A Characterization of Perfect Strategies for Mirror Games

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ABSTRACT

We associate mirror games with the universal game algebra and use the *-representation to describe quantum commuting operator strategies. We provide an algebraic characterization of whether or not a mirror game has perfect commuting operator strategies. This new characterization uses a smaller algebra introduced by Paulsen and others for synchronous games and the noncommutative Nullstellensatz developed by Cimpric, Helton and collaborators. An algorithm based on noncommutative Gröbner basis computation and semidefinite programming is given for certifying that a given mirror game has no perfect commuting operator strategies.

KEYWORDS

Mirror Game, Perfect Commuting Operator Strategies, Noncommutative Nullstellensatz, Noncommutative Gröbner basis, Sum of Squares

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1 INTRODUCTION

Quantum nonlocal games have been an active area of research for mathematicians, physicists, and computer scientists in past decades. The violation of Bell inequality has verified the non-locality of quantum mechanics [1], which can be explained in the framework of nonlocal games [8, 34]. A nonlocal game has two or multiple players and a verifier. The verifier sends a question to each player separately, and each player sends an answer back to the verifier without communicating with the others. The verifier determines whether the players win for the given questions and answers. We have a classical strategy if the players can only share classical information. We have a quantum strategy if we allow the players to share quantum information. Bell inequality violations have been proved in the CHSH game [7], where the winning probability using classical strategies is at most 3/4, while a quantum strategy using an entangled state shared by two players can achieve a success probability $\cos^2(\pi/8) \approx 0.85$. Noncommutative Positivstellensätze have been used to study nonlocal games in [11, 32].

A synchronous game is a nonlocal game with two players called Alice and Bob, where Alice and Bob are sent the same question and win if and only if they send the same response. Paulsen and his collaborators found a simpler formulation using a smaller algebra and hard zeroes to study synchronous games in [19, 35]. It has been shown that the success probability of a synchronous game is given by the trace of a bilinear function on a smaller algebra, see Theorem 5.5 in [35], and Theorem 3.2 in [19]. In [2, 19, 42], they give algebraic characterizations of perfect quantum commuting operator strategies for a general game using noncommutative Nullstellensätze [4–6] and Positivstellensätze [3, 17, 18, 29]. Theorem 8.3 and 8.7 in [2] provide a simplified version of the Nullstellensatz theorem for synchronous games.

In [27], Lupini, et al. introduce a new class of nonlocal games called imitation games, in which another player's answer completely determines each player's answer. Any synchronous game is an imitation game as the players send the same answers for the same questions. Some imitation games are not synchronous, such as mirror games, unique games [37], and variable assignment games [27]. Lupini, etc., associates a C*-algebra with any imitation game and characterizes perfect quantum commuting strategies in terms of the properties of this C*-algebra.

As an interesting subclass of imitation games, mirror games include unique games and synchronous games. Theorem 5.5 in [35] for synchronous games has been generalized to Theorem 6.1 in [27] for mirror games, and a representation of perfect quantum commuting strategies for mirror games in terms of traces is also given in the paper. It is natural to ask whether one can obtain similar results as Theorem 8.3 and 8.7 in [2] for mirror games. We answer the question in Theorem 3.1: we provide an algebraic characterization of whether or not a mirror game has perfect commuting operator strategies based on a noncommutative Nullstellensatz and sums of squares. This new characterization uses a smaller algebra introduced by Paulsen and others for synchronous games and the

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noncommutative Nullstellensatz developed by Cimpric, Helton, and collaborators [4–6]. An example is given to demonstrate how to use noncommutative Gröbner basis algorithm [30] and semidefinite programming [41] to verify that a given mirror game has no perfect commuting operator strategies. It would be interesting to see how to extend these results to imitation games.

The paper is organized as follows. Section 2 introduces some preliminary results and definitions of nonlocal games. Some background material on classical strategies and quantum strategies of nonlocal games are included. We also introduce the universal game algebra and its *-representation. Section 3 contains our main result on characterizing whether or not a mirror game has perfect commuting operator strategies based on a noncommutative Nullstellensatz and sums of squares. Finally, Section 4 shows how to use noncommutative Gröbner basis and semidefinite programming to verify that a given mirror game has no perfect commuting operator strategies. A running example is given to demonstrate the computations.

2 PRELIMINARIES

A nonlocal game $\mathcal{G}$ involves a verifier and two players, Alice and Bob. For fixed non-empty finite sets X, Y and A, B, there exists a distribution μ on $X \times Y$. After choosing a pair $(x, y) \in X \times Y$ randomly according to $\mu(x, y)$, the verifier sends elements x to Alice and y to Bob as questions. Alice and Bob send the verifier corresponding answers $a \in A$ and $b \in B$. After receiving an answer from each player, the verifier evaluates the scoring function

$$\lambda : X \times Y \times A \times B \longrightarrow \{0, 1\} \tag{2.1}$$

If $\lambda(x, y, a, b) = 1$, we say Alice and Bob win; otherwise, they lose the game. Alice and Bob know the sets X, Y, A, B and the scoring function λ, but they can't communicate during the game. Alice and Bob can make some arrangements before the game starts.

A deterministic strategy for the players consists of two functions:

$$a : X \longrightarrow A, \ \ b : Y \longrightarrow B, \tag{2.2}$$

and Alice sends $a(x)$ to the verifier if she receives x, and Bob sends $b(y)$ to the verifier if he receives y. Given a deterministic strategy, the players win the game $\mathcal{G}$ with an expectation

$$\sum_{x,y} \mu(x, y)\lambda(x, y, a(x), b(y)). \tag{2.3}$$

We can also give a probabilistic strategy for $\mathcal{G}$ as follows: for each pair $(x, y) \in X \times Y$, let Alice and Bob have mutually independent distributions $p_{x,a}, q_{y,b}$ for $a \in A, b \in B$. When the players receive the questions (x, y), Alice sends the answer a to the verifier with probability $p_{x,a}$ and Bob sends the answer b to the verifier with probability $q_{y,b}$. The winning expectation is

$$\sum_{x,y,a,b} \mu(x, y) p_{x,a} q_{y,b} \lambda(x, y, a, b). \tag{2.4}$$

All deterministic strategies and probabilistic strategies are collectively referred to as classical strategies. We record the set of all classical strategies as C_c, which is a closed set. Notice that any probabilistic strategy can be expressed as a convex combination of deterministic strategies so that the maximal winning expectation of a nonlocal game $\mathcal{G}$ with classical strategies is always obtained by some deterministic strategy. The classical value of $\mathcal{G}$ is defined as the maximal winning expectation

$$\omega_c(\mathcal{G}) = \max_{a,b} \sum_{x,y} \mu(x, y)\lambda(x, y, a(x), b(y)). \tag{2.5}$$

We use the Dirac notation in quantum information to represent the unit vector (a state) in Hilbert space. If Alice and Bob are allowed to share a quantum entangled state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, where both $\mathcal{H}_A$ and $\mathcal{H}_B$ are finite-dimensional Hilbert space, and then they can have a quantum strategy described as follows:

- If Alice receives x, she performs the projection-valued measure (PVM) $P_{x,a}$ on $\mathcal{H}_A$ part of $|\psi\rangle$ and sends the measurement result a to the verifier.
- If Bob receives y, he performs the PVM $Q_{y,b}$ on $\mathcal{H}_B$ part of $|\psi\rangle$ and sends the measurement result b to the verifier.

If we replace PVM by POVM (positive operator-valued measure), the results below will also hold [14, 36].

We record the set of all finite-dimensional quantum strategies as C_q. If we drop the requirement of finite dimension, i.e., $\mathcal{H}_A, \mathcal{H}_B$ can be infinite-dimensional Hilbert spaces, then we get a set of quantum strategies denoted as C_{qs}. Slofstra [38, 39] has proved that neither C_q nor C_{qs} is a closed set. We denote the closure of C_q as C_{qa}. It is evident that

$$C_c \subseteq C_q \subseteq C_{qs} \subseteq C_{qa}$$

Each of the above " $\subseteq$ " is strictly inclusive. The first strict inclusion comes from Bell's inequality, and the last two strict inclusions come from results in [9, 12, 38, 39].

The winning expectation for the given quantum strategy is

$$\sum_{x,y,a,b} \mu(x, y) \cdot \langle\psi|P_{x,a} \otimes Q_{y,b}|\psi\rangle \cdot \lambda(x, y, a, b). \tag{2.6}$$

If we take all of the quantum strategies, the supremum of winning expectations is

$$\omega_q(\mathcal{G}) = \sup_{\substack{\mathcal{H}_A, \mathcal{H}_B, \psi, \\ P_{x,a}, Q_{y,b}}} \sum_{x,y,a,b} \mu(x, y) \cdot \langle\psi|P_{x,a} \otimes Q_{y,b}|\psi\rangle \cdot \lambda(x, y, a, b), \tag{2.7}$$

which is called the quantum value of $\mathcal{G}$. The quantum value can certainly be attained in C_{qa}, but not necessarily in C_q or C_{qs}.

Now we give a quantum commuting operator strategy for $\mathcal{G}$ as follows. Let $\mathcal{H}$ be a (perhaps infinite-dimensional) Hilbert space, $|\psi\rangle \in \mathcal{H}$, and for every $(x, y) \in X \times Y$, Alice and Bob have PVMs $\{E(1)^x_a, \ a \in A\}$ and $\{E(2)^y_b, \ b \in B\}$, respectively. Those two sets of PVMs satisfy the following conditions:

$$E(1)^x_a E(2)^y_b = E(2)^y_b E(1)^x_a, \ \forall \ (x, y, a, b) \in X \times Y \times A \times B. \tag{2.8}$$

When Alice receives an input x, she performs $\{E(1)^x_a, \ a \in A\}$ on $|\psi\rangle$ and sends the result a to the verifier; Similarly, when Bob receives an input y, he performs $\{E(2)^y_b, \ b \in B\}$ on $|\psi\rangle$ and sends the result b to the verifier.

We denote the set of all the quantum commuting operator strategies as C_{qc}. We know that C_{qc} is closed [13]. Given a quantum commuting operator strategy of $\mathcal{G}$, the winning expectation is

$$\sum_{x,y,a,b} \mu(x, y) \cdot \langle\psi|E(1)^x_a \cdot E(2)^y_b|\psi\rangle \cdot \lambda(x, y, a, b). \tag{2.9}$$

Then the supremum of winning expectation (note that it can certainly be obtained) is

$$\omega_{co}(\mathcal{G}) = \sup_{\substack{\mathcal{H},\psi,\\ E(1)^x_a, E(2)^y_b}} \sum_{x,y,a,b} \mu(x,y)\cdot\langle\psi|E(1)^x_a\cdot E(2)^y_b|\psi\rangle\cdot\lambda(x,y,a,b) \tag{2.10}$$

which is called the quantum commuting operator value of $\mathcal{G}$.

It is easy to see that $C_{qa} \subseteq C_{qc}$ [13], so that we have $\omega_c(\mathcal{G}) \leq \omega_q(\mathcal{G}) \leq \omega_{co}(\mathcal{G})$. If we restrict the Hilbert space $\mathcal{H}$ to be finite-dimensional in the commuting operator strategies, then $\omega_q(\mathcal{G}) = \omega_{co}(\mathcal{G})$ (see [38, 40]). There exist games $\mathcal{G}$ for which $\omega_q(\mathcal{G}) < \omega_{co}(\mathcal{G})$ in the infinite-dimensional case, see [13, 39]. The problem of whether $C_{qc} = C_{qa}$ is the famous Tsirelson's problem, and it is true if and only if the Connes' embedding conjecture is true [10]. Kirchberg shows that Connes' conjecture has several equivalent reformulations in operator algebras and Banach space theory [22]. In [25], Klep and Schweighofer show that Connes' embedding conjecture on von Neumann algebras is equivalent to the tracial version of the Positivstellensatz. In 2020, Ji and his collaborators proved $MIP^* = RE$, which implies that Connes' embedding conjecture is false [20]. But we still don't know an explicit counterexample. See [13, 16, 33] for recent results on the Connes' embedding problem. This is the main motivation for us to study quantum nonlocal games.

We say a strategy is perfect if and only if the players can certainly win the game with this strategy. A natural problem is to ask whether there exists a perfect strategy in C_c (or $C_q, C_{qs}, C_{qa}, C_{qc}$) for a given game $\mathcal{G}$.

In [27], the authors introduce a new class of nonlocal games called imitation games, and they provide an algebraic characterization of perfect commuting operator strategies for these games. In this paper, we mainly discuss the mirror game, which is a special subclass of imitation games.

Definition 2.1 (mirror game). *Let $\mathcal{G}$ be a nonlocal game with a question set $X \times Y$, an answer set $A \times B$ and a scoring function $\lambda : X \times Y \times A \times B \to \{0,1\}$. The distribution on $X \times Y$ is the uniform distribution. We say G is a mirror game if there exist functions $\xi : X \to Y$ and $\eta : Y \to X$ such that:*

$$\lambda(x,\xi(x),a,b)\lambda\left(x,\xi(x),a',b\right) = 0,\ \forall x \in X, a \neq a' \in A, b \in B, \tag{2.11}$$

$$\lambda(\eta(y),y,a,b)\lambda\left(\eta(y),y,a,b'\right) = 0,\ \forall y \in Y, a \in A, b \neq b' \in B. \tag{2.12}$$

Example 2.1. Let $X = Y = A = B = \{0,1\}$ and the scoring function λ be given as follows:

λ \ (x,y) / (a,b)	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	1	0	1	0
(0,1)	0	0	1	1
(1,0)	0	1	0	0
(1,1)	1	0	0	1

We can check that $\mathcal{G} = (X, Y, A, B, \lambda)$ is a mirror game with

$$\xi : 0 \mapsto 0,\ 1 \mapsto 0,\ \eta : 0 \mapsto 0,\ 1 \mapsto 1.$$

We use the universal game algebra and representation defined in [2] to describe the relations between the PVMs in the commuting operator strategy below.

Definition 2.2 (universal game algebra). *Let*

$$\mathbf{e} = (e(1)^x_a)_{x\in X, a\in A} \cup (e(2)^y_b)_{y\in Y, b\in B} \tag{2.13}$$

and $\mathbb{C}\langle\mathbf{e}\rangle$ be the noncommutative free algebra generated by the tuple $\mathbf{e}$. Let $\mathscr{I}$ be the two-sided ideal generated by the following polynomials:

$$\begin{aligned}&\left\{e(1)^x_a e(2)^y_b - e(2)^y_b e(1)^x_a \mid \forall x,y,a,b\right\}\\ &\cup \left\{(e(1)^x_a)^2 - e(1)^x_a \mid \forall x,a\right\} \cup \left\{(e(2)^y_b)^2 - e(2)^y_b \mid \forall y,b\right\}\\ &\cup \left\{e(1)^x_{a_1} e(1)^x_{a_2} \mid \forall x,\ a_1 \neq a_2\right\} \cup \left\{e(2)^y_{b_1} e(2)^y_{b_2} \mid \forall y,\ b_1 \neq b_2\right\}\\ &\cup \left\{\sum_{a\in A} e(1)^x_a - 1 \mid \forall x\right\} \cup \left\{\sum_{b\in B} e(2)^y_b - 1 \mid \forall y\right\}.\end{aligned} \tag{2.14}$$

Then we define $\mathcal{U} = \mathbb{C}\langle\mathbf{e}\rangle/\mathscr{I}$ and equip $\mathcal{U}$ with the involution induced by

$$(e(1)^x_a)^* = e(1)^x_a,\ (e(2)^y_b)^* = e(2)^y_b. \tag{2.15}$$

where the " $*$ " of a complex number is its conjugate. We call $\mathcal{U}$ the universal game algebra of $\mathcal{G}$.

For the universal game algebra $\mathcal{U}$, we can use *-representation to describe a commuting operator strategy. A *-representation of $\mathcal{U}$ is a unital *-homomorphism

$$\pi : \mathcal{U} \to \mathcal{B}(\mathcal{H}), \tag{2.16}$$

where $\mathcal{B}(\mathcal{H})$ denotes the set of bounded linear operators on a Hilbert space $\mathcal{H}$ and π satisfies $\pi(u^*) = \pi(u)^*, \forall u \in U$. It is obvious that any commutative PVMs $\{E(1)^x_a,\ a \in A\}$ and $\{E(2)^y_b,\ b \in B\}$ can be obtained by the unital *-homomorphism

$$\pi : e(1)^x_a \mapsto E(1)^x_a,\ e(2)^y_b \mapsto E(2)^y_b, \tag{2.17}$$

and given an arbitrary unital *-homomorphism, the image of $\mathcal{U}$'s generators is commutative PVMs. Therefore, each commuting operator strategy corresponds to a pair $(\pi, |\psi\rangle)$, where $\pi : \mathcal{U} \to \mathcal{B}(\mathcal{H})$ is a *-representation and $|\psi\rangle \in \mathcal{H}$ is a state (a unit vector). We can use the language of representation to rewrite $\omega_{co}(\mathcal{G})$ as follows:

$$\omega_{co}(\mathcal{G}) = \sup_{\pi,\psi}\langle\psi|\pi(\Phi_{\mathcal{G}})|\psi\rangle, \tag{2.18}$$

where

$$\Phi_{\mathcal{G}} = \sum_{x,y}\sum_{a,b}\mu(x,y)\lambda(x,y,a,b)e(1)^x_a e(2)^y_b, \tag{2.19}$$

and the supremum is taken over all *-representations π of $\mathcal{U}$ into bounded operators on a Hilbert space $\mathcal{H}$ and state $|\psi\rangle \in \mathcal{H}$.

Since we assume that μ is a uniform distribution, $\Phi_{\mathcal{G}}$ can be simplified to

$$\Phi_{\mathcal{G}} = \frac{1}{|X|\cdot|Y|}\sum_{x,y}\sum_{a,b}\lambda(x,y,a,b)e(1)^x_a e(2)^y_b. \tag{2.20}$$

It's obvious that a game $\mathcal{G}$ has a perfect commuting operator strategy if and only if

$$\omega_{co}(\mathcal{G}) = 1. \tag{2.21}$$

We also need the concept of tracial linear functional and tracial state.

Definition 2.3. *A linear mapping $\tau : \mathcal{A} \to \mathbb{C}$ on an algebra $\mathcal{A}$ is said to be tracial if and only if*

$$\tau(ab) = \tau(ba),\ \forall a, b \in \mathcal{A}. \tag{2.22}$$

Given a Hilbert space $\mathcal{H}$ and an operator algebra $\mathcal{A}$ acting on $\mathcal{H}$, a state $|\psi\rangle \in \mathcal{H}$ is called a tracial state if the linear mapping it induces is tracial, i.e.

$$\langle\psi|ab|\psi\rangle = \langle\psi|ba|\psi\rangle,\ \forall a, b \in \mathcal{A}. \tag{2.23}$$

Especially if $\mathcal{A}$ is a von Neumann algebra, and there exists such a tracial linear mapping τ on $\mathcal{A}$, we say $(\mathcal{A}, \tau)$ is a tracial von Neumann algebra.

The definition of determining set is given in [2].

Definition 2.4 (determining set). *Let $\mathcal{G}$ be a nonlocal game; its universal game algebra is $\mathcal{U}$. A set $\mathcal{F} \subseteq \mathcal{U}$ is denoted as a determining set of $\mathcal{G}$ if it satisfies that a pair $(\pi, |\psi\rangle)$ is a perfect commuting operator strategy if and only if $\pi(\mathcal{F})|\psi\rangle = \{0\}$.*

According to Theorem 3.5 in [2], given any nonlocal game, we have a natural determining set:

Proposition 2.1. *Let $\mathcal{G} = (X, Y, A, B, \lambda)$ be a nonlocal game, the set of invalid elements*

$$\mathcal{N} = \{e(1)^x_a e(2)^y_b \mid \lambda(x, y, a, b) = 0\} \tag{2.24}$$

is a determining set. We call it the invalid determining set.

Corollary 2.2. *The left ideal $\mathcal{L}(\mathcal{N})$ generated by $\mathcal{N}$ is also a determining set.*

For a mirror game $\mathcal{G}$, suppose its universal game algebra is $\mathcal{U}$, and we define:

$$f^{\eta(y)}_{y,b} = \textstyle\sum_{a\in A, \lambda(\eta(y),y,a,b)=1} e(1)^{\eta(y)}_a, \tag{2.25}$$

$$g^{\xi(x)}_{x,a} = \textstyle\sum_{b\in B, \lambda(x,\xi(x),a,b)=1} e(2)^{\xi(x)}_b. \tag{2.26}$$

Definition 2.5. *Let $\mathcal{G} = (X, Y, A, B, \lambda)$ be a nonlocal game. For $x \in X, y \in Y, a \in A$ and $b \in B$, denote*

$$E^a_{x,y} = \{b \in B : \lambda(x, y, a, b) = 1\} \tag{2.27}$$

$$E^b_{x,y} = \{a \in A : \lambda(x, y, a, b) = 1\}. \tag{2.28}$$

We define a mirror game as regular if and only if

$$\cup_{a\in A} E^a_{x,\xi(x)} = B \text{ and } \cup_{b\in B} E^b_{\eta(y),y} = A,\ \forall x \in X, y \in Y. \tag{2.29}$$

Remark that this condition appeared in [27] firstly, but they didn't name it.

Lemma 2.1. *A mirror game $\mathcal{G}$ is regular if and only if the universal game algebra satisfies:*

$$\sum_{a\in A} g^{\xi(x)}_{x,a} = 1,\ \forall x \in X \text{ and } \sum_{b\in B} f^{\eta(y)}_{y,b} = 1,\ \forall y \in Y. \tag{2.30}$$

Proof. By the definition of regularity and the universal game algebra. □

Example 2.1 (continued). *For the mirror game $\mathcal{G}$ defined in Example 2.1, we can compute that*

$$f^{\eta(0)}_{0,0} = e(1)^0_0,\ f^{\eta(0)}_{0,1} = e(1)^0_1,\ f^{\eta(1)}_{1,0} = 0,\ f^{\eta(1)}_{1,1} = e(1)^0_0 + e(1)^0_1 = 1.$$

It is easy to check that $\sum_{b\in B} f^{\eta(y)}_{y,b} = 1,\ \forall y \in Y$. Similarly, we have

$$g^{\xi(0)}_{0,0} = e(2)^0_0,\ g^{\xi(0)}_{0,1} = e(2)^0_1,\ g^{\xi(1)}_{1,0} = 1,\ g^{\eta(1)}_{1,1} = 0.$$

It is true that $\sum_{a\in A} g^{\xi(x)}_{x,a} = 1,\ \forall x \in X$. Hence, $\mathcal{G}$ is a regular mirror game.

In the following sections, we'll only consider regular mirror games.

3 MAIN RESULT

Given a universal game algebra $\mathcal{U}$ of a nonlocal game $\mathcal{G}$, a general noncommutative Nullstellensatz developed by Cimpric, Helton, and their collaborators [5, 6] has been adapted to Theorem 4.1 and 4.3 in [2] to show that $\mathcal{G}$ has a perfect commuting operator strategy if and only if there exists a *-representation $\pi : \mathcal{U} \to \mathcal{B}(\mathcal{H})$ and a state $|\psi\rangle \in \mathcal{H}$ satisfying

$$\pi(\mathcal{L}(\mathcal{N}))|\psi\rangle = \{0\}, \tag{3.1}$$

which is also equivalent to

$$-1 \notin \mathcal{L}(\mathcal{N}) + \mathcal{L}(\mathcal{N})^* + \mathrm{SOS}_{\mathcal{U}}, \tag{3.2}$$

where

$$\mathrm{SOS}_{\mathcal{U}} = \left\{ \sum_{i=1}^{n} u_i^* u_i \mid u_i \in \mathcal{U},\ n \in \mathbb{N} \right\}, \tag{3.3}$$

$\mathcal{L}(\mathcal{N})$ is the left ideal generated by the invalid determining set $\mathcal{N}$.

For synchronous games, the authors use a smaller algebra $\mathcal{U}(1)$ which is the subalgebra of $\mathcal{U}$ generated by $e(1)^x_a$, and prove that a synchronous game has a perfect commuting operator strategy if and only if there exists a *-representation $\pi' : \mathcal{U}(1) \to \mathcal{B}(\mathcal{H})$ and a tracial state $|\psi\rangle \in \mathcal{H}$ satisfying

$$\pi'(\mathcal{J}(\mathrm{synch}\mathcal{B}(1)))|\psi\rangle = \{0\}, \tag{3.4}$$

where $\mathcal{J}(\mathrm{synch}\mathcal{B}(1))$ is a two-sided ideal in $\mathcal{U}(1)$, see Theorem 8.3 and 8.7 in [2].

In Theorem 3.1, we generalize Theorem 8.3 and 8.7 in [2] for mirror games and provide a characterization of whether or not a mirror game has perfect commuting operator strategies using smaller algebras $\mathcal{U}(1)$ and $\mathcal{U}(2)$, where $\mathcal{U}(1)$ is the subalgebra of $\mathcal{U}$ generated by $e(1)^x_a$ only, and $\mathcal{U}(2)$ is the subalgebra of $\mathcal{U}$ generated by $e(2)^y_b$ only.

Let $\mathcal{J}(\mathrm{mir1})$ be the two-sided ideal of $\mathcal{U}(1)$ generated by

$$\left\{ e(1)^x_a f^{\eta(y)}_{y,b} \mid \lambda(x, y, a, b) = 0 \right\}, \tag{3.5}$$

and $\mathcal{J}(\mathrm{mir2})$ be the two-sided ideal of $\mathcal{U}(2)$ generated by

$$\left\{ e(2)^y_b g^{\xi(x)}_{x,a} \mid \lambda(x, y, a, b) = 0 \right\}. \tag{3.6}$$

Example 2.1 (continued). *Let's continue the computation in Example 2.1. The two-sided ideal $\mathcal{J}(\mathrm{mir1})$ is generated by the following elements:*

$$\{e(1)^0_0 e(1)^0_1,\ e(1)^0_1 e(1)^0_0,\ 0,\ e(1)^0_0,\ e(1)^0_1,\ e(1)^1_1 e(1)^0_0,\ e(1)^1_1 e(1)^0_1, 0, 0\}.$$

It is clear that $\mathcal{J}(\mathrm{mir1})$ is generated by $\{e(1)^0_0,\ e(1)^0_1\}$ in $\mathcal{U}(1)$.

Theorem 3.1 (main result). *A regular mirror game with its universal game algebra $\mathcal{U}$ and invalid determining set $\mathcal{N}$ has a perfect commuting operator strategy if and only if any of the equivalent conditions are satisfied:*

(1) *There exists a *-representation* $\pi : \mathcal{U} \to \mathcal{B}(\mathcal{H})$ *and a state* $|\psi\rangle \in \mathcal{H}$ *satisfying*

$$\pi(\mathcal{L}(\mathcal{N}))|\psi\rangle = \{0\}; \tag{3.7}$$

(2) *There exists a *-representation* $\pi' : \mathcal{U}(1) \to \mathcal{B}(\mathcal{H})$ *and a tracial state* $|\psi\rangle \in \mathcal{H}$ *satisfying*

$$\pi'(\mathcal{J}(\mathrm{mir1}))|\psi\rangle = \{0\}; \tag{3.8}$$

(3) *There exists a *-representation* $\pi'' : \mathcal{U}(2) \to \mathcal{B}(\mathcal{H})$ *and a tracial state* $|\phi\rangle \in \mathcal{H}$ *satisfying*

$$\pi''(\mathcal{J}(\mathrm{mir2}))|\phi\rangle = \{0\}; \tag{3.9}$$

(4) *There exists a *-representation* π_0' *of* $\mathcal{U}(1)$ *mapping into a tracial von Neumann algebra* $\mathcal{W} \subseteq \mathcal{B}(\mathcal{H})$ *satisfying*

$$\pi_0'(\mathcal{J}(\mathrm{mir1})) = \{0\}; \tag{3.10}$$

(5) *There exists a *-representation* π_0'' *of* $\mathcal{U}(2)$ *mapping into a tracial von Neumann algebra* $\mathcal{W}' \subseteq \mathcal{B}(\mathcal{H})$ *satisfying*

$$\pi_0''(\mathcal{J}(\mathrm{mir2})) = \{0\}. \tag{3.11}$$

To prove our main theorem, we introduce several lemmas.

Lemma 3.1. *For every* $x \in X$ *and* $a \in A$, *we have* $e(1)^x_a - g^{\xi(x)}_{x,a} \in \mathcal{L}(\mathcal{N})$. *Similarly, for every* $y \in Y$ *and* $b \in B$, *we have* $e(2)^y_b - f^{\eta(y)}_{y,b} \in \mathcal{L}(\mathcal{N})$.

Proof. Firstly, by $\sum_{a'} e(1)^x_{a'} = 1$, we have

$$\begin{aligned}
&e(1)^x_a - g^{\xi(x)}_{x,a}\\
&= e(1)^x_a - \left(\sum_{a'\in A} e(1)^x_{a'}\right) g^{\xi(x)}_{x,a} = e(1)^x_a - e(1)^x_a \cdot g^{\xi(x)}_{x,a}\\
&\quad + \left(\sum_{a'\neq a} e(1)^x_{a'}\right)\cdot\left(\sum_{b\in B,\lambda(x,\xi(x),a,b)=1} e(2)^{\xi(x)}_b\right)\\
&= e(1)^x_a \cdot \left(1 - \sum_{b\in B,\lambda(x,\xi(x),a,b)=1} e(2)^{\xi(x)}_b\right)\\
&\quad + \sum_{a'\neq a}\sum_{b\in B,\lambda(x,\xi(x),a,b)=1} e(1)^x_{a'} e(2)^{\xi(x)}_b.
\end{aligned}$$

Notice that

$$1 - \sum_{b\in B,\lambda(x,\xi(x),a,b)=1} e(2)^{\xi(x)}_b = \sum_{b\in B,\lambda(x,\xi(x),a,b)=0} e(2)^{\xi(x)}_b.$$

By the definition of $\mathcal{L}(\mathcal{N})$, we have

$$\begin{aligned}
&e(1)^x_a\left(1 - \textstyle\sum_{b\in B,\lambda(x,\xi(x),a,b)=1} e(2)^{\xi(x)}_b\right)\\
&= \textstyle\sum_{b\in B,\lambda(x,\xi(x),a,b)=0} e(1)^x_a e(2)^{\xi(x)}_b \in \mathcal{L}(\mathcal{N}).
\end{aligned} \tag{3.12}$$

On the other hand, it is known by the definition of mirror games that $\lambda(x,\xi(x),a',b) = 0$ when $a' \neq a$ and $\lambda(x,\xi(x),a,b) = 1$. Hence we have $e(1)^x_{a'} e(2)^{\xi(x)}_b \in \mathcal{N}$ which implies

$$\sum_{a'\neq a}\sum_{b\in B,\lambda(x,\xi(x),a,b)=1} e(1)^x_{a'} e(2)^{\xi(x)}_b \in \mathcal{L}(\mathcal{N}). \tag{3.13}$$

Therefore, we have

$$e(1)^x_a - g^{\xi(x)}_{x,a} \in \mathcal{L}(\mathcal{N}). \tag{3.14}$$

Similarly, $e(2)^y_b - f^{\eta(y)}_{y,b}$ can be rewritten as follows:

$$\begin{aligned}
e(2)^y_b - f^{\eta(y)}_{y,b} &= e(2)^y_b - f^{\eta(y)}_{y,b}\left(\sum_{b'\in B} e(2)^y_{b'}\right)\\
&= e(2)^y_b - f^{\eta(y)}_{y,b} e(2)^y_b + \sum_{b'\neq b} f^{\eta(y)}_{y,b} e(2)^y_{b'}\\
&= \left(1 - \sum_{a\in A,\lambda(\eta(y),y,a,b)=1} e(1)^{\eta(y)}_a\right)\cdot e(2)^y_b\\
&\quad + \sum_{b'\neq b}\sum_{a\in A,\lambda(\eta(y),y,a,b)=1} e(1)^{\eta(y)}_a e(2)^y_{b'}.
\end{aligned}$$

We still have

$$\begin{aligned}
&\left(1 - \textstyle\sum_{a\in A,\lambda(\eta(y),y,a,b)=1} e(1)^{\eta(y)}_a\right)\cdot e(2)^y_b\\
&= \textstyle\sum_{a\in A,\lambda(\eta(y),y,a,b)=0} e(1)^{\eta(y)}_a e(2)^y_b \in \mathcal{L}(\mathcal{N}),
\end{aligned} \tag{3.15}$$

and by the definition of the mirror game, we have

$$\sum_{b'\neq b}\sum_{a\in A,\lambda(\eta(y),y,a,b)=1} e(1)^{\eta(y)}_a e(2)^y_{b'} \in \mathcal{L}(\mathcal{N}). \tag{3.16}$$

Therefore, we have

$$e(2)^y_b - f^{\eta(y)}_{y,b} \in \mathcal{L}(\mathcal{N}). \tag{3.17}$$

□

Lemma 3.2. *We have the following inclusion relations:*

$$\{e(1)^x_a f^{\eta(y)}_{y,b} \mid \lambda(x,y,a,b) = 0\} \subseteq \mathcal{L}(\mathcal{N}); \tag{3.18}$$

$$\{e(2)^y_b g^{\xi(x)}_{x,a} \mid \lambda(x,y,a,b) = 0\} \subseteq \mathcal{L}(\mathcal{N}). \tag{3.19}$$

Proof. Notice that

$$e(1)^x_a f^{\eta(y)}_{y,b} = e(1)^x_a e(2)^y_b - e(1)^x_a\left(e(2)^y_b - f^{\eta(y)}_{y,b}\right)$$

As $\lambda(x,y,a,b) = 0$, we know $e(1)^x_a e(2)^y_b \in \mathcal{N} \subseteq \mathcal{L}(\mathcal{N})$. By Lemma 3.1, we have $e(2)^y_b - f^{\eta(y)}_{y,b} \in \mathcal{L}(\mathcal{N})$. Therefore, we have

$$e(1)^x_a f^{\eta(y)}_{y,b} \in \mathcal{L}(\mathcal{N}). \tag{3.20}$$

For $e(2)^y_b g^{\xi(x)}_{x,a}$, we have

$$\begin{aligned}
e(2)^y_b g^{\xi(x)}_{x,a} &= e(2)^y_b e(1)^x_a - e(2)^y_b\left(e(1)^x_a - g^{\xi(x)}_{x,a}\right)\\
&= e(1)^x_a e(2)^y_b - e(2)^y_b\left(e(1)^x_a - g^{\xi(x)}_{x,a}\right)\\
&\quad (\text{as } e(1)^x_a \text{ always commutes with } e(2)^y_b)
\end{aligned}$$

We still have $e(1)^x_a e(2)^y_b \in \mathcal{N} \subseteq \mathcal{L}(\mathcal{N})$ by $\lambda(x,y,a,b) = 0$, and $e(1)^x_a - g^{\xi(x)}_{x,a} \in \mathcal{L}(\mathcal{N})$ by Lemma 3.1. Then we have

$$e(2)^y_b g^{\xi(x)}_{x,a} \in \mathcal{L}(\mathcal{N}). \tag{3.21}$$

□

Lemma 3.3. *We have* $\mathcal{J}(\mathrm{mir1}) \subseteq \mathcal{L}(\mathcal{N})$ *and* $\mathcal{J}(\mathrm{mir2}) \subseteq \mathcal{L}(\mathcal{N})$.

Proof. Firstly let us consider a monomial

$$w(e(1)) = e(1)^{x_1}_{a_1}\cdots e(1)^{x_t}_{a_t} \in \mathcal{U}(1),$$

we have:

$$\begin{aligned}
& e(1)^{x_1}_{a_1}\cdots e(1)^{x_t}_{a_t} - g^{\xi(x_t)}_{x_t,a_t} e(1)^{x_1}_{a_1}\cdots e(1)^{x_{t-1}}_{a_{t-1}} \\
&= e(1)^{x_1}_{a_1}\cdots e(1)^{x_t}_{a_t} - e(1)^{x_1}_{a_1}\cdots e(1)^{x_{t-1}}_{a_{t-1}} g^{\xi(x_t)}_{x_t,a_t} \\
&= e(1)^{x_1}_{a_1}\cdots e(1)^{x_{t-1}}_{a_{t-1}}\left(e(1)^{x_t}_{a_t} - g^{\xi(x_t)}_{x_t,a_t}\right) \in \mathcal{L}(\mathcal{N}).
\end{aligned}$$

Then we have

$$\begin{aligned}
& e(1)^{x_1}_{a_1}\cdots e(1)^{x_{t-1}}_{a_{t-1}} e(1)^{x_t}_{a_t} - g^{\xi(x_t)}_{x_t,a_t} g^{\xi(x_{t-1})}_{x_{t-1},a_{t-1}}\cdots g^{\xi(x_1)}_{x_1,a_1} \\
&= e(1)^{x_1}_{a_1}\cdots e(1)^{x_t}_{a_t} - g^{\xi(x_t)}_{x_t,a_t} e(1)^{x_1}_{a_1}\cdots e(1)^{x_{t-1}}_{a_{t-1}} \\
&+ g^{\xi(x_t)}_{x_t,a_t}\left(e(1)^{x_1}_{a_1}\cdots e(1)^{x_{t-1}}_{a_{t-1}} - g^{\xi(x_{t-1})}_{x_{t-1},a_{t-1}} e(1)^{x_1}_{a_1}\cdots e(1)^{x_{t-2}}_{a_{t-2}}\right) \\
&+ \cdots + g^{\xi(x_t)}_{x_t,a_t}\cdots g^{\xi(x_2)}_{x_2,a_2}\left(e(1)^{x_1}_{a_1} - g^{\xi(x_1)}_{x_1,a_1}\right) \in \mathcal{L}(\mathcal{N}).
\end{aligned} \tag{3.22}$$

It's known that $g^{\xi(x_t)}_{x_t,a_t}\cdots g^{\xi(x_1)}_{x_1,a_1} = w^*(g)$. Then equation (3.22) can be written as

$$w(e(1)) - w^*(g) \in \mathcal{L}(\mathcal{N}). \tag{3.23}$$

Suppose

$$p = \sum_{\substack{x,y,a,b,\ \lambda(x,y,a,b)=0\\ u,w}} u(e(1))\cdot e(1)^x_a f^{\eta(y)}_{y,b}\cdot w(e(1)) \in \mathcal{J}(\text{mir1}),$$

where $u(e(1)), w(e(1))$ are monomials in $\mathcal{U}(1)$ and $\lambda(x,y,a,b)=0$, we compute:

$$\begin{aligned}
p &= \sum\Big(u(e(1))\cdot e(1)^x_a f^{\eta(y)}_{y,b}\cdot\big(w(e(1)) - w^*(g)\big) \\
&\quad + u(e(1))\cdot e(1)^x_a f^{\eta(y)}_{y,b} w^*(g)\Big) \\
&= \sum\Big(u(e(1))\cdot e(1)^x_a f^{\eta(y)}_{y,b}\cdot\big(w(e(1)) - w^*(g)\big) \\
&\quad + w^*(g)\, u(e(1))\cdot e(1)^x_a f^{\eta(y)}_{y,b}\Big).
\end{aligned}$$

The second " = " is true because $w^*(g)$ is a polynomial in $\mathcal{U}(2)$, which commutes with all of elements in $\mathcal{U}(1)$. By the equation (3.23), we know that

$$u(e(1))\cdot e(1)^x_a f^{\eta(y)}_{y,b}\cdot\big(w(e(1)) - w^*(g)\big) \in \mathcal{L}(\mathcal{N}), \tag{3.24}$$

and from Lemma 3.2 we know

$$w^*(g)u(e(1))\cdot e(1)^x_a f^{\eta(y)}_{y,b} \in \mathcal{L}(\mathcal{N}). \tag{3.25}$$

Then we conclude that every $p \in \mathcal{J}(\text{mir1})$ satisfies $p \in \mathcal{L}(\mathcal{N})$, which means $\mathcal{J}(\text{mir1}) \subseteq \mathcal{L}(\mathcal{N})$.

Similarly, we have

$$\begin{aligned}
& e(2)^{y_1}_{b_1}\cdots e(2)^{y_t}_{b_t} - f^{\eta(y_t)}_{y_t,b_t} e(2)^{y_1}_{b_1}\cdots e(2)^{y_{t-1}}_{y_{t-1}} \\
&= e(2)^{y_1}_{b_1}\cdots e(2)^{y_t}_{b_t} - e(2)^{y_1}_{b_1}\cdots e(2)^{y_{t-1}}_{b_{t-1}} f^{\eta(y_t)}_{y_t,b_t} \\
&= e(2)^{y_1}_{b_1}\cdots e(2)^{y_{t-1}}_{b_{t-1}}(e(2)^{y_t}_{b_t} - f^{\eta(y_t)}_{y_t,b_t}) \in \mathcal{L}(\mathcal{N}).
\end{aligned} \tag{3.26}$$

It is also true that

$$\begin{aligned}
w(e(2)) - w^*(f) = {} & e(2)^{y_1}_{b_1}\cdots e(2)^{y_{t-1}}_{b_{t-1}} e(2)^{y_t}_{b_t} \\
& - f^{\eta(y_t)}_{y_t,b_t} f^{\eta(y_{t-1})}_{y_{t-1},b_{t-1}}\cdots f^{\eta(y_1)}_{y_1,y_1} \in \mathcal{L}(\mathcal{N}).
\end{aligned} \tag{3.27}$$

Then for

$$q = \sum u(e(2))\cdot e(2)^y_b g^{\xi(x)}_{x,a}\cdot w(e(2)) \in \mathcal{J}(\text{mir2}),$$

we also have

$$\begin{aligned}
q &= \sum\Big(u(e(2))\cdot e(2)^y_b g^{\xi(x)}_{x,a}\cdot(w(e(2)) - w^*(f)) \\
&\qquad + u(e(2))\cdot e(2)^y_b g^{\xi(x)}_{x,a} w^*(f)\Big) \\
&= \sum\Big(u(e(2))\cdot e(2)^y_b g^{\xi(x)}_{x,a}\cdot(w(e(2)) - w^*(f)) \\
&\qquad + w^*(f)u(e(2))\cdot e(2)^y_b g^{\xi(x)}_{x,a}\Big) \in \mathcal{L}(\mathcal{N}).
\end{aligned} \tag{3.28}$$

Therefore we have $\mathcal{J}(\text{mir2}) \subseteq \mathcal{L}(\mathcal{N})$. □

Lemma 3.4. *Let $(\pi, |\psi\rangle)$ be a perfect commuting operator strategy of a regular mirror game $\mathcal{G}$, then $|\psi\rangle$ is a tracial state on both $\pi(\mathcal{U}(1))$ and $\pi(\mathcal{U}(2))$.*

Proof. For the case $\pi(\mathcal{U}(1))$, it suffices to show that for any different $e(1)^{x_1}_{a_1}$ and $e(1)^{x_2}_{a_2}$, we have

$$\langle\psi|\pi(e(1)^{x_1}_{a_1})\pi(e(1)^{x_2}_{a_2})|\psi\rangle = \langle\psi|\pi(e(1)^{x_2}_{a_2})\pi(e(1)^{x_1}_{a_1})|\psi\rangle,$$

and we can complete the proof by using inductions on the length of monomials and linearity.

In fact, since $(\pi, |\psi\rangle)$ is a perfect commuting operator strategy of $\mathcal{G}$, we have $\pi(\mathcal{L}(\mathcal{N}))|\psi\rangle = \{0\}$ according to Definition 2.4 and Proposition 2.1. Lemma 3.1 tells us that every $e(1)^x_a - g^{\xi(x)}_{x,a} \in \mathcal{L}(\mathcal{N})$, so we have

$$\pi(e(1)^{x_1}_{a_1} - g^{\xi(x_1)}_{x_1,a_1})|\psi\rangle = 0,\ \text{and}\ \pi(e(1)^{x_2}_{a_2} - g^{\xi(x_2)}_{x_2,a_2})|\psi\rangle = 0.$$

Therefore, we have

$$\begin{aligned}
& \langle\psi|\pi(e(1)^{x_1}_{a_1})\pi(e(1)^{x_2}_{a_2})|\psi\rangle \\
&= \langle\psi|\pi(e(1)^{x_1}_{a_1})\pi(g^{\xi(x_2)}_{x_2,a_2})|\psi\rangle \\
&= \langle\psi|\pi\left(e(1)^{x_1}_{a_1} g^{\xi(x_2)}_{x_2,a_2}\right)|\psi\rangle \qquad (\pi \text{ is a representation}) \\
&= \langle\psi|\pi\left(g^{\xi(x_2)}_{x_2,a_2} e(1)^{x_1}_{a_1}\right)|\psi\rangle \qquad (g^{\xi(x_2)}_{x_2,a_2} \text{ commutes with } e(1)^{x_1}_{a_1}) \\
&= \langle\psi|\pi(g^{\xi(x_2)}_{x_2,a_2})\pi(e(1)^{x_1}_{a_1})|\psi\rangle \\
&= \langle\psi|\pi(e(1)^{x_2}_{a_2})\pi(e(1)^{x_1}_{a_1})|\psi\rangle.
\end{aligned}$$

This shows that $|\psi\rangle$ is a tracial state on $\pi(\mathcal{U}(1))$.

Similarly, using $e(2)^y_b - f^{\eta(y)}_{y,b} \in \mathcal{L}(\mathcal{N})$, we can prove that $|\psi\rangle$ is a tracial state on $\pi(\mathcal{U}(2))$. □

Now we can prove our main theorem.

Proof of Theorem3.1. We show that (1) $\iff$ (2) $\iff$ (4) and (1) $\iff$ (3) $\iff$ (5).

Firstly, (1) is equivalent to the existence of a perfect commuting operator strategy by the definition of the determining set.

(1) $\Longrightarrow$ (2): Suppose $(\pi, |\psi\rangle)$ is a pair that satisfies the conditions in (1), and we let π' be the restriction of π to $\mathcal{U}(1)$. It is obvious

$$\pi'(\mathcal{J}(\text{mir1}))|\psi\rangle = \pi(\mathcal{J}(\text{mir1}))|\psi\rangle \subseteq \pi(\mathcal{L}(\mathcal{N}))|\psi\rangle = \{0\},$$

where the first " = " comes from the restriction, the " ⊆ " is derived from Lemma 3.3, and the second " = " is derived from Proposition 2.1. By Lemma 3.4, We know $|\psi\rangle$ is a tracial state. Then (1) $\Longrightarrow$ (2) has been proved.

(2) $\Longrightarrow$ (1): Using (π', ψ), we define the following positive linear functional

$$\ell' : \mathcal{U}(1) \to \mathbb{C}, \quad h \mapsto \langle\psi|\pi'(h)|\psi\rangle.$$

Since $|\psi\rangle$ is a tracial state, we know ℓ' is tracial. Next extend ℓ' to a linear functional ℓ on $\mathcal{U}$ by mapping a monomial

$$\ell : \ w(e(1))u(e(2)) \mapsto \ell'\left(w(e(1))u^*(f)\right),$$

where

$$u^*(f) = f_{y_t,b_t}^{\eta(y_t)} f_{y_{t-1},b_{t-1}}^{\eta(y_{t-1})} \cdots f_{y_1,b_1}^{\eta(y_1)}$$

if $u(e(2)) = e(2)_{b_1}^{y_1} \cdots e(2)_{b_{t-1}}^{y_{t-1}} e(2)_{b_t}^{y_t}$. We show that ℓ is well-defined. It is sufficient to show that ℓ is well defined on $\mathbb{C}$. Notice that the regularity ensures that:

$$\sum_{b\in B} f_{y,b}^{\eta(y)} = 1, \ \forall y \in Y. \text{ (by Lemma 2.1)}$$

Then we have

$$\ell\left(\sum_{b\in B} e(2)_b^y\right) = \ell'\left(\sum_{b\in B} f_{y,b}^{\eta(y)}\right) = \langle\psi|\psi\rangle = 1, \ \forall y \in Y.$$

On the other hand,

$$\ell\left(\sum_{b\in B} e(2)_b^y\right) = \ell(1) = \ell'(1) = 1.$$

Therefore, ℓ is really well defined.

Next we show that ℓ can distinguish -1 and $\mathrm{SOS}_{\mathcal{U}} + \mathcal{L}(\mathcal{N}) + \mathcal{L}(\mathcal{N})^*$. The motivation of this proof is similar to the proof of Theorem 8.3 in [2].

Since $|\psi\rangle$ is a tracial state, we know that ℓ is symmetric in a sense that $\ell(h^*) = \ell(h)^*$ for all $h \in \mathcal{U}(1)$. To check that ℓ is positive, let $h = \sum_{i,j} \beta_{ij} w_i(e(1))u_j(e(2)) \in \mathcal{U}$, then we have

$$h^*h = \sum_{i,j}\sum_{k,s} \beta_{ij}^*\beta_{ks} \cdot w_i^*(e(1))w_k(e(1))u_j^*(e(2))u_s(e(2)),$$

whence

$$\ell\left(h^*h\right) = \sum_{i,j}\sum_{k,s} \beta_{ij}^*\beta_{ks} \cdot \ell'\left(w_i^*(e(1))w_k(e(1))u_s^*(f)u_j(f)\right). \tag{3.29}$$

Set

$$\check{h} = \sum_{i,j} \beta_{ij} w_i(e(1))u_j^*(f) \in \mathcal{U}(1)$$

Then we have

$$\check{h}^*\check{h} = \sum_{i,j}\sum_{k,s} \beta_{ij}^*\beta_{ks} u_j(f) w_i^*(e(1))w_k(e(1))u_s^*(f),$$

$$\ell'\left(\check{h}^*\check{h}\right) = \sum_{i,j}\sum_{k,s} \beta_{ij}^*\beta_{ks} \ell'\left(u_j(f) w_i^*(e(1))w_k(e(1))u_s^*(f)\right). \tag{3.30}$$

Since ℓ' is tracial, we have

$$\begin{aligned}&\ell'\left(w_i^*(e(1))w_k(e(1))u_s^*(f)u_j(f)\right)\\ &= \ell'\left(u_j(f)w_i^*(e(1))w_k(e(1))u_s^*(f)\right).\end{aligned}$$

This implies that the values in Equation (3.29) and Equation (3.30) are the same. Therefore, we have

$$\ell\left(h^*h\right) = \ell'\left(\check{h}^*\check{h}\right) \geq 0,$$

which implies $\ell(\mathrm{SOS}_{\mathcal{U}}) \geq 0$.

It remains to show that $\ell(\mathcal{L}(\mathcal{N})) = \{0\}$. Elements in $\mathcal{L}(\mathcal{N})$ are linear combinations of monomials of the form

$$w(e(1))u(e(2))e(1)_a^x e(2)_b^y = w(e(1))e(1)_a^x u(e(2))e(2)_b^y \tag{3.31}$$

with $\lambda(x, y, a, b) = 0$. Applying ℓ to Equation (3.31) gives that

$$\ell\left(w(e(1))e(1)_a^x u(e(2))e(2)_b^y\right) = \ell'\left(w(e(1))e(1)_a^x f_{y,b}^{\eta(y)} u^*(f)\right).$$

But $e(1)_a^x f_{y,b}^{\eta(y)} \in \mathcal{J}(\mathrm{mir1})$, whence

$$w(e(1))e(1)_a^x f_{y,b}^{\eta(y)} u^*(f) \in \mathcal{J}(\mathrm{mir1}).$$

Hence we have

$$\ell'\left(w(e(1))e(1)_a^x f_{y,b}^{\eta(y)} u^*(f)\right) = 0.$$

We have proved that $\ell(-1) = -1$ and

$$\ell\left(\mathrm{SOS}_{\mathcal{U}} + \mathfrak{L}(\mathcal{N}) + \mathfrak{L}(\mathcal{N})^*\right) \subseteq \mathbb{R}_{\geq 0},$$

whence $-1 \notin \mathrm{SOS}_{\mathcal{U}} + \mathfrak{L}(\mathcal{N}) + \mathfrak{L}(\mathcal{N})^*$, which implies (1) by Theorem 4.3 in [2].

(2) $\Longrightarrow$ (4): Now we have the pair $(\pi', |\psi\rangle)$, where $\pi' : \mathcal{U}(1) \to \mathcal{B}(\mathcal{H})$ is a *-representation and $|\psi\rangle$ is a tracial state. Now we construct the von Neumann algebra $\mathcal{W}$ and $\pi_0' : \mathcal{U}(1) \to \mathcal{W}$.

We denote the completion of $\{\pi'(\mathcal{U}(1))|\psi\rangle\} \subseteq \mathcal{H}$ as $\check{\mathcal{H}}$, and it's obvious that $\check{\mathcal{H}}$ is a closed subspace of $\mathcal{H}$. Then we have

$$\pi'(\mathcal{U}(1))\check{\mathcal{H}} \subseteq \check{\mathcal{H}},$$

and π' induces a *-representation $\check{\pi}' : \mathcal{U}(1) \to \mathcal{B}(\check{\mathcal{H}})$ naturally. We let $\mathcal{W} = \mathcal{B}(\check{\mathcal{H}})$ and $\pi_0' = \check{\pi}'$ as what we desire. Next we'll prove that $\mathcal{W}$ and π_0' satisfy the requirement of the item (4).

Firstly notice that $\mathcal{B}(\check{\mathcal{H}})$ is a von Neumann algebra because it is closed in the weak operator topology.

Secondly, since $|\psi\rangle$ is a tracial state on $\mathcal{B}(\mathcal{H})$, $|\psi\rangle$ is also a tracial state on $B(\check{\mathcal{H}})$. Thus $\tau : \mathcal{B}(\check{\mathcal{H}}) \to \mathbb{C}, a \mapsto \langle\psi|a|\psi\rangle$ is a tracial linear functional on $\mathcal{B}(\check{\mathcal{H}})$, and $(\mathcal{B}(\check{\mathcal{H}}), \tau)$ is a tracial von Neumann algebra.

Lastly, to show $\check{\pi}'(\mathcal{J}(\mathrm{mir1})) = \{0\}$, it suffices to show the following claim:
For any $u \in \mathcal{U}(1)$ and $|\phi\rangle = \pi'(u)|\psi\rangle \in \check{\mathcal{H}}$, we have

$$\check{\pi}'(\mathcal{J}(\mathrm{mir1}))|\phi\rangle = \{0\}.$$

We have

$$\begin{aligned}&\check{\pi}'(\mathcal{J}(\mathrm{mir1}))|\phi\rangle\\ &= \pi'(\mathcal{J}(\mathrm{mir1})u)|\psi\rangle && \text{(the definition of } \phi)\\ &= \pi'(\mathcal{J}(\mathrm{mir1}))|\psi\rangle && (\mathcal{J}(\mathrm{mir1}) \text{ is a two-sided ideal})\\ &= \{0\} && \text{(by item (2)).}\end{aligned}$$

(4) $\Longrightarrow$ (2): We start with the tracial von Neumann algebra $\mathcal{W}$ with trace τ defined in (4) and perform a Gelfand-Naimark-Segal (GNS) construction [21]. There is a Hilbert space $\mathcal{K}$, a unit vector $\rho \in \mathcal{K}$, and a *-representation $\pi_1' : \mathcal{W} \to \mathcal{B}(\mathcal{K})$ such that

$$\tau(a) = \left\langle \pi_1'(a)\rho, \rho \right\rangle, \quad a \in \mathcal{W}.$$

Since τ is a trace, ρ is a tracial state for $\pi_1'(\mathcal{W})$. Then the *-representation $\pi_1' \circ \pi_0' : \mathcal{U}(1) \to \mathcal{B}(\mathcal{K})$ together with $\rho \in \mathcal{K}$ satisfy (2).

(1) $\Longrightarrow$ (3) is similar to (1) $\Longrightarrow$ (2), but using another side of Lemma 3.3 and Lemma 3.4.

(3) $\Longrightarrow$ (1) is similar to (2) $\Longrightarrow$ (1). Here, we extend ℓ' : $\mathcal{U}(2) \to \mathbb{C}$ to a linear functional ℓ_1 from algebra $\mathcal{U}$ to $\mathbb{C}$:

$$\ell_1 : \ u(e(2))w(e(1)) \mapsto \ell' \left(u(e(2))w^*(g)\right).$$

As $\sum_{a\in A} g_{x,a}^{\xi(x)} = 1, \ \forall x \in X$, we know that ℓ_1 is well defined. The proof of $\ell_1(\mathrm{SOS}_{\mathcal{U}(1)}) \geq 0$ is similar to the corresponding part in (2) $\Longrightarrow$ (1).

For the proof of $\ell_1(\mathcal{L}(\mathcal{N})) = \{0\}$, since elements in $\mathcal{U}(1)$ commute with those in $\mathcal{U}(2)$, elements in $\mathcal{N}$ can also be written as $e(2)_b^y e(1)_a^x$. Then elements in $\mathcal{L}(\mathcal{N})$ are linear combinations of monomials of the form

$$w(e(1))u(e(2))e(2)_b^y e(1)_a^x = u(e(2))e(2)_b^y w(e(1))e(1)_a^x \tag{3.32}$$

with $\lambda(x, y, a, b) = 0$. Applying the new ℓ_1 to (3.32) gives that

$$\ell_1 \left(u(e(2))e(2)_b^y w(e(1))e(1)_a^x\right) = \ell' \left(u(e(2))e(2)_b^y g_{x,a}^{\xi(x)} w^*(g)\right).$$

But $e(2)_b^y g_{x,a}^{\xi(x)} \in \mathcal{J}(\mathrm{mir2})$, whence $u(e(2))e(2)_b^y g_{x,a}^{\xi(x)} w^*(g) \in \mathcal{J}(\mathrm{mir2})$. Therefore we derive that

$$\ell' \left(u(e(2))e(2)_b^y g_{x,a}^{\xi(x)} w^*(g)\right) = 0.$$

i.e. we get $\ell_1(\mathcal{L}(\mathcal{N})) = \{0\}$. Using Theorem 4.3 in [2], we can show (3) $\Longrightarrow$ (1).

Finally, the proofs of (3) $\Longrightarrow$ (5) and (5) $\Longrightarrow$ (3) are similar to the proofs of (2) $\Longrightarrow$ (4) and (4) $\Longrightarrow$ (2). □

Remark 1. *Theorem 3.1 may not hold if a mirror game is not regular. For example, let the scoring function $\lambda = 0$ for all questions and answers. It is not a regular mirror game since $\mathcal{J}(\mathrm{mir1}) = \mathcal{J}(\mathrm{mir2}) = \{0\}$. Items (2),(4) in Theorem 3.1 always hold. However, we can easily verify that $\mathcal{G}$ can't have a perfect commuting operator strategy. Therefore, Theorem 3.1 is only true for regular mirror games.*

In [25], Klep and Schweighofer show that Connes' embedding conjecture on von Neumann algebras is equivalent to the tracial version of the Positivstellensatz. See [3, 23, 24] for more recent progress in tracial optimizations. It has been shown in Theorem 8.7 [2] that, given a *-algebra $\mathcal{A}$ satisfying the condition of Archimedean, i.e. for every $a \in \mathcal{A}$, there is an $\varepsilon \in \mathbb{N}$ with $\varepsilon - a^*a \in \widetilde{\mathrm{SOS}}_{\mathcal{A}}$, where

$$\widetilde{\mathrm{SOS}}_{\mathcal{A}} = \{a \in \mathcal{A} \mid \exists b \in \mathrm{SOS}_{\mathcal{A}}, \ a - b \text{ is a sum of commutators}\}. \tag{3.33}$$

Then there exists a $*$-representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ and a tracial state $0 \neq |\psi\rangle \in \mathcal{H}$ satisfying

$$\pi(f)|\psi\rangle = 0, \text{ for all } f \in \mathfrak{L}, \tag{3.34}$$

if and only if there exists a $*$-representation $\pi : \mathcal{A} \to \mathcal{F}$ into a tracial von Neumann algebra $(\mathcal{F}, \tau)$ satisfying

$$\tau(\pi(f)) = 0, \text{ for all } f \in \mathfrak{L}; \tag{3.35}$$

which is also equivalent to

$$-1 \notin \widetilde{\mathrm{SOS}}_{\mathcal{A}} + \mathfrak{L} + \mathfrak{L}^*, \tag{3.36}$$

By cyclic unitary generators defined in [2], we can show that both $\mathcal{U}(1)$ and $\mathcal{U}(2)$ are group algebra. And by Example 4.4 of [2] we know $\mathcal{U}(1)$ and $\mathcal{U}(2)$ are Archimedean. Hence, we can combine the above equivalent condition (3.34), (3.36) with item (2), (4) of our Theorem 3.1. Then we have the following corollary:

Corollary 3.2. *A regular mirror game with its universal game algebra $\mathcal{U}$ and invalid determining set $\mathcal{N}$ has a perfect commuting operator strategy if and only if $-1 \notin \widetilde{\mathrm{SOS}}_{\mathcal{U}(1)} + \mathcal{J}(\mathrm{mir1}) + \mathcal{J}(\mathrm{mir1})^*$. For a special case, if a mirror game satisfies*

$$-1 \in \mathrm{SOS}_{\mathcal{U}(1)} + \mathcal{J}(\mathrm{mir1}) + \mathcal{J}(\mathrm{mir1})^*, \tag{3.37}$$

then it cannot have a perfect commuting strategy. Similar results hold for $\mathcal{J}(\mathrm{mir2})$.

Notice that $\mathcal{J}(\mathrm{mir1})$ is a two-sided ideal, so that $\mathcal{J}(\mathrm{mir1}) + \mathcal{J}(\mathrm{mir1})^*$ is still a two-sided ideal, generated by

$$\left\{e(1)_a^x f_{y,b}^{\eta(y)}, \ f_{y,b}^{\eta(y)} e(1)_a^x \mid \lambda(x, y, a, b) = 0\right\}. \tag{3.38}$$

Then we can use the noncommutative Gröbner basis method to solve this ideal membership problem [26, 28, 30, 43]

4 A PROCEDURE FOR PROVING NONEXISTENCE OF PERFECT STRATEGY

According to Corollary 3.2, we can prove that a regular mirror game $\mathcal{G}$ doesn't have a perfect commuting operator strategy using noncommutative Gröbner basis and semidefinite programming.

The main steps of the procedure are listed as follows.

(1) Let $\mathbb{C}\langle e(1)\rangle$ be the free algebra generated by $\{e(1)_a^x \mid x \in X, \ a \in A\}$, and Π be the canonical projection from $\mathbb{C}\langle e(1)\rangle$ onto $\mathcal{U}(1)$. Then $\Pi^{-1}(\mathcal{J}(\mathrm{mir1}))$ is a two-sided ideal in $\mathbb{C}\langle e(1)\rangle$, generated by

$$\begin{aligned} &\left\{e(1)_a^x f_{y,b}^{\eta(y)} \mid \lambda(x, y, a, b) = 0\right\} \\ \cup &\left\{(e(1)_a^x)^2 - e(1)_a^x, \ e(1)_{a_1}^x e(1)_{a_2}^x, \ \sum_{a\in A} e(1)_a^x - 1\right\}. \end{aligned} \tag{4.1}$$

Therefore $\Pi^{-1}(\mathcal{J}(\mathrm{mir1})) + \Pi^{-1}(\mathcal{J}(\mathrm{mir1}))^*$ is a two-sided ideal generated by

$$\begin{aligned} &\left\{e(1)_a^x f_{y,b}^{\eta(y)}, \ f_{y,b}^{\eta(y)} e(1)_a^x \mid \lambda(x, y, a, b) = 0\right\} \\ \cup &\left\{(e(1)_a^x)^2 - e(1)_a^x, \ e(1)_{a_1}^x e(1)_{a_2}^x, \ \sum_{a\in A} e(1)_a^x - 1\right\}. \end{aligned} \tag{4.2}$$

(2) We compute the noncommutative Gröbner basis GB of $\Pi^{-1}(\mathcal{J}(\mathrm{mir1})) + \Pi^{-1}(\mathcal{J}(\mathrm{mir1}))^*$.
   (a) If $1 \in$ GB, then we have

$$-1 \in \mathrm{SOS}_{\mathbb{C}\langle e(1)\rangle} + \Pi^{-1}(\mathcal{J}(\mathrm{mir1})) + \Pi^{-1}(\mathcal{J}(\mathrm{mir1}))^*. \tag{4.3}$$

   Hence, we have

$$-1 \in \mathrm{SOS}_{\mathcal{U}(1)} + \mathcal{J}(\mathrm{mir1}) + \mathcal{J}(\mathrm{mir1})^*, \tag{4.4}$$

   which implies that the game can't have a perfect strategy.
   (b) Otherwise, we check whether there exist polynomials $s_j \in \mathscr{U}(1)$ such that

$$1 + \sum_{j=1}^{k} s_j^* s_j \in \Pi^{-1}(\mathcal{J}(\mathrm{mir1})) + \Pi^{-1}(\mathcal{J}(\mathrm{mir1}))^*.$$

   Let W_d be the column vector composed of monomials in $\mathbb{C}\langle e(1)\rangle$ having a total degree less than or equal to d. Using an SDP solver to test whether there exists a positive semidefinite matrix G such that

$$1 + W_d^* G W_d \to_{\mathrm{GB}} 0 \tag{4.5}$$

- If (4.5) has a solution G, then the mirror game can't have a perfect strategy.
- Otherwise, set $d := d + 1$ and go back.

REMARK 2. *Since a free algebra generated by two or more variables is non-Noetherian, Buchberger's procedure for computing a non-commutative Gröbner basis may not terminate [31, 43]. Thus our procedure may not terminate in finite steps.*

If the procedure stops at some degree d, we can verify that the mirror game has no perfect commuting operator strategy. Otherwise, we do not know whether the mirror game has a perfect commuting operator strategy.

In fact, according to [27, Theorem 5.1], an imitation game $\mathcal{G}$ has a perfect commuting operator strategy if and only if a tracial state exists on the C^*-algebra $C^*(\mathcal{G})$. By [15, Remark 2.21], $C^*(\mathcal{G})$ is a free hypergraph C^*-algebra, and there is no algorithm to determine whether a free hypergraph C^*-algebra has a tracial state [15, Theorem 3.6]. Hence, there is no algorithm that terminates in finite steps to determine whether a mirror game (an imitation game) has a perfect commuting operator strategy.

Example 2.1 (continued). *Let $\mathbb{C}\langle e(1)\rangle$ be the free algebra generated by $\{e(1)^i_j \mid (i,j) \in \{0,1\}^2\}$, and $\mathcal{U}(1)$ be the subalgebra of the universal game algebra $\mathcal{U}$ generated by $\{e(1)^i_j \mid (i,j) \in \{0,1\}^2\}$. Then we have the natural projection $\Pi : \mathbb{C}\langle e(1)\rangle \to \mathcal{U}(1)$.*

*Notice that $\mathcal{J}(\text{mir1})$ is *-closed. Hence, we have*

$$\mathcal{J}(\text{mir1}) + \mathcal{J}(\text{mir1})^* = \mathcal{J}(\text{mir1}),$$

and

$$\begin{aligned}
&\Pi^{-1}(\mathcal{J}(\text{mir1})) \\
&= \{e(1)^0_0,\ e(1)^0_1,\ e(1)^0_0 + e(1)^0_1 - 1,\ e(1)^1_0 + e(1)^1_1 - 1, \\
&\quad e(1)^0_0 e(1)^0_1,\ e(1)^0_1 e(1)^0_0,\ e(1)^1_0 e(1)^1_1,\ e(1)^1_1 e(1)^1_0, \\
&\quad (e(1)^0_0)^2 - e(1)^0_0,\ (e(1)^0_1)^2 - e(1)^0_1, \\
&\quad (e(1)^1_0)^2 - e(1)^1_0,\ (e(1)^1_1)^2 - e(1)^1_1\}
\end{aligned}$$

is a two-sided ideal in $\mathscr{U}(1)$. It is evident that

$$-1 \in \text{SOS}_{\mathcal{U}(1)} + \mathcal{J}(\text{mir1}) \iff -1 \in \text{SOS}_{\mathbb{C}\langle e(1)\rangle} + \Pi^{-1}(\mathcal{J}(\text{mir1}))$$

Using the software NCAlgebra (https://github.com/NCAlgebra/), we can show that 1 is in the Gröbner basis of $\Pi^{-1}(\mathcal{J}(\text{mir1}))$ i.e.,

$$-1 \in \Pi^{-1}(\mathcal{J}(\text{mir1})).$$

Therefore, this game doesn't have a perfect commuting operator strategy.

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