linear map π from \mathbb{R}^m to \mathbb{R}^n such that $C = \pi(K \cap L)$, we will have $0^+C = \pi(K \cap 0^+L)$. It is clear that if we define Q to be $K \cap L$, then Q is a polyhedron. For $\forall x \in Q$, there exist extreme points $\alpha_1, \dots, \alpha_t$ in Q and non-zero extreme directions $\alpha_{t+1}, \dots, \alpha_{t+s}$ in 0^+Q such that $x = \lambda_1\alpha_1 + \dots + \lambda_t\alpha_t + \lambda_{t+1}\alpha_{t+1} + \dots + \lambda_{t+s}\alpha_{t+s}$, where $\lambda_1 + \dots + \lambda_t = 1$ and $\lambda_i \geq 0$ for $i = 1, \dots, t+s$. Then we have $\pi(x) = \lambda_1\pi(\alpha_1) + \dots + \lambda_t\pi(\alpha_t) + \lambda_{t+1}\pi(\alpha_{t+1}) + \dots + \lambda_{t+s}\pi(\alpha_{t+s})$. So 0^+C is generated by $\pi(\alpha_{t+1}), \dots, \pi(\alpha_{t+s})$. On the other hand, since $\alpha_{t+1}, \dots, \alpha_{t+s}$ generate $0^+Q = K \cap 0^+L$, thus $\pi(\alpha_{t+1}), \dots, \pi(\alpha_{t+s})$ generate $\pi(K \cap 0^+L)$. Hence, $0^+C = \pi(K \cap 0^+L)$ and our proof is completed.

Theorem 4.5 *Let $C \subset \mathbb{R}^n$ be a full dimensional polyhedron containing no lines and $K \subset \mathbb{R}^m$ a full dimensional polyhedral cone. If C is not a translated convex cone, then C has a K -lift defined by (3.1) if and only if the slack matrix of C has a K -factorization. If C is a translated convex cone, we can have the same result if we replace the K -lift (3.1) by the K -lift (3.6).*

Proof According to Lemma 4.4, the polyhedron C has a K -lift defined by (3.1) if and only if it has a K -lift defined by (3.4). The properness condition that L intersects $\text{int}(K)$ is

used to guarantee the strong duality in the proof Theorem 4.3 (see the proof of Theorem 1 in [6]). When K is a polyhedral cone, the minimization problem involved in the proof becomes a linear-programming problem and the strong duality holds if $K \cap L \neq \emptyset$.

Example 4.6 Consider the polyhedron $C \subset \mathbb{R}^2$ defined by

$$C = \left\{ (x_1, x_2) \in \mathbb{R}^2 : \begin{pmatrix} 0 & 1 \\ -2 + \sqrt{3} & 1 \\ 1 - \sqrt{3} & \sqrt{3} - 1 \\ -1 & 2 - \sqrt{3} \\ -1 & -2 + \sqrt{3} \\ 1 - \sqrt{3} & 1 - \sqrt{3} \\ -2 + \sqrt{3} & -1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq \begin{pmatrix} 2 \\ \sqrt{3} \\ 1 \\ 2 - \sqrt{3} \\ \sqrt{3} - 2 \\ -2\sqrt{3} + 3 \\ \sqrt{3} - 2 \\ 0 \end{pmatrix} \right\}.$$

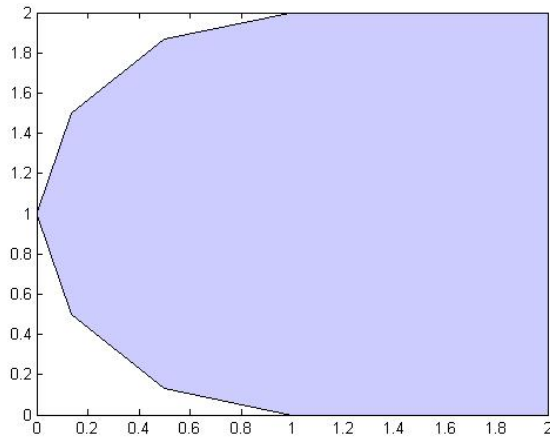


Figure 4: Example 4.6

By Theorem 4.5, C has a \mathbb{R}_+^6 -lift if and only if the slack matrix S has a \mathbb{R}_+^6 -factorization. We

denote the coefficient matrix by H and the right hand side vector by d . The slack matrix S is

$$S = \begin{bmatrix} 0 & 1 - \frac{\sqrt{3}}{2} & \frac{1}{2} & 1 & \frac{3}{2} & 1 + \frac{\sqrt{3}}{2} & 2 & 0 \\ 0 & 0 & 2 - \sqrt{3} & \sqrt{3} - 1 & 3 - \sqrt{3} & \sqrt{3} & 2 & 2 - \sqrt{3} \\ 2 - \sqrt{3} & 0 & 0 & 2 - \sqrt{3} & \sqrt{3} - 1 & 3 - \sqrt{3} & \sqrt{3} & \sqrt{3} - 1 \\ \sqrt{3} - 1 & 2 - \sqrt{3} & 0 & 0 & 2 - \sqrt{3} & \sqrt{3} - 1 & 3 - \sqrt{3} & 1 \\ 3 - \sqrt{3} & \sqrt{3} - 1 & 2 - \sqrt{3} & 0 & 0 & 2 - \sqrt{3} & \sqrt{3} - 1 & 1 \\ \sqrt{3} & 3 - \sqrt{3} & \sqrt{3} - 1 & 2 - \sqrt{3} & 0 & 0 & 2 - \sqrt{3} & \sqrt{3} - 1 \\ 2 & \sqrt{3} & 3 - \sqrt{3} & \sqrt{3} - 1 & 2 - \sqrt{3} & 0 & 0 & 2 - \sqrt{3} \\ 2 & 1 + \frac{\sqrt{3}}{2} & \frac{3}{2} & 1 & \frac{1}{2} & 1 - \frac{\sqrt{3}}{2} & 0 & 0 \end{bmatrix}.$$

We compute a \mathbb{R}_+^6 -factorization of S as $S = U \cdot V$ where

$$U = \begin{bmatrix} 1 & 1 & 0 & 1 - \frac{\sqrt{3}}{2} & 0 & 0 \\ 1 & -2\sqrt{3} + 4 & 2 - \sqrt{3} & 0 & 0 & 0 \\ \sqrt{3} - 1 & 0 & \frac{\sqrt{3}}{2} - \frac{1}{2} & 0 & 0 & \frac{\sqrt{3}}{2} - \frac{1}{2} \\ 2 - \sqrt{3} & 0 & 0 & 2 - \sqrt{3} & 0 & 1 \\ 0 & 0 & 0 & 2 - \sqrt{3} & 2 - \sqrt{3} & 1 \\ 0 & 0 & \frac{\sqrt{3}}{2} - \frac{1}{2} & 0 & \sqrt{3} - 1 & \frac{\sqrt{3}}{2} - \frac{1}{2} \\ 0 & -2\sqrt{3} + 4 & 2 - \sqrt{3} & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 - \frac{\sqrt{3}}{2} & 1 & 0 \end{bmatrix},$$

$$V = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & \sqrt{3} & 2 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} - 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & \sqrt{3} & 1 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{3} - 1 & 0 & 0 & 0 & 0 & 0 & \sqrt{3} - 1 & 1 \end{bmatrix}.$$

The \mathbb{R}_+^6 -lift of C is:

$$C = \{(x_1, x_2) \mid \exists y \in \mathbb{R}_+^6 \text{ s.t. } Hx + Uy = d\}.$$

If we eliminate x_1 and x_2 from $Hx + Uy = d$, we have

$$\{y \in \mathbb{R}_+^6 \mid y_1 = 1 + (\sqrt{3} - 1)y_4 + \frac{\sqrt{3} + 1}{2}y_6 - \frac{\sqrt{3} + 1}{2}y_3 - y_5, \\ y_2 = \frac{1}{2} - \frac{1}{2}y_4 - \frac{\sqrt{3} + 1}{4}y_6 + \frac{\sqrt{3} + 1}{4}y_3\}.$$

4.2 K is a positive semidefinite cone

The *positive semidefinite rank* (psd rank) of a polytope C is the smallest k such that C has an S_+^k -lift [6, 7], we refer to [2] for a nice survey on the mathematical properties of psd rank. The definition can be extended to the case that C is a polyhedron. Since a positive semidefinite cone is a nice cone, the psd rank of C is equal to the psd rank of the slack matrix of C . The following lemma extends the result in [7, Proposition 3.8].

Lemma 4.7 *Assume C is a full dimensional polyhedron containing no lines. The polyhedron $C \subset \mathbb{R}^n$ has a facet of psd rank k , then the psd rank of C is at least $k + 1$.*

Proof Let F be a facet of C . Assume the slack matrix S_F of F has psd rank k . Let $\alpha_1, \dots, \alpha_s$ be vertices of F and $\alpha_{s+1}, \dots, \alpha_{s+t}$ extreme directions of F . Suppose the facets of F correspond to the facets F_1, \dots, F_r of C other than F . Since $F \neq C$, there exists a vertex or an extreme direction denoted by α which does not belong to F and $F(\alpha) > 0$. The slack matrix S_C of C contains a $(r + 1) \times (s + t + 1)$ submatrix which is indexed by F_1, \dots, F_r, F in the row and $\alpha_1, \dots, \alpha_s, \alpha_{s+1}, \dots, \alpha_{s+t}, \alpha$ in the column and has the following form

$$S' = \begin{pmatrix} S_F & w \\ 0 & F(\alpha) \end{pmatrix} \text{ where } w \in \mathbb{R}_+^r \text{ and } F(\alpha) > 0.$$

According to [7, Proposition 2.6], we know that the psd rank of S' is $k + 1$. Hence S_C has psd rank at least $k + 1$.

The lower bound on the psd rank of a polytope given in [7, Proposition 3.2] can be extended to the case of a polyhedron.

Theorem 4.8 *If $C \subset \mathbb{R}^n$ is a full dimensional polyhedron that contains no lines, then the psd rank of C is at least n .*

Proof The proof is similar to the one given in [7, Proposition 3.2]. The only difference is that if $n = 1$, C can be a half line. Hence there exists a slack matrix whose size is 1 by 2. Obvious, the psd rank of this slack matrix is 1. Assume the statement holds up to dimension $n - 1$. We select a facet F of C which has dimension $n - 1$ and its psd rank is at least $n - 1$. By Lemma 4.7, the psd rank of C is at least n .

Remark 4.9 There exists a full dimensional polyhedron $C \subset \mathbb{R}^n$ that contains no lines such that the psd rank of C is n . For example, consider the n -dimensional nonnegative orthant $\mathbb{R}_+^n = \{x \mid x_i \geq 0, i = 1, \dots, n\}$. The slack matrix of \mathbb{R}_+^n is $(0, I_n)$ where I_n is a unit matrix and 0 is a zero vector and its psd rank is n .

4.3 Identifying the slack matrix of a polyhedron

Gouveia et al. proposed in [5] algorithmic methods to identify whether a nonnegative matrix is a slack matrix of a polyhedral cone or a polytope. These results can be generalized to characterize the slack matrix of a polyhedron. Similar to [5, Lemma 10], we have the following lemma:

Lemma 4.10 *A nonnegative matrix S is a slack matrix of a polyhedron C if and only if it is a slack matrix of a full dimensional polyhedron which contains no lines.*

Proof Since the slack matrix of a polyhedron is also a slack matrix of its translation. We can assume that the polyhedron contains the origin. Assume that C contains lines, and can be decomposed as $C = C_0 + L_1$, where $C_0 = C \cap L_1^\perp$ is a convex set containing no lines and L_1^\perp is the orthogonal complement of L_1 .

Let C be a polyhedron defined by a set of linear inequalities $f_i(x) \leq \alpha_i$, $g_j(x) \leq 0$ where $\alpha_i > 0$, $1 \leq i \leq k_1$ and $1 \leq j \leq k_2$. Every point in C can be expressed by the convex combinations of a set of points c_1, \dots, c_t and directions r_1, \dots, r_s . According to Definition 4.1, the slack matrix S of a polyhedron C can be factorized as

$$S = U \cdot V = \begin{pmatrix} \alpha_1 & -f_1 \\ \vdots & \vdots \\ \alpha_{k_1} & -f_{k_1} \\ 0 & -g_1 \\ \vdots & \vdots \\ 0 & -g_{k_2} \end{pmatrix} \begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ c_1 & \cdots & c_t & r_1 & \cdots & r_s \end{pmatrix}. \quad (4.3)$$

Since the linear functions corresponding to f_i and g_j are bounded above on C , f_i and g_j are orthogonal to L_1 . Let Q be the orthogonal basis of L_1^\perp , then we have $f_i \cdot (I - QQ^T) = 0$ and $g_j \cdot (I - QQ^T) = 0$ where I is an identity matrix and 0 is a zero vector. We have the following equalities:

$$\begin{aligned} S &= U \cdot \begin{pmatrix} 1 & 0 \\ 0 & I - QQ^T + QQ^T \end{pmatrix} \cdot V \\ &= U \cdot \begin{pmatrix} 0 & 0 \\ 0 & I - QQ^T \end{pmatrix} \cdot V + U \cdot \begin{pmatrix} 1 & 0 \\ 0 & QQ^T \end{pmatrix} \cdot V \\ &= U \cdot \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & Q^T \end{pmatrix} \cdot V. \end{aligned}$$

Let $U' = U \cdot \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix}$ and $V' = \begin{pmatrix} 1 & 0 \\ 0 & Q^T \end{pmatrix} \cdot V$, it is easy to see that $S = U' \cdot V'$ is the slack matrix of $Q^T C_0$, which is a polyhedron that contains no lines.

If C_0 is not full dimensional, $\text{aff}(C_0)$ is a nontrivial linear space. By similar transformations used above, we can show that S is the slack matrix of C_0 in $\text{aff}(C_0)$.

The following theorem and its proof is similar to [5, Theorem 6].

Theorem 4.11 *A nonnegative matrix $S \in \mathbb{R}_+^{p \times q}$ with $\text{rank}(S) \geq 2$ is a slack matrix of a polyhedron if and only if S is a slack matrix of a polyhedral cone and there exists a vector whose component consists of only 0 and 1 contained in the row space of S .*

In [5, Theorem 14], [4, Corollary 5] and [7, Lemma 3.1], they characterized the rank of a slack matrix in terms of the dimension of a polytope. When C is a pointed polyhedral cone, its dimension is equal to the rank of its slack matrix [5, Lemma 13]). These results can be extended to the case that C is a polyhedron.

Theorem 4.12 *Let $C \subset \mathbb{R}^n$ be a n -dimensional polyhedron containing no lines. If C is not a translated convex cone, then the rank of the slack matrix S is $n + 1$.*

Proof Suppose C is not a translated convex cone. Since the rank of its slack matrix does not change after the translation of C and all the slack matrices of C have the same rank, we can assume that C contains the origin and its canonical slack matrix can be written as (4.3). We

show that the matrix U is of full column rank. Otherwise, there exists a vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ such

that $U \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$. If $x_1 \neq 0$, set $x_1 = 1$. Then $1 - f(x_2) = 0$ for all $f \in D_1$ and $g(x_2) = 0$

for all $g \in D_2$. By Theorem 3.6, C is a translated convex cone. This leads to a contradiction. If

$x_1 = 0$, since for every vector $y \in C^\circ$, there exist $\lambda_i^1 \geq 0$, $1 \leq i \leq k_1$ and $\lambda_j^2 \geq 0$, $1 \leq j \leq k_2$ such

that $y = \sum_{i=1}^{k_1} \lambda_i^1 f_i + \sum_{j=1}^{k_2} \lambda_j^2 g_j$ for $f_i \in D_1$ and $g_j \in D_2$, we have $\langle x_2, y \rangle = 0$. Since $\dim(C^\circ) = n$,

C° contains an interior. We derive $x_2 = 0$ since $\langle y, x_2 \rangle = 0$ for each y in C° . Hence, U is a

full column rank matrix. Moreover, since C is a n -dimensional polyhedron, the dimension of

the cone in \mathbb{R}^{n+1} generated by vectors $\begin{pmatrix} 1 \\ c_1 \end{pmatrix}, \dots, \begin{pmatrix} 1 \\ c_t \end{pmatrix}, \begin{pmatrix} 0 \\ r_1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ r_s \end{pmatrix}$ is $n + 1$.

Hence, the matrix V has rank $n + 1$. Therefore, the rank of S is $n + 1$.

Corollary 4.13 *Let C be a polyhedron such that $C = C_0 + L_1$ where L_1 is the lineality space of C and $C_0 = C \cap L_1^\perp$. If C is not a translated convex cone, then the rank of its slack matrix S is $\dim(C_0) + 1$.*

In [10, Theorem 3.2], they gave an upper bound $\lceil 6 \min\{m, n\}/7 \rceil$ of the nonnegative rank for a rank-three nonnegative matrix in $\mathbb{R}^{m \times n}$. By Theorem 4.12 and Corollary 4.13, the slack matrix $S \in \mathbb{R}^{m \times n}$ of every polyhedron in \mathbb{R}^2 is rank-three except that the polyhedron is a translated convex cone. It is easy to show that when $\min(m, n) \geq 7$, every such slack matrix has a nontrivial nonnegative factorization. This fact motivates us to compute a \mathbb{R}_+^6 -factorization of the slack matrix $S \in \mathbb{R}^{8 \times 8}$ in Example 4.6.

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