A Noncommutative Nullstellensatz Derived from Two-Answer Quantum Nonlocal Games

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ABSTRACT

This paper introduces a noncommutative version of the Nullstellensatz, motivated by the study of quantum nonlocal games. It has been proved that a two-answer game with a perfect quantum strategy also admits a perfect classical strategy. We generalize this result to the infinite-dimensional case, showing that a two-answer game which has a perfect commuting operator strategy also admits a perfect classical strategy. And this result induces a special case of noncommutative Nullstellensatz.

KEYWORDS

Noncommutative Nullstellensatz, Sum of Squares, GNS construction, Quantum nonlocal games

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1 INTRODUCTION

The Nullstellensatz and Positivstellensatz are among the core results of classical algebraic geometry. Investigating their generalizations in the noncommutative setting holds significant theoretical value and has important applications in operator algebras and quantum information. [6] firstly showed that all noncommutative positive free polynomials are sums of squares, and [7] explored the positivity of noncommutative polynomials and its implications for zero sets in matrix spaces. [3] presented a directional Nullstellensatz version of the noncommutative Nullstellensatz, and then [2] used this result to characterize whether a general nonlocal game admits a perfect commuting operator strategy.

Quantum nonlocal games play an important role in quantum information theory. They are a class of games based on quantum entanglement, where the players' strategies and outcomes show significant differences between classical and quantum communication. Quantum nonlocal games were first introduced by [4], and through the non-classicality of quantum entanglement, they demonstrate the fundamental differences between quantum and classical theories.

In this research area, the mathematical models of quantum nonlocal games are often described using algebraic structures. These structures typically involve complex relationships between generators, particularly in defining and proving related theoretical results, where ideals, algebras, and representation theory often play key roles.

This paper proposes a noncommutative Nullstellensatz, inspired by the study of two-answer quantum nonlocal games. Specifically, we construct an algebraic structure associated with these games to explore its zero-point problem Jianting: ? and study the properties of this structure from an algebraic perspective. The background of this problem comes from a deep understanding of operator algebras in quantum nonlocal games, and our goal is to use noncommutative algebraic techniques to reveal the deep connections between quantum and classical strategies for nonlocal games.

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2 PRELIMINARIES

2.1 Definitions

Let X, Y, A, B be finite sets, where $A = B = \{0, 1\}$, and $\mathbb{C}\langle\{e_a^x, f_b^y\}\rangle$ be the free algebra generated by $\{e_a^x, f_b^y : (x, y, a, b) \in X\times Y\times A\times B\}$. Define the two sided ideal

$$\begin{aligned}\mathcal{I} = \langle (e_a^x)^2 - e_a^x,\ (f_b^y)^2 - f_b^y;\\ \sum_{a\in A} e_a^x - 1,\ \sum_{b\in B} f_b^y - 1;\\ e_a^x f_b^y - f_b^y e_a^x \mid x \in X, y \in Y, a \in A, b \in B\rangle\end{aligned}$$

and let $\mathcal{A} = \mathbb{C}\langle\{e_a^x, f_b^y\}\rangle/\mathcal{I}$. Note that through simple computation we can see that $e_0^x e_1^x,\ \forall x \in X$ and $f_0^y f_1^y,\ \forall y \in Y$ are in $\mathcal{I}$. The elements in $\mathcal{I}$ can be seen as the relations that the generators satisfy. We can also equip $\mathcal{A}$ with the natural involution "$*$" induced by $(e_a^x)^* = e_a^x$ and $(f_b^y)^* = f_b^y$. Then $\mathcal{A}$ is a complex $*-$algebra. Moreover, $\mathcal{A}$ is a group algebra. Let $A_x = e_0^x - e_1^x,\ B_y = f_0^y - f_1^y$ for any $x \in X,\ y \in Y$, and we have

$$A_x^2 = B_y^2 = 1, A_x = A_x^*,\ B_y = B_y^*,$$

$$e_a^x = \frac{1 + (-1)^a A_x}{2},\ f_b^y = \frac{1 + (-1)^b B_y}{2}.$$

Define G to be the group generated by all the elements $A_x,\ x \in X$ and $B_y,\ y \in Y$, and equip the group algebra of G with the natural involution $*:\ g^* = g^{-1}$ and $(g_1 g_2)^* = g_2^* g_1^*,\ \forall g, g_1, g_2 \in G$, then we can see that $\mathcal{A} = \mathbb{C}[G]$.

We denote

$$\mathrm{SOS}_{\mathcal{A}} := \{\sum_{i=1}^{n} \alpha_i^* \alpha_i \mid n \in \mathbb{N},\ \alpha_i \in \mathcal{A}\}.$$

It is well known that $\mathcal{A}$ is Archimedean, that is to say, for every $\alpha \in \mathcal{A}$, there exists $\eta \in \mathbb{N}$ such that $\eta - \alpha^*\alpha \in \mathrm{SOS}_{\mathcal{A}}$ [2, Example 4.4].

2.2 Motivations

The motivation for studying the Nullstellensatz on this algebra originates from quantum nonlocal games. If the reader is familiar with this field, they can skip the content of this subsection.

A quantum nonlocal game can be discribed as a scoring function λ from the finite set $X \times Y \times A \times B$ to $\{0, 1\}$, where the player Alice has a question set X and an answer set A, while the player Bob has a question set Y and an answer set B. The players cannot communicate during the game, but they can make some arrangements before they play the game. The players are said to win the game when $\lambda(x, y, a, b) = 1$, and they lose otherwise. A (deterministic) classical strategy involves two mappings $u : X \to A$ and $v : Y \to B$, when Alice receives a question $x \in X$ she responds with $u(x)$, and similarly, Bob responds with $v(y)$ when he receives $y \in Y$. If the players share a quantum state ϕ on a (perhaps infinite dimensional) Hilbert space $\mathcal{H}$, and for every question pair $(x, y) \in X \times Y$ Alice and Bob perform commuting projection-valued measurements $\{E_a^x \in \mathcal{B}(\mathcal{H}) : \sum_{a \in A} E_a^x = \mathrm{id}\}$ and $\{F_b^y \in \mathcal{B}(\mathcal{H}) : \sum_{b \in B} F_b^y = \mathrm{id}\}$ respectively to determine their answers, then the game is said to have a commuting operator strategy. Furthermore, if we restrict the quantum state ϕ to be a tensor $\phi_1 \otimes \phi_2$, where ϕ_1 and ϕ_2 are in finite dimensional Hilbert space $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, then we get a (finite dimensional) quantum strategy. We call a strategy perfect if the players can always win the game using this strategy.

Nonlocal games have been extensively studied in quantum information theory due to their profound implications for understanding quantum entanglement, quantum complexity theory, and the foundations of quantum mechanics. In 2020 Ji et al [8] used nonlocal games to prove that "MIP*=RE", which implies the famous Connes' embedding conjecture is not true.

It is easy to prove that if a nonlocal game has a perfect classical strategy, then it must have a perfect quantum strategy, and a perfect quantum strategy must be a perfect commuting operator strategy. However, the converse does not hold. For example, the famous Magic Square game admits a perfect quantum strategy but has no perfect classical strategy [5]. But if we additionally require $A = B = \{0, 1\}$, then a game with answer set equals to $A \times B$ that admits a perfect quantum strategy also have a perfect classical strategy [5, Theorem 3]. Our contribution is to extend this proof to the infinite-dimensional case and, in combination with [2], derive a form of Nullstellensatz.

3 MAIN RESULT

Theorem 3.1. *Let $\mathcal{A}$ be the complex $*-$algebra defined above. Let $\Lambda \subseteq X \times Y \times A \times B$ and $\mathcal{N} = \{e_a^x f_b^a \mid (x, y, a, b) \in \Lambda\}$, and $\mathcal{L}(\mathcal{N})$ be the left ideal generated by $\mathcal{N}$. Then*

$$-1 \notin \mathrm{SOS}_{\mathcal{A}} + \mathcal{L}(\mathcal{N}) + \mathcal{L}(\mathcal{N})^* \iff$$

there exists a $*-$representation $\rho : \mathcal{A} \to \mathbb{C}$ such that $\rho(\mathcal{N}) = \{0\}$.

Proof. ($\Longleftarrow$) is easy. Otherwise if we sssume that this direction does not hold, i.e $-1 \in \mathrm{SOS} + \mathcal{L}(\mathcal{N}) + \mathcal{L}(\mathcal{N})^*$ and there exists such a $*-$representation ρ that $\rho(\mathcal{N}) = \{0\}$, then we have

$$-1 = \rho(-1) \in \rho(\mathrm{SOS}_{\mathcal{A}}) \geqslant 0,$$

which is a contradiction!

($\Longrightarrow$) Suppose we have $-1 \notin \mathrm{SOS} + \mathcal{L}(\mathcal{N}) + \mathcal{L}(\mathcal{N})^*$, we need to construct a one-dimensional representation ρ which satisfies the condition. Our approach is as follows: first, construct an appropriate (possibly infinite-dimensional) representation, and then use this representation to construct the desired one-dimensional representation.

At the beginning, using the Nullstellensatz given by Watts, Helton and Klep [2, Theorem 4.3] we know there exists a $*-$representation $\sigma : \mathcal{A} \to \mathcal{B}(H)$ and a vector $\psi \in \mathcal{H}$ such that $\sigma(\mathcal{L}(\mathcal{N})) = \{0\}$. Moreover, we can get that $\mathcal{H}$ is a separable Hilbert space. For completeness we briefly write its proof. By the Hahn-Banach theorem [1, Theorem III.1.7] there exists a functional $f : \mathcal{A} \to \mathbb{C}$ which separate -1 and $\mathrm{SOS}_{\mathcal{A}} + \mathcal{L}(\mathcal{N}) + \mathcal{L}(\mathcal{N})^*$, i.e

$$f(-1) \leqslant 0,\ f(\mathrm{SOS}_{\mathcal{A}} + \mathcal{L}(\mathcal{N}) + \mathcal{L}(\mathcal{N})^*) \subseteq \mathbb{R}_{\geqslant 0}.$$

Since $\mathcal{A}$ is a group algebra, we know $\mathrm{SOS}_{\mathcal{A}}$ is Archimedean, and thus the separation is strict, i.e. we can suppose $f(-1) = -1$.

Since $\mathcal{L}(\mathcal{N})$ is a subspace of $\mathcal{A}$, we can get that $f(\mathrm{SOS}_{\mathcal{A}} + \mathcal{L}(\mathcal{N}) + \mathcal{L}(\mathcal{N})^*) \geqslant 0$ means $f(\mathcal{L}(\mathcal{N})) = \{0\}$ and $f(\mathrm{SOS}_{\mathcal{A}}) \subseteq \mathbb{R}_{\geqslant 0}$. Thus for every self-adjoint $g = g^* \in \mathcal{A}$, we can take the decomposition $g = g_1 - g_2$, where $g_1 = (\frac{g+1}{2})^*(\frac{g+1}{2})$ and $g_2 = \frac{1}{4}g^*g + \frac{1}{4}$ are in $\mathrm{SOS}_{\mathcal{A}}$ [9, Example 1]. By Archimedeanity we know there exists a $\eta \in \mathbb{N}$ such that $\eta - g_1 \in \mathrm{SOS}_{\mathcal{A}}$, so $\eta - g \in \mathrm{SOS}_{\mathcal{A}}$, and $f(\eta - g) \in \mathbb{R}_{\geqslant 0}$, i.e $f(g) \in \mathbb{R}$. Then, for every $h \in \mathcal{A}$, using

$$h = \frac{h + h^*}{2} + \mathrm{i}\frac{h - h^*}{2\mathrm{i}}$$

we can get $f(h^*) = f(h)^*$.

Now perform the GNS construction. Define the sesquilinear form on $\mathcal{A}$

$$\langle \alpha \mid \beta \rangle = f(\beta^* \alpha)$$

and $M = \{\alpha \in \mathcal{A} :\ f(\alpha^*\alpha) = 0\}$. By Cauchy-Schwarz inequality we know M is a left ideal of $\mathcal{A}$. Form the quotient space $\widetilde{H} := H/M$, and equip it with the inner product $\langle \cdot \mid \cdot \rangle$. Then we can complete it to the Hilbert space $\mathcal{H}$.

It should be noted that we can require $\mathcal{H}$ to be a separable Hilbert space. This is because $\mathcal{A}$ has only a finite number of generators, which allows us to generate a countable dense subset of $\mathcal{A}$ using these generators with rational coefficients. By transferring this to the quotient space, we achieve the separability of $\mathcal{H}$.

Define the quotient map $\phi : \mathcal{A} \to \mathcal{H},\ \alpha \mapsto \alpha + M$, the cyclic vector $\psi := \phi(1) = 1 + M$, and the left regular representation

$$\sigma : \mathcal{A} \to \mathcal{B}(\mathcal{H}),\ \alpha \mapsto (p + M \mapsto \alpha p + M)\,.$$

By Archimedeanity, it is easy to verify that $\sigma(\alpha)$ is bounded for every $\alpha \in \mathcal{A}$, and thus σ is a $*$−representation.

Now we prove $\sigma(\mathcal{L}(\mathcal{N}))\psi = \{0\}$. Since $f(\mathcal{L}(\mathcal{N})) = \{0\}$ and $\mathcal{L}(\mathcal{N})^*\mathcal{L}(\mathcal{N}) \subseteq \mathcal{L}(\mathcal{N})$, we know that $\mathcal{L}(\mathcal{N}) \subseteq M$. Therefore, for any $\beta \in \mathcal{L}(N) \subseteq M$, we have

$$\sigma(\beta)\psi = \sigma(\beta)(1+M) = \beta + M \in M, \text{ i.e } \sigma(\beta)\psi = 0 \in \mathcal{H}$$

as desired.

Now we construct the one-dimensional representation ρ. Since

$$\sum_{a\in A}\sum_{b\in B} \psi^*\sigma(e^x_a f^y_b)\psi = 1$$

for every fixed pair (x, y), we know that there exist $(x, y, a, b) \in X \times Y \times A \times B$ such that $\psi^*\sigma(e^x_a f^y_b)\psi \neq 0$. Let

$$\Pi = \{(x, y, a, b) \in X \times Y \times A \times B : \ \psi^*\sigma(e^x_a f^y_b)\psi \neq 0\},$$

and we have $\Pi \subseteq X \times Y \times A \times B \setminus \Lambda$ since $\sigma(\mathcal{L}(\mathcal{N}))\psi = \{0\}$ and thus $\psi^*\sigma(e^x_a f^y_b)\psi = 0$ for any $(x, y, a, b) \in \Lambda$.

Using the generators A_x and B_y we can rewrite:

$$\begin{aligned}\psi^*\sigma(e^x_a f^y_b)\psi = &\frac{1}{4}\\ &+ \frac{1}{4}(-1)^a\psi^*\sigma(A_x)\psi\\ &+ \frac{1}{4}(-1)^b\psi^*\sigma(B_y)\psi\\ &+ \frac{1}{4}(-1)^{a+b}\psi^*\sigma(A_xB_y)\psi.\end{aligned} \tag{3.1}$$

Since $\mathcal{H}$ is separable, we can choose an orthogonal basis of $\mathcal{H}$ named

$$\{\psi_1, \psi_2, \ldots\ldots\},$$

where $\psi_1 = \psi$. Define

$$\begin{aligned}k : X &\to \mathbb{N}\\ x &\mapsto \min\{j \in \mathbb{N} : \ \psi_j^*\sigma(A_x)\psi \neq 0\};\\ l : Y &\to \mathbb{N}\\ y &\mapsto \min\{j \in \mathbb{N} : \ \psi_j^*\sigma(B_y)\psi \neq 0\}.\end{aligned}$$

Note that for every $x \in X$, $k(x)$ is well defined because $\psi \neq 0$ and $\sigma(A_x)^2 = 1$, thus there must exist a $j \in \mathbb{N}$ such that $\psi_j^*\sigma(A_x)\psi \neq 0$ (otherwise $\sigma(A_x)\psi = 0$ a contradiction!). Similarly for the case of $l(y)$.

Let

$$\begin{aligned}u : X &\to A\\ x &\mapsto \begin{cases}0, \ 0 \leqslant \arg\psi_{k(x)}\sigma(A_x)\psi < \pi;\\ 1, \ \pi \leqslant \arg\psi_{k(x)}\sigma(A_x)\psi < 2\pi,\end{cases};\\ v : Y &\to B\\ y &\mapsto \begin{cases}0, \ 0 \leqslant \arg\psi_{l(y)}\sigma(B_y)\psi < \pi;\\ 1, \ \pi \leqslant \arg\psi_{l(y)}\sigma(B_y)\psi < 2\pi,\end{cases}.\end{aligned}$$

We have the following claim:

Claim 1. *For every $(x, y, u(x), v(y)) \in X \times Y \times A \times B$, we have $(x, y, u(x), v(y)) \in \Pi$. That is to say, $\psi^*\sigma(e^x_{u(x)} f^y_{v(y)})\psi \neq 0$.*

We'll leave the proof of Claim 1 to the end. Using this claim we can construct the one-dimensional $*$−representation ρ as follows: for every $x \in X$,

$$\rho(e^x_{u(x)}) = 1, \ \rho(e^x_{1-u(x)}) = 0;$$

and for every $y \in Y$,

$$\rho(f^y_{v(y)}) = 1, \ \rho(f^y_{1-v(y)}) = 0.$$

Then, by linearity and homogeneity, we extend ρ to the entire $\mathcal{A}$. It is obvious that $\rho(e^x_a)$ and $\rho(f^y_b)$ satisfy all the relations of $\mathcal{A}$, thus ρ is indeed a $*$−representation. Since

$$\rho(e^x_a f^y_b) = 1 \iff (a = u(x)) \wedge (b = l(y))$$

we have $\rho(e^x_a f^y_b) = 1 \Longrightarrow (x, y, a, b) \in \Pi$. Since $\Pi \cap \Lambda = \emptyset$, this means that for every $(x, y, a, b) \in \Lambda$, i.e $e^x_a f^y_b \in \mathcal{N}$, $\rho(e^x_a f^y_b) = 0$ holds, which completes the proof.

Finally we prove Claim 1.

Proof of Claim 1. We take $a = u(x)$ and $b = v(y)$ in equation (3.1), and then

$$\begin{aligned}\psi^*\sigma(e^x_{u(x)} f^y_{v(y)})\psi = &\frac{1}{4}\\ &+ \frac{1}{4}(-1)^{u(x)}\psi^*\sigma(A_x)\psi\\ &+ \frac{1}{4}(-1)^{v(y)}\psi^*\sigma(B_y)\psi\\ &+ \frac{1}{4}(-1)^{u(x)+v(y)}\psi^*\sigma(A_xB_y)\psi.\end{aligned} \tag{3.2}$$

Notice that $\sigma(A_x)$ and $\sigma(B_y)$ are commutative self-adjoint operators, so $\psi^*\sigma(A_x)\psi$, $\psi^*\sigma(B_y)\psi$ and $\psi^*\sigma(A_xB_y)\psi$ are all real numbers.

If $\psi^*\sigma(A_x)\psi \neq 0$, since $\psi_1 = \psi$ we know $k(x) = 1$ and

$$(-1)^{u(x)}\psi^*\sigma(A_x)\psi > 0$$

because of the construction of u. Similarly, if $\psi^*\sigma(B_y)\psi \neq 0$, we have

$$(-1)^{v(y)}\psi^*\sigma(B_y)\psi > 0.$$

Therefore, either $\psi^*\sigma(A_x)\psi$ or $\psi^*\sigma(B_y)\psi$ is nonzero, we have

$$\frac{1}{4}(-1)^{u(x)}\psi^*\sigma(A_x)\psi + \frac{1}{4}(-1)^{v(y)}\psi^*\sigma(B_y)\psi > 0,$$

and since $\frac{1}{4} + \frac{1}{4}(-1)^{u(x)+v(y)}\psi^*\sigma(A_xB_y)\psi \geqslant 0$, we have

$$\psi^*\sigma(e^x_{u(x)} f^y_{v(y)})\psi > 0.$$

Then we only need to consider the case

$$\psi^*\sigma(A_x)\psi = \psi^*\sigma(B_y)\psi = 0.$$

We need to prove that $\frac{1}{4} + \frac{1}{4}(-1)^{u(x)+v(y)}\psi^*\sigma(A_xB_y)\psi > 0$ in this case. Assume the contrary happens, i.e

$$(-1)^{u(x)+v(y)}\psi^*\sigma(A_xB_y)\psi = -1.$$

By Parseval's identity we can get that

$$\mathrm{id} = \sum_{i=1}^{\infty} \psi_i\psi_i^*.$$

Then

$$(-1)^{u(x)+v(y)}\psi^*\sigma(A_xB_y)\psi = \sum_{i=1}^{\infty}(-1)^{u(x)}\psi^*\sigma(A_x)\psi_i\cdot(-1)^{v(y)}\psi_i^*\sigma(B_y)\psi. \tag{3.3}$$

We can observe that the right hand side of (3.3) is the inner product of two infinite-dimensional unit vectors

$$((-1)^{u(x)}\psi_1^*\sigma(A_x)\psi, (-1)^{u(x)}\psi_2^*\sigma(A_x)\psi, \ldots\ldots)$$

and

$$((-1)^{v(y)}\psi_1^*\sigma(B_y)\psi, (-1)^{v(y)}\psi_2^*\sigma(B_y)\psi, \ldots\ldots).$$

By Cauchy-Schwarz's inequality we know that

$$(-1)^{u(x)+v(y)}\psi^*\sigma(A_xB_y)\psi = -1$$

if and only if

$$(-1)^{u(x)}\psi_j^*\sigma(A_x)\psi = -(-1)^{v(y)}\psi_j^*\sigma(B_y)\psi \tag{3.4}$$

holds for every $j \in \mathbb{N}$. However, (3.4) must fail to hold for $j = \min\{k(x), l(y)\}$, because if $k(x) \neq l(y)$ it obvious fails; otherwise we find $\arg\left((-1)^{u(x)}\psi_j^*\sigma(A_x)\psi\right)$ and $\arg\left((-1)^{v(y)}\psi_j^*\sigma(B_y)\psi\right)$ are both in the range $[0, \pi)$, which is contradict to (3.4) again!

Therefore, when $\psi^*\sigma(A_x)\psi = \psi^*\sigma(B_y)\psi = 0$ we have proved that $\frac{1}{4} + \frac{1}{4}(-1)^{u(x)+v(y)}\psi^*\sigma(A_xB_y)\psi > 0$. That is to say,

$$\psi^*\sigma(e^x_{u(x)}f^y_{v(y)})\psi > 0$$

always holds, which proves the claim. □

□

4 SOME DISCUSSIONS

Here are some remarks and discussions about this result.

(1) It's easy to see that $\mathcal{A}$ is the universal game algebra defined in [2] for a two-answer nonlocal game, and the set $\mathcal{N}$ can be viewed as the invalid deteermining set of the game. Then our result shows that for a two answer nonlocal game, if it has a perfect commuting operator strategy, it must has a perfect classical strategy, which generlizes the result of [5, Theorem 3].

(2) Our proof is not contained in [2, Section 5], because the elements in the set $\mathcal{N}$ cannot necessarily be expressed in the form of $\beta g - 1, \beta \in \mathbb{C}, g \in G$, thus a two-answer game is not necessarily a torically determined game.

(3) If A or B has three or more elements, it is well known that this result will fail to hold, because there exists a nonlocal game which has a perfect commuting-operator strategy but no perfect classical strategies. From another perspective, equation (3.1) no longer holds in this case, which prevents us from reaching a similar conclusion.

(4) The condition "there exists a $*-$representation $\rho : \mathcal{A} \to \mathbb{C}$ such that $\rho(\mathcal{N}) = \{0\}$ does not imply that 1 is in the two sided ideal generated by $\mathcal{N}$. This is because $\mathcal{A}$ is a noncommutative algebra and the Hilbert Nullstellensatz does not hold on it.

(5) The algebra $\mathcal{A}$ is finite generated, and the set $\mathcal{N}$ is also a finite set. Howerver, the proof of our theorem uses infinite-dimensional space. Therefore, we are interested in whether there exists a purely algebraic proof of this theorem, without using infinite-dimension.

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