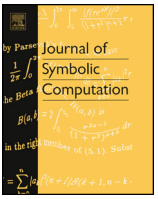
Journal of Symbolic Computation 140 (2027) 102630



Contents lists available at ScienceDirect

Journal of Symbolic Computation

journal homepage: www.elsevier.com/locate/jsc



A Vergleichsstellensatz of Strassen's type for a noncommutative preordered semialgebra through the semialgebra of its fractions



Tao Zheng [a], Lihong Zhi [b,c]

[a] School of Mathematics and Statistics, Xidian University, Xi'an, 710126, China
[b] Key Laboratory of Mathematics Mechanization, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, China
[c] University of Chinese Academy of Sciences, Beijing, 100049, China

ARTICLE INFO

Article history:
Received 22 May 2023
Received in revised form 26 January 2026
Accepted 6 October 2026
Available online 7 October 2026

Keywords:
Semiring
Strassen's theorem
Positivstellensatz

ABSTRACT

Preordered semialgebras and semirings are two kinds of algebraic structures frequently occurring in real algebraic geometry. Strassen's Vergleichsstellensatz and its generalized versions are analogs of those well-known Positivstellensätze for rings. While these Vergleichsstellensätze accept only a commutative setting (for the semirings in question), we prove in this paper a noncommutative version of one of the generalized Vergleichsstellensätze proposed by Fritz (2021b). The most crucial step in our proof is to define the semialgebra of the fractions of a noncommutative semialgebra, which generalizes (some of) the definitions in the literature. Our new theorem characterizes the relaxed preorder on a noncommutative semialgebra induced by all monotone homomorphisms to $\mathbb{R}_+$ via three other equivalent conditions on the semialgebra of its fractions.

© 2026 Elsevier Ltd. All rights are reserved, including those for text and data mining, AI training, and similar technologies.

E-mail address: zhengtao@xidian.edu.cn (T. Zheng).

https://doi.org/10.1016/j.jsc.2026.102630
0747-7171/© 2026 Elsevier Ltd. All rights are reserved, including those for text and data mining, AI training, and similar technologies.

1. Introduction

A (noncommutative) *semiring* S is a set together with binary operations

$$+, * : S \times S \to S$$

respectively called *addition* and *multiplication*, and with elements $0_S, 1_S \in S$ such that $(S, +, 0_S)$ is a commutative monoid, $(S, *, 1_S)$ is a noncommutative monoid and such that the multiplication distributes over the addition (Golan, 1999).

In (Fritz, 2021a, Definition 3.5), the definition of a commutative semialgebra is given, which can be easily modified to a noncommutative version, although the noncommutative case is not considered in Fritz (2021a) at all: A (noncommutative) *semialgebra* S is a (noncommutative) semiring together with a (nonnegative) scalar multiplication

$$\mathbb{R}_+ \times S \to S, (r, x) \mapsto r \cdot x$$

that is a commutative monoid homomorphism from $(\mathbb{R}_+, +)$ or $(S, +)$ to $(S, +)$ in the first or the second argument respectively, and satisfies the following additional laws: *i*) $1 \cdot x = x$, *ii*) $(r \cdot x) * (s \cdot y) = (rs) \cdot (x * y)$. These laws imply that

$$(r \cdot 1_S) * x = (r \cdot 1_S) * (1 \cdot x) = r \cdot (1_S * x) = r \cdot x,$$

from which we have

$$\begin{aligned} r \cdot (s \cdot x) &= (r \cdot 1_S) * (s \cdot x) \\ &= (rs) \cdot (1_S * x) \\ &= (rs) \cdot x. \end{aligned}$$

This is the module property, which will be used from time to time in the following arguments.

Definition 1.1. A semialgebra S is *zero-sum-free* if for any $a, b \in S$, $a + b = 0_S$ implies $a = b = 0_S$. It is *zero-divisor-free* if for any $a, b \in S$, $a * b = 0_S$ implies $a = 0_S$ or $b = 0_S$.

In a zero-divisor-free semialgebra S, $r \cdot a = 0_S$ implies $r = 0$ or $a = 0_S$ for any $r \in \mathbb{R}_+$ and $a \in S$: $r \cdot a = 0_S \Rightarrow (r \cdot 1_S) * a = 0_S \Rightarrow r \cdot 1_S = 0_S$ or $a = 0_S$. If $a \neq 0_S$ and $r \neq 0$, then $1_S = r^{-1} \cdot (r \cdot 1_S) = r^{-1} \cdot 0_S = 0_S$. Hence $a = a * 1_S = a * 0_S = 0_S$, which is a contradiction.

A *preorder relation* (or, *preorder*) $\leq$ on a set X is a binary relation that is both reflexive and transitive (Golan (1999), pp. 119). Set $x, y \in X$, we sometimes write "$x \geq y$" instead of "$y \leq x$", and write "$x < y$" or "$y > x$" instead of the condition "$x \leq y$ and $x \neq y$" throughout the paper. A preorder relation $\leq$ on X is considered *trivial* if $x \leq y$ holds for any $x, y \in X$.

A *preordered semiring* (Fritz, 2021b, Definition 2.2) is a semiring with a preorder relation $\leq$ such that for all a, x, y in the semiring,

$$x \leq y \text{ implies } \begin{cases} a + x \leq a + y, \\ a * x \leq a * y, \\ x * a \leq y * a. \end{cases} \tag{1}$$

Similarly, a *preordered semialgebra* is a semialgebra with a preorder relation $\leq$ such that $x \leq y$ implies the inequalities in (1). Note that it also implies the inequality

$$r \cdot x \leq r \cdot y \tag{2}$$

for any $r \in \mathbb{R}_+$, since $r \cdot x = (r \cdot 1_S) * x \leq (r \cdot 1_S) * y = r \cdot y$.

Example 1.2. Let $\mathbb{R}_+\langle x_1,\ldots,x_n\rangle$ be the set of all noncommutative polynomials in variables $x_1,\ldots,x_n$ with coefficients in $\mathbb{R}_+$. Define

$$\begin{aligned} S &= \mathbb{R}_+\langle x_1,\ldots,x_n\rangle' \\ &= \{0\}\cup\{p\in\mathbb{R}_+\langle x_1,\ldots,x_n\rangle \mid p \text{ has nonzero constant term}\}, \end{aligned}$$

then S is a noncommutative semialgebra with "$+$", "$*$" and "$\cdot$" being the usual addition, multiplication and (nonnegative) scalar multiplication for noncommutative polynomials in $\mathbb{R}_+\langle x_1,\ldots,x_n\rangle$.

For any polynomial $p\in\mathbb{R}_+\langle x_1,\ldots,x_n\rangle$ and any term (word) T generated by the noncommutative variables $x_1,\ldots,x_n$, we denote by $\mathrm{coe}(T,p)$ the coefficient of p with respect to the term T if T is a term of p, and define $\mathrm{coe}(T,p)=0$ otherwise. The coefficient-wise preorder $\le$ on S is then defined by

$$p\le q \text{ if and only if } \forall T,\ \mathrm{coe}(T,p)\le\mathrm{coe}(T,q),$$

for any $p,q\in S$. For instance, set

$$\begin{cases} p &= 2x_1^2x_2^3x_1+x_2x_1x_2+7 \\ q &= 3x_1^2x_2^3x_1+5x_2x_1x_2+x_3x_1+9 \end{cases},$$

then we have $p\le q$ since

$$\begin{cases} \mathrm{coe}(x_1^2x_2^3x_1,p) &= 2\le 3 &= \mathrm{coe}(x_1^2x_2^3x_1,q) \\ \mathrm{coe}(x_2x_1x_2,p) &= 1\le 5 &= \mathrm{coe}(x_2x_1x_2,q) \\ \mathrm{coe}(x_3x_1,p) &= 0\le 1 &= \mathrm{coe}(x_3x_1,q) \\ \mathrm{coe}(1,p) &= 7\le 9 &= \mathrm{coe}(1,q) \end{cases}.$$

It is straightforward that S becomes a preordered semialgebra if it is equipped with the coefficient-wise preorder (i.e., the implications in (1) hold).

Let E and K be two semialgebras. A map f from E to K is a *semialgebra homomorphism* (or, simply *homomorphism*) if $f(0_E)=0_K$, $f(1_E)=1_K$ and for any $x,y\in E$ and any $r\in\mathbb{R}_+$, the following equations hold

$$f(x+y)=f(x)+f(y),\ f(x*y)=f(x)*f(y) \text{ and } f(r\cdot x)=r\cdot f(x).$$

If $\le_E$ and $\le_K$ are two preorder relations on E and K respectively, then a map f from E to K is *monotone* (*w.r.t.* $\le_E$ and $\le_K$) if for any $x,y\in E$, $x\le_E y$ implies $f(x)\le_K f(y)$.

The following definition of a *power universal* element in a noncommutative preordered semialgebra is a natural generalization of (Fritz, 2021b, Definition 2.8).

Definition 1.3. Let S be a noncommutative preordered semialgebra with $1_S\ge 0_S$. An element $u\in S$ with $u\ge 1_S$ is *power universal* (*w.r.t.* the preorder $\le$) if for every nonzero $x\in S$ there is a number $k\in\mathbb{N}$ such that

$$u^k*x\ge 1_S,\ \ x*u^k\ge 1_S \text{ and } u^k\ge x. \tag{3}$$

Recently, Strassen's separation theorem (Strassen, 1988; Zuiddam, 2019) for preordered semirings (called "Strassen's Vergleichsstellensatz" Fritz (2021c)) has been generalized tremendously by Fritz and Vrana (Fritz, 2021a,b,c; Vrana, 2021). This is motivated by its various applications to real algebraic geometry, probability theory, theoretical computer science, and quantum information theory (Fritz, 2021b; Zuiddam, 2019; Li and Zuiddam, 2020; Perry et al., 2021; Bunth and Vrana, 2021; Robere and Zuiddam, 2022; Fritz, 2021d). Strassen's Vergleichsstellensatz and its generalizations are analogs of those Positivstellensätze in real algebraic geometry focusing mainly on ordered rings and

fields. Among them, Fritz's results (Fritz, 2021b,c) recover the classical Positivstellensatz of Krivine-Kadison-Dubois, which suggests that the theory of semirings and semialgebras can provide us with new insights into real algebraic geometry.

While Strassen's Vergleichsstellensatz and its generalizations accept only the commutative setting for the semirings under consideration, we prove in this paper a noncommutative version of one of Fritz's generalized Vergleichsstellensätze in Fritz (2021b). As in the commutative case, the main assumption on the preordered semialgebra S considered in our theorem is:

Assumption 1.4. The inequality "$1_S \geq 0_S$" holds in the preordered semialgebra S, and there is a power universal element $u \in S$.

The assumption of the existence of a power universal element is similar to the Archimedean condition in the traditional real algebraic geometry. Moreover, the semialgebra S considered in this paper also satisfies the following assumption:

Assumption 1.5. The semialgebra S is both zero-sum-free and zero-divisor-free, and the inequality "$1_S \neq 0_S$" holds.

Then our main result can be stated as follows:

Theorem 1.6. *Let $(S, \leq)$ be a preordered semialgebra satisfying Assumptions 1.4 and 1.5, with a power universal element $u \in S$. Then, for every nonzero $x, y \in S$, the following are equivalent:*

(a) $f(x) \geq f(y)$ for every monotone semialgebra homomorphism $f : S \to \mathbb{R}_+$.

(b) For every real number $\epsilon > 0$, there is a finite number $m \in \mathbb{N}$ such that

$$\overline{x} + \sum_{j=0}^{m} \epsilon^{j+1} \cdot \overline{u}^j \succcurlyeq \overline{y}. \tag{4}$$

(c) For every real number $r \in \mathbb{R}_+$ and every real number $\epsilon > 0$, there is a polynomial $p \in \mathbb{Q}_+[X]$ such that $p(r) \leq \epsilon$ and

$$\overline{x} + p(\overline{u}) \succcurlyeq \overline{y}.$$

(d) For every real number $r \in \mathbb{R}_+$ and every real number $\epsilon > 0$, there is a polynomial $p \in \mathbb{Q}_+[X]$ such that $p(r) \leq 1 + \epsilon$ and

$$p(\overline{u}) * \overline{x} \succcurlyeq \overline{y}. \tag{5}$$

The elements $\overline{x}, \overline{y}$ and $\overline{u}$ above are the images of x, y and u for the canonical map (Definition 2.10) from S to the semialgebra of its fractions, and the preorder "$\preccurlyeq$" is a preorder on the semialgebra of the fractions which is directly derived from the original preorder "$\leq$" on S (Definition 3.1).

Example 1.7. Let $S = \mathbb{R}_+\langle x_1, \ldots, x_n\rangle'$ be the semialgebra in Example 1.2 equipped with the coefficient-wise preorder. Then it is straightforward that it satisfies Assumptions 1.4–1.5 with the element $u = 2 + \sum_{i=1}^{n} 2 \cdot x_i$ being power universal. Theorem 1.6 can therefore be applied.

The main technical difficulty in proving Theorem 1.6 is to define properly the semialgebra of the fractions of a noncommutative semialgebra: In the commutative case, every fraction of elements in a commutative semialgebra/semiring can be written in the simple form $\frac{a}{b}$ for two elements a, b in the semialgebra/semiring. However, fractions in the noncommutative case can be of more complicated forms, *e.g.*, $d^{-1} * g * h^{-1}$ and $(d + g^{-1})^{-1}$, which can no longer be written in the form $\frac{a}{b}$. Our definition for the semialgebra of the fractions of a noncommutative semialgebra in Subsection 2.3 generalizes some of the definitions in the literature, and it coincides with the usual definition in the commutative

case (see Proposition B.4 in Zheng and Zhi (2023)). Now that theories on commutative semirings and semialgebras help us better understand commutative real algebraic geometry, it is reasonable to expect the same thing in the noncommutative case.

The main difference between Theorem 1.6 and Fritz's Vergleichsstellensatz (Fritz, 2021b, Theorem 2.12) is that, in the commutative case, the corresponding inequalities in conditions (b) - (d) can be rewritten as inequalities in the original semialgebra S (see Proposition C.2 in Zheng and Zhi (2023)), while in Theorem 1.6, one may find it non-trivial to do the same thing. Luckily, in Theorem 6.7, a variation of Theorem 1.6, we express those equivalence conditions in a form only involving the original preorder $\leq$ and some equations in the semialgebra of fractions.

In the next section, we define the semialgebra of the fractions of a noncommutative semialgebra that satisfies Assumption 1.5. This definition is the cornerstone of the main theorem of our paper. Section 3 indicates how a preorder relation on a semialgebra derives another preorder relation on the semialgebra of its fractions. Section 4 is devoted to interpreting how an $\mathbb{R}_+$-valued semialgebra homomorphism can be extended from a semialgebra to the semialgebra of its fractions. In Section 5, we explain how an $\mathbb{R}$-linear space can be constructed from a (noncommutative) semialgebra. This is mentioned in Fritz (2021b), but we include it for rigorousness and completeness. Finally, in Section 6, we present the proof of Theorem 1.6 and illustrate it with some examples.

2. The semialgebra of the fractions of a noncommutative semialgebra

Throughout the paper, the letter "S" stands for a noncommutative semialgebra that satisfies Assumption 1.5.

In the following example, we introduce a simple method in which one may define the semialgebra of nonnegative rational numbers $\mathbb{Q}_+$ from the semialgebra of natural numbers $\mathbb{N}$. Then we define the semialgebra of the fractions of S similarly.

Example 2.1. To define $\mathbb{Q}_+$ from $\mathbb{N}$, there are 2 steps:

i) Define the set of formal expressions $U = \{n \oslash m \,|\, n, m \in \mathbb{N}\}$ with $\oslash$ the formal division. There are obviously some "illegal" expressions in U like $3 \oslash 0$ and $0 \oslash 0$, but we only care about the set of "legal" expressions $W = \{n \oslash m \,|\, n, m \in \mathbb{N}, m > 0\}$.

ii) Since there are some expressions, *e.g.*, $2 \oslash 6$ and $3 \oslash 9$, which clearly stand for the same rational number, we need an equivalence relation R on W telling whether two expressions "equal". Here is a simple way to define R: $n \oslash m \overset{R}{\sim} i \oslash j$ if and only if $nj = mi$. We then define $\mathbb{Q}_+ = W/R$, which can be regarded as the semialgebra of the fractions of $\mathbb{N}$.

We will find that we can define the semialgebra of the fractions of S in the same way.

2.1. Defining the set of formal rational expressions of the elements in S

The first step is to construct the set of all formal expressions of fractions (as the set U in Example 2.1). Since S is noncommutative, the "fractions" of S may have more complicated forms than just a/b. For instance, we should allow expressions like $(2 \cdot a * b + c^{-1})^{-1}$, which contains multiplication, scale multiplication, addition, and inversion. Thus, we define U to be the set of all finite formal expressions in the elements in S with the formal addition "$\oplus$", the formal multiplication "$\circledast$," the formal scalar multiplication "$\odot$" and the formal inversion "$^{-1}$" ($\oplus$, $\circledast$, and $\odot$ are binary operations while $^{-1}$ is unary). To be precise, we have

Definition 2.2. The set $U(S)$ (or simply U, if it won't cause any ambiguity) of *formal rational expressions* consisting of finitely many formal operations and elements in a semialgebra S, refers to the set determined exactly by the following rules:

i) $S \subset U$,

ii) $a \circledast b \in U \Leftrightarrow a \in U \wedge b \in U$,

iii) $a \oplus b \in U \Leftrightarrow a \in U \wedge b \in U$,

iv) $r \odot a \in U \Leftrightarrow r \in \mathbb{R}_+ \wedge a \in U$,
v) $a^{-1} \in U \Leftrightarrow a \in U$.

The number of formal operations (i.e., "$\circledast$", "$\oplus$", "$\odot$" or "$^{-1}$") contained in any $x \in U$, which will be used later, can also be defined inductively as follows:

Definition 2.3. The *number of formal operations* contained in $x \in U$, denoted by $|x|$, is given by the following inductive rules:
i) $|x| = 0$ if $x \in S$,
ii) $|x| = |a| + |b| + 1$ if $x = a \circledast b$ or $x = a \oplus b$ with $a, b \in U$,
ii) $|x| = |a| + 1$ if $x = r \odot a$ or $x = a^{-1}$ with $a \in U$.
Sometimes we simply say "x has $|x|$ (formal) operations".

Note that there is no such element as $a \oplus b \oplus c$ in U since $\oplus$ is a binary operation. But there are elements of U of the form $(a \oplus b) \oplus c$ or $a \oplus (b \oplus c)$. By Definition 2.2, every element of U either contains no formal operations (*i.e.*, it is in S) or is of exactly one of the following forms: $r \odot a, a \circledast b, a \oplus b, a^{-1}$. To illustrate, set $s_i \in S$, then these are respectively some elements of U of those forms:

$$2 \odot (s_1 \oplus s_2)^{-1}, \ \ 0_S \circledast s_1, \ \ (0_S \circledast s_1) \oplus (0_{\mathbb{R}} \odot s_2^{-1}), \ \ (0_S \circledast s_1)^{-1}.$$

The numbers of formal operations contained in those elements are, respectively, 3, 1, 4 and 2, by Definition 2.3. Moreover, it is clear that the last expression is "illegal" since it is the inverse of an expression that "equals" to zero (called *a null expression*). Therefore, the set of "legal" expressions in U are such that whenever they contain a sub-expression of the form a^{-1}, a is not a null expression. On the other hand, there are many null expressions in U, as the second and the third ones shown above. The set of all "legal" expressions (denoted by W) is not as clear as in the commutative case in Example 2.1, since the set of null expressions (denoted by $\mathcal{O}$) is more complicated. We define these two sets together now that they tangle with each other:

Definition 2.4. The sets of *legal* and *null* formal rational expressions of the elements in a semialgebra S, denoted by $W(S)$ (or W) and $\mathcal{O}(S)$ (or $\mathcal{O}$) respectively, refer to the subsets of U which are determined exactly by the following rules:
i) $S \subset W$ and $\mathcal{O} \cap S = \{0_S\}$;
for all $a, b \in U$,
ii) $a \oplus b \in W \Leftrightarrow a \in W \wedge b \in W$;
$a \oplus b \in \mathcal{O} \Leftrightarrow a \in \mathcal{O} \wedge b \in \mathcal{O}$;
iii) $a \circledast b \in W \Leftrightarrow a \in W \wedge b \in W$;
$a \circledast b \in \mathcal{O} \Leftrightarrow (a \in \mathcal{O} \wedge b \in W) \vee (b \in \mathcal{O} \wedge a \in W)$;
for all $a \in U$ and $r \in \mathbb{R}_+$,
iv) $r \odot a \in W \Leftrightarrow a \in W$;
$r \odot a \in \mathcal{O} \Leftrightarrow a \in \mathcal{O} \vee (r = 0_{\mathbb{R}} \wedge a \in W)$;
for all $a \in U$,
v) $a^{-1} \notin \mathcal{O}$;
$a^{-1} \in W \Leftrightarrow a \in W \backslash \mathcal{O}$.

All these rules are easy to understand: Rule i) means elements of S are legal expressions and 0_S is the only null expression in S. The first conditions of Rules ii) – iv) mean that the legal expressions are closed *w.r.t.* the formal operations $\oplus, \circledast$ and $\odot$ and that only legal expressions can generate other legal expressions. The second condition in Rule v) means that only the inverses of those legal-but-not-null expressions are legal inverses. The second condition in Rule ii) says the set of null expressions is closed *w.r.t.* the formal addition and the zero-sum-free property holds. The second condition in Rules iii) – iv) means a null expression times any legal expression is again a null expression, and that the

zero-divisor-free property holds. The condition $a^{-1} \notin \mathcal{O}$ in Rule v) means anything of the form a^{-1} cannot be a null expression.

Definition 2.4 does uniquely characterize two subsets of U, since there is an algorithm deciding whether a given expression $a \in U$ is in W (or $\mathcal{O}$) or not. Some examples are sufficient for the readers to understand that:

Example 2.5. Let s_i be some nonzero elements of S. Then we have

$$\begin{aligned} & (s_1 \oplus s_2)^{-1} \in W \\ \Leftrightarrow\ & s_1 \oplus s_2 \in W \backslash \mathcal{O} \\ \Leftrightarrow\ & s_1 \oplus s_2 \in W \wedge s_1 \oplus s_2 \notin \mathcal{O} \\ \Leftrightarrow\ & (s_1 \in W \wedge s_2 \in W) \wedge (s_1 \notin \mathcal{O} \vee s_2 \notin \mathcal{O}) \\ \Leftrightarrow\ & \textbf{\textit{True}}. \end{aligned}$$

The reasoning above uses only rules in Definition 2.4 several times. Whenever a rule is used, an operation in the expression $(s_1 \oplus s_2)^{-1}$ is reduced. When there are no operations, Rule i) in Definition 2.4 helps decide whether the statement is true or false. Here is another example:

$$\begin{aligned} & (0_S \circledast s_1)^{-1} \in W \\ \Leftrightarrow\ & 0_S \circledast s_1 \in W \backslash \mathcal{O} \\ \Leftrightarrow\ & 0_S \circledast s_1 \in W \wedge 0_S \circledast s_1 \notin \mathcal{O} \\ \Leftrightarrow\ & (0_S \circledast s_1 \in W) \wedge 0_S \notin \mathcal{O} \wedge s_1 \notin \mathcal{O} \\ \Leftrightarrow\ & \textbf{\textit{False}}, \end{aligned}$$

which means $(0_S \circledast s_1)^{-1}$ is not legal. We can also decide whether an element of U is in $\mathcal{O}$. For instance, we have $(s_1 \oplus s_2)^{-1} \notin \mathcal{O}$ according to Rule v) in Definition 2.4, and

$$\begin{aligned} & s_1 \oplus s_2 \in \mathcal{O} && && 0_S \circledast s_1 \in \mathcal{O} \\ \Leftrightarrow\ & s_1 \in \mathcal{O} \wedge s_2 \in \mathcal{O}\,, && \text{and} \quad \Leftrightarrow\ & 0_S \in \mathcal{O} \vee s_1 \in \mathcal{O}\,. \\ \Leftrightarrow\ & \textbf{\textit{False}} && && \textbf{\textit{True}} \end{aligned}$$

The following proposition shows that all null expressions are legal, which is important.

Proposition 2.6. *The set $\mathcal{O}$ is a subset of W.*

Proof. It suffices to show that

$$\forall i \in \mathbb{N}, \forall a \in \mathcal{O},\ |a| = i \Rightarrow a \in W,$$

where $|a|$ is the number of formal operations contained in a (Definition 2.3). But this can be proved inductively (for i) using Definition 2.4. □

Remark 2.7. Combining Proposition 2.6 with the second rule in Conditions iii) of Definition 2.4, one obtains the implication

$$a \in \mathcal{O} \wedge b \in \mathcal{O} \Rightarrow a \circledast b \in \mathcal{O}.$$

Since Proposition 2.6 itself is derived from Definition 2.4, so is the implication above. Thus there is no need to include that implication in Definition 2.4.

2.2. Constructing an equivalence relation on the set of legal expressions

As in Example 2.1, we define in this subsection an equivalence relation $R(S)$ (or written as R) on the set W so that the quotient W/R becomes a semialgebra of the fractions of S with every nonzero element invertible. This is why it is called the *semialgebra-oriented* equivalence relation (Definition 2.9). The principal is to ensure that R includes only necessary relations, such that R is as small as possible, and the semialgebra of fractions W/R can then be as general as possible.

On the one hand, there are no semialgebra laws in the set of legal expressions W, *e.g.*, $w_1 = (a \oplus b) \oplus c$ and $w_2 = a \oplus (b \oplus c)$ are not the same. On the other hand, from Example 2.1, we know that $(a, b) \in R$ means $a = b$ in the semialgebra of fractions W/R. Therefore, we will have $w_1 = w_2$ (and thus the associative law of addition) in W/R if and only if we include the pair (w_1, w_2) into the equivalence relation R. Since we expect W/R to be a semialgebra, all pairs (like (w_1, w_2) mentioned above) which stand for any semialgebra axioms should necessarily be included in R. These pairs are in the sets listed below:

$$\begin{aligned}
A_1 &= \{((a \oplus b) \oplus c, a \oplus (b \oplus c)), ((a \circledast b) \circledast c, a \circledast (b \circledast c)),\\
&\quad (a \circledast (b \oplus c), (a \circledast b) \oplus (a \circledast c)), ((b \oplus c) \circledast a, (b \circledast a) \oplus (c \circledast a)),\\
&\quad (a \oplus b, b \oplus a), (0_S \oplus a, a), (a \oplus 0_S, a) \mid a, b, c \in W\},\\
A_2 &= \{(1_S \circledast a, a), (a \circledast 1_S, a), (0_S \circledast a, 0_S), (a \circledast 0_S, 0_S) \mid a \in W\},\\
A_3 &= \{(r \odot (a \oplus b), (r \odot a) \oplus (r \odot b)), ((r +_{\mathbb{R}} t) \odot a, (r \odot a) \oplus (t \odot a)),\\
&\quad (r \odot (a \circledast b), (r \odot a) \circledast b), (r \odot (a \circledast b), a \circledast (r \odot b)),\\
&\quad ((rt) \odot a, r \odot (t \odot a)), (1_{\mathbb{R}} \odot a, a), (0_{\mathbb{R}} \odot a, 0_S) \mid r, t \in \mathbb{R}_+, a, b \in W\}.
\end{aligned} \tag{6}$$

Similarly, since $(a \circledast b)^{-1}$ and $b^{-1} \circledast a^{-1}$ are different expressions in W, we need to include into R those axioms involving inversion:

$$\begin{aligned}
A_4 &= \{(a^{-1} \circledast a, 1_S), (a \circledast a^{-1}, 1_S), ((a \circledast b)^{-1}, b^{-1} \circledast a^{-1}), (1_S^{-1}, 1_S),\\
&\quad (s \odot (a^{-1}), (\tfrac{1}{s} \odot a)^{-1}) \mid a, b \in W \backslash \mathcal{O}, s \in \mathbb{R}_{>0}\}.
\end{aligned} \tag{7}$$

Besides, we also expect that the identities in S remain valid in W/R. For instance, suppose $s_1 = x_1 * x_2 + 2$, $s_2 = x_1 * x_2 * x_1 * x_2 + 2$ and $s_3 = 2x_1 * x_2 + 1$ are elements in the semialgebra $S = \mathbb{R}_+\langle x_1, \ldots, x_n\rangle'$ defined in Example 1.2, then $s_1^2 = s_2 + 2s_3$ is an identity in S. This identity should also be true in the semialgebra of the fractions of S. To explain that explicitly, we define

$$Q = \{w \in W \mid w \text{ does not contain the formal inversion}\},$$

then any $w \in Q$ can be evaluated in the semialgebra S: one takes off all the "circles" of the symbols $\oplus, \circledast$ and $\odot$, turning them into $+, *$ and $\cdot$ (the operations in S) respectively, whenever they occur in the expression w. In this way, the expression w results in an element of S, denoted by w_S. To illustrate, let s_1 and s_2 be as above, then $w = s_2 \oplus (2 \cdot s_3)$ is an expression without the formal inversion and $w_S = s_2 + 2s_3$, which is an element in S. Now the condition "identities in S remains valid in W/R" require us to include into R the following set of pairs:

$$A_5 = \{(a, b) \in Q \times Q \mid a_S = b_S\}. \tag{8}$$

Finally, the expressions in the set $\mathcal{O}$ defined in Definition 2.4 should be regarded as zero. Thus we need to include into R the pairs in the set below:

$$A_6 = \{(a, 0_S) \mid a \in \mathcal{O}\} = \mathcal{O} \times \{0_S\}. \tag{9}$$

From above, the pairs in the set $\cup_{i=1}^{6} A_i$ should be included in R. However, to define operations in the semialgebra of fractions W/R, R needs to be closed under formal operations $\oplus, \circledast, \odot$ and $^{-1}$. Precisely, we have

Definition 2.8. Suppose that R is an equivalence relation on the set of legal expressions W. Then R is called to be *closed under the formal operations* $\oplus, \circledast, \odot$ and $^{-1}$ if the following conditions hold:

i) $(a,b) \in R \wedge (c,d) \in R \Rightarrow (a \oplus c, b \oplus d) \in R \wedge (a \circledast c, b \circledast d) \in R$;

ii) $(a,b) \in R \Rightarrow (r \odot a, r \odot b) \in R$;

iii) $(a,b) \in R \wedge a \notin \mathcal{O} \wedge b \notin \mathcal{O} \Rightarrow (a^{-1}, b^{-1}) \in R$;

for any $a,b,c,d \in W$ and any $r \in \mathbb{R}_+$.

To see why the set R has to meet these conditions, suppose $\overline{a}, \overline{b} \in W/R$ are the equivalence classes of $a, b \in W$, respectively. Then it is natural to define the addition in W/R by

$$\overline{a} + \overline{b} = \overline{a \oplus b}.$$

Now "the relation R is closed under the formal addition $\oplus$" ensures that this addition is well-defined. Otherwise, it seems impossible to define an addition in W/R.

Note that if $R_1, R_2 \subset W \times W$ are two equivalence relations on W which are both closed under all the formal operations, then so is their intersection $R_1 \cap R_2$. Therefore we have the following definition

Definition 2.9. The *semialgebra-oriented* equivalence relation $R(S) \subset W \times W$ on the set W of legal formal rational expressions in elements of a semialgebra S is defined to be the minimal subset of $W \times W$ such that (1) it contains $\cup_{i=1}^{6} A_i$, that (2) it is an equivalence relation and that (3) it is closed under the formal operations $\oplus, \circledast, \odot$ and $^{-1}$. In other words, if $\mathscr{R}$ is the set of all subsets of $W \times W$ which meet these three conditions, then

$$R = \bigcap_{T \in \mathscr{R}} T.$$

The equivalence relation R is well-defined since $W \times W \in \mathscr{R}$ and $\mathscr{R}$ is not empty.

Definition 2.10. For any $w \in W$, we denote by $\overline{w}$ the R-equivalence class containing w. The map $\overline{\ }: W \to W/R, w \mapsto \overline{w}$ is called *the generalized canonical map* while its restriction to S, *i.e.*, the map $\overline{\ }: S \to W/R, s \mapsto \overline{s}$, is called *the canonical map*.

The following proposition is important, but its proof is tedious. The details of the proof can be found in Zheng and Zhi (2023) Proposition 2.11.

Proposition 2.11. *These two sets coincide:* $\overline{0}_S = \mathcal{O}$.

2.3. Defining the semialgebra of the fractions of S

Set $F = W/R$ to be the set of equivalence classes in W, to turn it into a semialgebra, we can define an addition, a multiplication, an inversion, and a scalar multiplication in F as follows,

$$\left.\begin{array}{rcl} \overline{a} + \overline{b} & = & \overline{a \oplus b} \\ \overline{a} * \overline{b} & = & \overline{a \circledast b} \\ r \cdot \overline{a} & = & \overline{r \odot a} \\ \overline{c}^{-1} & = & \overline{c^{-1}} \end{array}\right\}, \ \forall a,b \in W, \ \forall r \in \mathbb{R}_+, \ \forall c \in W \backslash \mathcal{O}.$$

Proposition 2.12. *The operations given above are well-defined, and they turn F into a semialgebra with any nonzero element invertible.*

Proof. The definitions of addition, multiplication, and scalar multiplication are all well-defined since R is closed under the formal operations $\oplus, \circledast$, and $\odot$. The relations in the set $A_1 \cup A_2 \cup A_3$ ensure

that all semialgebra axioms are valid in F. Moreover, $\overline{0}_S$ and $\overline{1}_S$ are the identities of addition and multiplication.

We must show that the inversion is well-defined and is "the inversion" for the multiplication defined here.

Set $e \in W$ and suppose that $\overline{e} = \overline{c}$. By Proposition 2.11, $c \notin \mathcal{O}$ implies $\overline{c} \neq \overline{0}_S$. Hence $\overline{e} \neq \overline{0}_S$, meaning that $e \notin \mathcal{O}$. Thus $e^{-1} \in W$. Moreover, $\overline{e^{-1}} = \overline{c^{-1}}$ since R is closed under inversion. Therefore, the inversion is well-defined for any $c \notin \mathcal{O}$, *i.e.*, for any $\overline{c} \neq \overline{0}_S$. Finally, the relations in the set A_4 guarantee that the following equations hold for any $\overline{a}, \overline{b} \neq \overline{0}_S$: $\overline{a}^{-1} * \overline{a} = \overline{1}_S = \overline{a} * \overline{a}^{-1}$, $(\overline{a} * \overline{b})^{-1} = \overline{b}^{-1} * \overline{a}^{-1}$, meaning that this is exactly the inversion for the multiplication "$*$". □

Therefore, we have the following important definition:

Definition 2.13. The *semialgebra of the fractions* of a semialgebra S satisfying Assumption 1.5, denoted by $F(S)$ (or simply by F), refers to the semialgebra $W(S)/R(S)$ proposed in Proposition 2.12.

The proposition below is clear since $A_5 \subset R$:

Proposition 2.14. *The canonical map from S to F is a semialgebra homomorphism.*

Remark 2.15. The semialgebra F is also zero-sum-free and zero-divisor-free:

- $\overline{a} + \overline{b} = \overline{0}_S$ implies that $a \oplus b \in \mathcal{O}$. Hence $a, b \in \mathcal{O}$, that is, $\overline{a} = \overline{b} = \overline{0}_S$;
- $\overline{a} * \overline{b} = \overline{0}_S$ implies that $a \circledast b \in \mathcal{O}$. Hence $a \in \mathcal{O}$ or $b \in \mathcal{O}$, that is, $\overline{a} = \overline{0}_S$ or $\overline{b} = \overline{0}_S$;
- $r \cdot \overline{b} = \overline{0}_S$ implies that $r \odot b \in \mathcal{O}$. Hence $r = 0_{\mathbb{R}}$ or $b \in \mathcal{O}$, that is, $r = 0_{\mathbb{R}}$ or $\overline{b} = \overline{0}_S$.

The following proposition is natural. Its proof can be found in Zheng and Zhi (2023) Proposition 2.16.

Proposition 2.16. *Suppose that S is a semialgebra satisfying Assumption 1.5 and that $E \subset S$ is a sub-semialgebra with 0_S and 1_S its additive and multiplicative identities. Let $W(S)$ and $R(S)$ be the sets defined in Definitions 2.4 and 2.9 for the semialgebra S, and $W(E)$ and $R(E)$ the corresponding sets defined for the sub-semialgebra E. Then, we have $W(E) \subset W(S)$ and $R(E) \subset R(S)$. Moreover, there is a natural semialgebra homomorphism between the following two semialgebra of fractions: $F(E) \to F(S) : \overline{a}^E \mapsto \overline{a}^S$, where $a \in W(E)$ is any legal formal expressions in the elements of E, and $\overline{\ }^E : W(E) \to F(E)$ and $\overline{\ }^S : W(S) \to F(S)$ are the generalized canonical maps.*

2.4. Related work

The new definition of the semialgebra of fractions in this section generalizes the one in (Golan, 1999, Proposition 11.5), where elements in the semiring of the fractions of a noncommutative semiring are only allowed to be of the form $a^{-1} * b$. While our definition allows the inversion to appear in any position or even to be nested in an element, *e.g.*, we have elements of the form $a^{-1} * b * c^{-1} + d * p^{-1} * q$ and $(a + b^{-1})^{-1}$, *etc.*, which are apparently much more complicated.

In (Golan, 1999, Chap. 18), the author also generalized the definition of the semiring of fractions in (Golan, 1999, Chap. 11) via the *Gabriel filter*, which is a classical and powerful tool in algebra. We are not clear how our generalization and the one given in (Golan, 1999, Chap. 18) could possibly be related.

Another topic related to the content of this section is the theory of *skew fields* Cohn (1995), which deals primarily with rings in which every nonzero element has a multiplicative inverse. This property also holds for the semialgebra of fractions defined in this section (Proposition 2.12). The difference is that subtraction is allowed in a skew field, but not in the semialgebra of fractions introduced here.

In Proposition B.4 of Zheng and Zhi (2023), we show that when the semialgebra S is commutative, the semialgebra F of its fractions defined in this section coincides with the corresponding concept

in the commutative case, which can be obtained naturally from (Golan, 1999, Example 11.7). Therefore, the new definition of the semialgebra of fractions generalizes the corresponding concept in the commutative case, indicating that our definition is reasonable.

3. Preordered semialgebra and the derived preorder relation on the semialgebra of its fractions

This section discusses how a preorder relation on a semialgebra S derives a relation on the semialgebra $F = F(S)$ of the fractions of S.

In a semialgebra, we are only concerned with preorders that are compatible with all the semialgebra operations (that is, which satisfy the implication in (1)). Denote by $\mathscr{P}$ the set of such preorders on the semialgebra $F(S)$, then the intersection of any elements of $\mathscr{P}$ is still an element of $\mathscr{P}$. On the other hand, an inequality in S is naturally expected to be true in F. Therefore, we have the following definition:

Definition 3.1. Let $(S, \leq)$ be a preordered semialgebra, and F be the semialgebra of the fractions of S. Set $\mathscr{I} = \{(\overline{x}, \overline{y}) \mid x \leq y \text{ in } S\}$. Then the *derived preorder* $\preccurlyeq$ on F *w.r.t.* the preorder $\leq$ is the minimal preorder which is compatible with all the semialgebra operations and contains the set $\mathscr{I}$. That is,

$$\{(a, b) \in F \times F \mid a \preccurlyeq b\} = \bigcap_{\mathscr{I} \subset P \in \mathscr{P}} P. \tag{10}$$

For any $a, b \in F$, if there are $k \in \mathbb{N}_{\geq 1}$, $g_1, \ldots, g_k, h_1, \ldots, h_k \in F$ and $A_1, \ldots, A_k, B_1, \ldots, B_k \in S$ such that $A_i \leq B_i$ for all i, and that

$$\begin{cases} a &= \sum_{i=1}^{k} g_i \overline{A}_i h_i \\ b &= \sum_{i=1}^{k} g_i \overline{B}_i h_i \end{cases} \tag{11}$$

in F, then we write "$a \lessdot b$". The proposition below gives another perspective into the derived preorder:

Proposition 3.2. *If $a, b \in F$, then $a \preccurlyeq b$ if and only if there are $c_1, \ldots, c_k \in F$ for some $k \in \mathbb{N}$ such that*

$$a \lessdot c_1 \lessdot \cdots \lessdot c_k \lessdot b. \tag{12}$$

The proof, consisting of some routine reasoning, can be found in Zheng and Zhi (2023) Proposition 3.3. While the following corollary is interesting:

Corollary 3.3. *If $a \preccurlyeq b \in F$, then there is $w \in F$ such that $a + w \lessdot b + w$. That is, there are $k \in \mathbb{N}_{\geq 1}$, $g_1, \ldots, g_k, h_1, \ldots, h_k \in F$ and $A_1, \ldots, A_k, B_1, \ldots, B_k \in S$ such that $A_i \leq B_i$ for all i, and that*

$$\begin{cases} a + w &= \sum_{i=1}^{k} g_i \overline{A}_i h_i, \\ b + w &= \sum_{i=1}^{k} g_i \overline{B}_i h_i. \end{cases}$$

Proof. One observes that $a \lessdot c \lessdot b$ implies $a + c \lessdot c + b$. Thus, when $a \lessdot c \lessdot b$ for some $c \in F$, there is some $w \in F$ (*e.g.*, $w = c$) such that $a + w \lessdot b + w$. By reasoning inductively, one finds that condition (12) also ensures the existence of such a w. □

Recall that, for a commutative zero-divisor-free semiring K with preorder relation "$\leq$", the usual semiring of its fractions refers to the semiring

$$K^{\mathrm{fr}} = \left\{ \frac{x}{a} \,\middle|\, x, a \in K, a \neq 0_K \right\}$$

(see (Golan, 1999, Example 11.7)) and the preorder $\leq^{\mathrm{fr}}$ on K^{fr} induced by $\leq$ is as follows (see, *e.g.*, Fritz 2021b, page 13): for any $x, y \in K$ and any $a, b \in K \backslash \{0_K\}$,

$$\frac{x}{a} \leq^{\mathrm{fr}} \frac{y}{b} \quad \text{if and only if} \quad \exists t \in K\backslash\{0_K\}, x*b*t \leq y*a*t. \tag{13}$$

If K is a commutative zero-sum-free semialgebra, then the definitions of K^{fr} and $\leq^{\mathrm{fr}}$ above work as well. On the other hand, we may define the semialgebra $F(K)$ of the fractions of K as in Definition 2.13. Proposition B.4 of Zheng and Zhi (2023) shows that the two semialgebras K^{fr} and $F(K)$ coincide with each other. Furthermore, in Zheng and Zhi (2023) Proposition C.2, we indicate that the preordered semialgebras $(F(K), \preccurlyeq)$ and $(K^{\mathrm{fr}}, \leq^{\mathrm{fr}})$ are also isomorphic to each other, with $\preccurlyeq$ the derived preorder of $\leq$ given in Definition 3.1. This indicates that Definition 3.1 of the derived preorder is a noncommutative generalization of the definition of $\leq^{\mathrm{fr}}$ in condition (13) for the commutative case.

The following proposition is important for the proof of the main theorem:

Proposition 3.4. *If $u \in S$ is power universal in S (with respect to $\leq$), then $\overline{u}$ is power universal in F (with respect to $\preccurlyeq$).*

Proof. We may assume that $u \neq 0_S$, since otherwise, the preorder relations $\leq$ on S and $\preccurlyeq$ on F would be trivial. In that case, the property we want to prove is trivially true. Since $(S\backslash\{0_S\}) \cap \mathcal{O} = \emptyset$, $u \neq 0_S$ clearly implies $\lambda \cdot \overline{u} \neq \overline{0}_S$ for any fixed real number and $\lambda > 1$. In the following, we first prove that $\lambda \cdot \overline{u}$ is power universal in F, then indicate that $\overline{u}$ is power universal, too.

For any $x \in W\backslash\mathcal{O}$ (or equivalently, any nonzero $\overline{x} \in F$), we need to show that there is a number $k \in \mathbb{N}$ such that

$$(\lambda \cdot \overline{u})^k * \overline{x} \succcurlyeq \overline{1}_S, \ \ \overline{x} * (\lambda \cdot \overline{u})^k \succcurlyeq \overline{1}_S \ \text{ and } \ (\lambda \cdot \overline{u})^k \succcurlyeq \overline{x}. \tag{14}$$

For any element $w \in W$, we denote by $|w|$ the number of formal operations in it, as in Definition 2.3. Suppose that $|x| = i$ for $i \in \mathbb{N}$, we prove inductively on i:

The "$i = 0$" case:

When $i = 0$, $x \in S$ and $x \neq 0_S$. Since u is power universal in S, there is an integer $k \in \mathbb{N}$ such that inequalities (3) hold. By the definition of $\preccurlyeq$, we also have

$$\overline{u}^k * \overline{x} \succcurlyeq \overline{1}_S, \ \ \overline{x} * \overline{u}^k \succcurlyeq \overline{1}_S \ \text{ and } \ \overline{u}^k \succcurlyeq \overline{x}.$$

Since $\lambda > 1$, $\lambda \cdot \overline{u} = \overline{u} + (\lambda - 1) \cdot \overline{u} \succcurlyeq \overline{u}$. Thus $(\lambda \cdot \overline{u})^k \succcurlyeq \overline{u}^k$, and we obtain the inequalities in (14).

The induction step:

Assume that for any $0 \leq i \leq \ell$ ($\ell \in \mathbb{N}$) and any $x \in W\backslash\mathcal{O}$ with $|x| = i$, there is an integer $k \in \mathbb{N}$ such that the inequalities in (14) hold. We prove that this is also valid for $i = \ell + 1$:

Set $x \in W\backslash\mathcal{O}$ with $|x| = \ell + 1$. Then there are several cases:

1) $x = b \oplus c$ for some $b, c \in W$: Then $\overline{x} = \overline{b} + \overline{c}$ with $|b| \leq \ell$ and $|c| \leq \ell$. Since $\overline{x} \neq \overline{0}_S$, $\overline{b} \neq \overline{0}_S$ or $\overline{c} \neq \overline{0}_S$. If one of $\overline{b}$ and $\overline{c}$ equals $\overline{0}_S$, say, $\overline{b} = \overline{0}_S$, then $\overline{x} = \overline{c}$. According to the inductive assumption, for $c \in W\backslash\mathcal{O}$, there is an integer $k \in \mathbb{N}$ such that (14) hold with "$\overline{x}$" replaced by "$\overline{c}$". Therefore, inequalities in (14) hold for $\overline{x}$ itself as well. Now, set $\overline{b} \neq \overline{0}_S$ and $\overline{c} \neq \overline{0}_S$, then we have $k_1, k_2 \in \mathbb{N}$ such that

$$\begin{cases} (\lambda \cdot \overline{u})^{k_1} * \overline{b} \succcurlyeq \overline{1}_S, \quad \overline{b} * (\lambda \cdot \overline{u})^{k_1} \succcurlyeq \overline{1}_S, \quad (\lambda \cdot \overline{u})^{k_1} \succcurlyeq \overline{b}, \\ (\lambda \cdot \overline{u})^{k_2} * \overline{c} \succcurlyeq \overline{1}_S, \quad \overline{c} * (\lambda \cdot \overline{u})^{k_2} \succcurlyeq \overline{1}_S, \quad (\lambda \cdot \overline{u})^{k_2} \succcurlyeq \overline{c}. \end{cases} \tag{15}$$

Set $\hat{k} = \max\{k_1, k_2\}$, and choose sufficiently large $k \geq \hat{k}$ satisfying $\lambda^k \geq 2\lambda^{\hat{k}}$, then

$$(\lambda \cdot \overline{u})^k * \overline{x} \succcurlyeq (\lambda \cdot \overline{u})^{\hat{k}} * \overline{x} = (\lambda \cdot \overline{u})^{\hat{k}} * (\overline{b} + \overline{c}) \succcurlyeq \overline{1}_S + \overline{1}_S \succcurlyeq \overline{1}_S.$$

Similarly, we have $\overline{x} * (\lambda \cdot \overline{u})^k \succcurlyeq \overline{1}_S$. On the other hand,

$$(\lambda \cdot \overline{u})^k = \lambda^k \cdot \overline{u}^k \succcurlyeq (2\lambda^{\hat{k}}) \cdot \overline{u}^{\hat{k}} = 2 \cdot (\lambda \cdot \overline{u})^{\hat{k}} \succcurlyeq \overline{b} + \overline{c} = \overline{x}.$$

2) $x = b \circledast c$ for some $b, c \in W$: Then $\overline{x} = \overline{b} * \overline{c}$ with $b, c \in W \backslash \mathcal{O}$ such that $|b| \leq \ell$ and $|c| \leq \ell$. Hence the inequalities (15) hold for some $k_1, k_2 \in \mathbb{N}$. Set $k = k_1 + k_2$. Multiplying the rightmost inequalities in each rows of (15), we get $(\lambda \cdot \overline{u})^k \succcurlyeq \overline{b} * \overline{c} = \overline{x}$. Multiplying the leftmost inequality in the first row and the middle inequality in the second row of (15), we obtain $(\lambda \cdot \overline{u})^{k_1} * \overline{b} * \overline{c} * (\lambda \cdot \overline{u})^{k_2} \succcurlyeq \overline{1}_S$. Then, by multiplying $(\lambda \cdot \overline{u})^{-k_1}$ and $(\lambda \cdot \overline{u})^{-k_2}$ from the left and the right sides, respectively, one derives $\overline{x} \succcurlyeq (\lambda \cdot \overline{u})^{-k}$. From that, one obtains both $(\lambda \cdot \overline{u})^k * \overline{x} \succcurlyeq \overline{1}_S$ and $\overline{x} * (\lambda \cdot \overline{u})^k \succcurlyeq \overline{1}_S$ by simply multiplying $(\lambda \cdot \overline{u})^k$ from the left and the right sides respectively.

3) $x = r \odot b$ for $r \in \mathbb{R}_+$ and $b \in W$: In this case, $\overline{x} = r \cdot \overline{b}$ with $b \in W \backslash \mathcal{O}$ and r a positive real number. Moreover, $|b| \leq \ell$. Hence there is an integer $k_1 \in \mathbb{N}$ such that the inequalities in the first row or (15) hold. Choose a sufficiently large integer $m \in \mathbb{N}$ satisfying $\lambda^m \geq \max\{r, 1/r\}$ and set $k = m + k_1$. Then we have

$$\begin{aligned}(\lambda \cdot \overline{u})^k * \overline{x} &\succcurlyeq (\lambda \cdot \overline{1}_S)^m * (\lambda \cdot \overline{u})^{k_1} * (r \cdot \overline{b}) \\ &\succcurlyeq (\lambda^m r) \cdot ((\lambda \cdot \overline{u})^{k_1} * \overline{b}) \\ &\succcurlyeq 1_{\mathbb{R}} \cdot \overline{1}_S \\ &= \overline{1}_S,\end{aligned}$$

and similarly

$$\begin{aligned}\overline{x} * (\lambda \cdot \overline{u})^k &\succcurlyeq (r \cdot \overline{b}) * (\lambda \cdot \overline{u})^{k_1} * (\lambda \cdot \overline{1}_S)^m \\ &\succcurlyeq (\lambda^m r) \cdot (\overline{b} * (\lambda \cdot \overline{u})^{k_1}) \\ &\succcurlyeq 1_{\mathbb{R}} \cdot \overline{1}_S \\ &= \overline{1}_S.\end{aligned}$$

Moreover, we have $(\lambda \cdot \overline{u})^k = (\lambda \cdot \overline{u})^m * (\lambda \cdot \overline{u})^{k_1} \succcurlyeq (\lambda \cdot \overline{1}_S)^m * \overline{b} = \lambda^m \cdot \overline{b} \succcurlyeq r \cdot \overline{b} = \overline{x}$.

6) $x = b^{-1}$ for some $b \in W \backslash \mathcal{O}$: We have $\overline{x} = \overline{b}^{-1}$. Since $|b| \leq \ell$, $\overline{b}$ satisfies the inequalities in the first row of (15) for some integer $k_1 \in \mathbb{N}$. From the rightmost inequality we obtain both $(\lambda \cdot \overline{u})^{k_1} * \overline{b}^{-1} \succcurlyeq \overline{1}_S$ and $\overline{b}^{-1} * (\lambda \cdot \overline{u})^{k_1} \succcurlyeq \overline{1}_S$ by multiplying $\overline{b}^{-1}$ from its right and left sides, respectively. From the leftmost one we obtain $(\lambda \cdot \overline{u})^{k_1} \succcurlyeq \overline{b}^{-1}$ by multiplying $\overline{b}^{-1}$ from its right side.

By now, we have shown that $\lambda \cdot \overline{u}$ is power universal in F since the inequalities in (14) hold. We show that $\overline{u}$ is also power universal. Apparently, $(\lambda \cdot 1_S)^k \neq 0_S$, hence there is $k_0 \in \mathbb{N}$ such that $u^{k_0} \geq (\lambda \cdot 1_S)^k$. Therefore $\overline{u}^{k_0} \succcurlyeq (\lambda \cdot \overline{1}_S)^k$ in F. Using the inequalities in (14), we have $\overline{u}^{k_0+k} \succcurlyeq (\lambda \cdot \overline{1}_S)^k * (\lambda^{-k} \cdot \overline{x}) = \overline{x}$, and $\overline{u}^{k_0+k} * \overline{x} \succcurlyeq (\lambda \cdot \overline{1}_S)^k * \overline{u}^k * \overline{x} = (\lambda \cdot \overline{u})^k * \overline{x} \succcurlyeq \overline{1}_S$. Similarly, we also have $\overline{x} * \overline{u}^{k_0+k} \succcurlyeq \overline{1}_S$. Hence $\overline{u}$ is indeed power universal. □

4. Extending $\mathbb{R}_+$-valued homomorphisms from a semialgebra to the semialgebra of its fractions

This section will show that a monotone homomorphism from the preordered semialgebra S to some other preordered semialgebra T (which shares some good properties) can be extended to the semialgebra F of the fractions of S. In particular, the extension is always guaranteed when $T = \mathbb{R}_+$.

The proofs of Propositions 4.1 and 4.2 in this section can be found in Section 4 of Zheng and Zhi (2023). These proofs are inductive and consist of routine but tedious checks. Skipping them does no harm to the completeness of the paper.

Suppose that $(S, \leq)$ is a preordered semialgebra satisfying Assumptions 1.4 and 1.5, and that $(T, \leq_T)$ is a zero-sum-free and zero-divisor-free preordered semialgebra with every nonzero element invertible, and a nontrivial preorder relation $\leq_T$ such that $1_T \geq_T 0_T$ (*e.g.*, if $T = \mathbb{R}_+$, then all the assumptions on the semialgebra T are satisfied).

For any monotone semialgebra homomorphism $f : S \to T$, the following proposition defines a map f^W from the set of legal formal rational expressions $W(S)$ (Definition 2.4) to the semialgebra T:

Proposition 4.1. *Suppose $f : S \to T$ is a monotone semialgebra homomorphism with preordered semialgebras S and T described above. Then, the recursive assignment below for f^W makes it a well-defined map from W to T:*

1) For any $x \in S$, set $f^W(x) = f(x)$;

2) If $x \in W \setminus S$ contains some formal operations, then set

$$f^W(x) = \begin{cases} f^W(a) + f^W(b), & \text{if} \quad x = a \oplus b \text{ with } a, b \in W, \\ f^W(a) * f^W(b), & \text{if} \quad x = a \circledast b \text{ with } a, b \in W, \\ r \cdot f^W(a), & \text{if} \quad x = r \odot a \text{ with } r \in \mathbb{R}_+, a \in W, \\ (f^W(a))^{-1}, & \text{if} \quad x = a^{-1} \text{ with } a \in W \setminus \mathcal{O}. \end{cases} \tag{16}$$

The following proposition indicates that we can obtain a monotone semialgebra homomorphism from F to T by restricting f^W to F.

Proposition 4.2. *For any monotone semialgebra homomorphism $f : S \to T$ and any $a, b \in W$ such that $\overline{a} = \overline{b}$, we have $f^W(a) = f^W(b)$. Therefore, the map $f^F : F \to T$, $\overline{x} \mapsto f^W(x)$ $(\forall x \in W)$ is well-defined. Moreover, it is also a monotone semialgebra homomorphism.*

Remark 4.3. Clearly we have $f^F(\overline{x}) = f^W(x) = f(x)$ for every $x \in S$, in which sense the map f^F extends f.

Remark 4.4. For any monotone semialgebra homomorphism $\eta : F \to T$ such that $\eta(\overline{x}) = f^F(\overline{x})$ for all $x \in S$ (*i.e.*, it extends the monotone semialgebra homomorphism f), we can prove inductively that $\eta = f^F$. This indicates the universal property of the semialgebra F of the fractions of S.

Remark 4.5. Set $T = \mathbb{R}_+$, then Proposition 4.2 indicate that any monotone semialgebra homomorphism $f : S \to \mathbb{R}_+$ can be extended to a monotone semialgebra homomorphism mapping from F to $\mathbb{R}_+$.

5. From a semialgebra to a linear space

This section defines a preordered $\mathbb{R}$-linear space based on the semialgebra F. This definition is already given by Fritz (2021b), while we rewrite it for the completeness of the present paper.

Define $V = F - F$ to be the Grothendieck group (Karoubi, 1978, Chap. II) of the commutative monoid $(F, +)$, with each element in V of the form $a - b$ for some $a, b \in F$, and with $a - b = a' - b'$ in V if and only if $\exists c \in F$ *s.t.* $a + b' + c = a' + b + c$ in F. The addition in V is given by the rule

$$(a - b) + (c - d) = (a + c) - (b + d),$$

with the identity $0_F - 0_F$, while the inverse of $a - b$ in V is given by

$$-(a - b) = b - a.$$

Then, V becomes an $\mathbb{R}$-linear space once we define a scalar multiplication on it as below:

$$r \cdot (a - b) = \begin{cases} r \cdot a - r \cdot b, & \text{if } r \in \mathbb{R}_{\geq 0}, \\ (-r) \cdot b - (-r) \cdot a, & \text{if } r \in \mathbb{R}_{<0}. \end{cases}$$

In the sequel, we define a preorder relation on V based on the preorder relation $\preccurlyeq$ on F.

For any $a, b, c, d \in F$ (or equivalently, for any $a - b, c - d \in V$), we write $a - b \trianglelefteq c - d$ if and only if there is a $g \in F$ so that $a + d + g \preccurlyeq b + c + g$ in F. It is straightforward to prove that the relation $\trianglelefteq$ is well-defined and a preorder. Furthermore, it is not hard to show that

$$a-b \trianglelefteq c-d \quad \text{implies} \begin{cases} (a-b)+(g-h) & \trianglelefteq & (c+d)+(g-h), \\ r\cdot(a-b) & \trianglelefteq & r\cdot(c-d), \end{cases} \tag{17}$$

for any $a, b, c, d, g, h \in F$ and any $r \in \mathbb{R}_+$. These inequalities are similar to those in implication (1).

To conclude, the $\mathbb{R}$-linear space V is preordered with the relation $\trianglelefteq$, which is compatible with the addition and the scalar multiplication defined in itself.

6. The main theorem and the proof

This section is devoted to the proof of the main theorem: The first subsection gives the framework of the proof, while the second one provides some details that are skipped in the first subsection.

6.1. The framework of the proof

We state the main theorem again for convenience:

Theorem 1.6. *Let $(S, \leq)$ be a preordered semialgebra satisfying Assumptions 1.4 and 1.5, with a power universal element $u \in S$. Suppose that $(F, \preccurlyeq)$ is the preordered semialgebra of the fractions of S defined in Subsection 2.3 with "$\preccurlyeq$" in Definition 3.1 and $\overline{\ }: S \to F, x \mapsto \overline{x}$ the canonical map in Definition 2.10. Then, for every nonzero $x, y \in S$, the following are equivalent:*

(a) $f(x) \geq f(y)$ for every monotone semialgebra homomorphism $f : S \to \mathbb{R}_+$.

(b) For every real number $\epsilon > 0$, there is $m \in \mathbb{N}$ such that

$$\overline{x} + \sum_{j=0}^{m} \epsilon^{j+1} \cdot \overline{u}^j \succcurlyeq \overline{y}. \tag{18}$$

(c) For every $r \in \mathbb{R}_+$ and every real number $\epsilon > 0$, there is a polynomial $p \in \mathbb{Q}_+[X]$ such that $p(r) \leq \epsilon$ and

$$\overline{x} + p(\overline{u}) \succcurlyeq \overline{y}. \tag{19}$$

(d) For every $r \in \mathbb{R}_+$ and every real number $\epsilon > 0$, there is a polynomial $p \in \mathbb{Q}_+[X]$ such that $p(r) \leq 1+\epsilon$ and

$$p(\overline{u}) * \overline{x} \succcurlyeq \overline{y}. \tag{20}$$

Proof. The proof of the implication "(b) $\Rightarrow$ (c)" is almost the same as its counterpart in the proof of Theorem 3.1 in Fritz (2021b): for given r, we choose $\epsilon < r^{-1}$ and m such that Equation (18) holds, and take $p(X) = \sum_{j=0}^{m} \epsilon^{j+1} \cdot X^j$. Then we have $p(r) = \sum_{j=0}^{m} \epsilon^{j+1} \cdot r^j \leq \frac{\epsilon}{1-\epsilon r}$, which indeed can be made arbitrarily small.

The proof of "(c) $\Rightarrow$ (d)" is also similar to the corresponding part of the proof of Theorem 3.1 in Fritz (2021b). But there are some tiny differences since we are dealing with noncommutative semialgebras. So we rewrite the proof of "(c) $\Rightarrow$ (d)" for clearness:

Since $x \in S$ and $x \neq 0_S$, $x \notin \mathcal{O}$, we have $\overline{x} \neq \overline{0}_S$. By Proposition 3.4, $\overline{u}$ is power universal in F. Hence there is an integer $k \in \mathbb{N}$ so that $\overline{u}^k * \overline{x} \succcurlyeq \overline{1}_S$. For any $r \in \mathbb{R}_+$ and any real number $\epsilon > 0$, set

$$\epsilon' = \begin{cases} \frac{\epsilon}{r^k}, & \text{if} \quad r > 0, \\ 1, & \text{if} \quad r = 0. \end{cases}$$

Applying (c) to the numbers r and ϵ', we obtain $p_0 \in \mathbb{Q}_+[X]$ such that $p_0(r) \leq \epsilon'$ and $p_0(\overline{u}) + \overline{x} \succcurlyeq \overline{y}$. Set $p = 1 + p_0 X^k$, then

$$p(r) = 1 + p_0(r) r^k \leq 1 + \epsilon' r^k \leq 1 + \epsilon$$

and

$$p(\overline{u}) * \overline{x} = \overline{x} + p_0(\overline{u}) * \overline{u}^k * \overline{x} \succcurlyeq \overline{x} + p_0(\overline{u}) \succcurlyeq \overline{y}.$$

Hence (d) is deduced.

"(d)⇒(a)": Let $f : S \to \mathbb{R}_+$ be a monotone semialgebra homomorphism, we need to show that $f(x) \geq f(y)$. By Remark 4.5, the map

$$f^F : F \to \mathbb{R}_+, \overline{a} \mapsto f^W(a) \quad (\forall a \in W)$$

is a well-defined monotone semialgebra homomorphism. Setting $r = f^F(\overline{u})$ and applying f^F to inequality (20), we obtain

$$p(r) f^F(\overline{x}) \geq f^F(\overline{y}),$$

where $p(r) \leq 1 + \epsilon$, $f^F(\overline{x}) = f^W(x) = f(x)$ and $f^F(\overline{y}) = f(y)$. Thus, we have $(1+\epsilon) f(x) \geq f(y)$ for any real number $\epsilon > 0$. Taking $\epsilon \to 0$, we obtain $f(x) \geq f(y)$.

"(a)⇒(b)": This part of the proof is almost the same as the proof of Theorem 3.1 in Fritz (2021b). This is mainly due to the well-defined semialgebra of the fractions of a noncommutative semialgebra given in previous sections. We only sketch the proof of this part below, while the skipped details can be found in the next subsection.

Assume that F is *order-cancellative*, i.e. $g + w \succcurlyeq h + w$ implies $g \succcurlyeq h$ $(\forall g, h, w \in F)$. Let V be the linear space defined in Section 5. Define

$$C = \{v \in V \mid v \trianglerighteq 0_V\} = \{g - h \in V \mid g, h \in F, g \succcurlyeq h\},$$

then C is a cone in V by (17). For any real number $\epsilon > 0$, set

$$\mathcal{N}_\epsilon = \{a \in F \mid \exists m \in \mathbb{N} \ s.t. \ a \preccurlyeq \sum_{j=0}^{m} \epsilon^{j+1} \cdot \overline{u}^j\},$$

and define $\mathcal{N}_\epsilon - \mathcal{N}_\epsilon = \{a - b \in V \mid a, b \in \mathcal{N}_\epsilon\}$. Then, as in Fritz (2021b), using the set $\mathcal{N}_\epsilon - \mathcal{N}_\epsilon$ as ϵ varies a basis of neighborhoods of 0_V makes V a locally convex topological vector space. Denoting by $\overline{C}$ the closure of C in that topology, we have

$$\overline{x} - \overline{y} \in \overline{C} \ \Leftrightarrow \text{ the condition (b) holds}, \tag{21}$$

where the proof of the direction "⇒" relies on the assumption that $\preccurlyeq$ is order-cancellative. Denote by V^* the dual space of V, *i.e.*, the topological vector space consisting of all continuous linear (real-valued) functions on V and equipped with the weak-* topology. Define

$$C^* = \left\{\ell \in V^* \mid \ell(C) \subset [0, \infty)\right\},$$

then C^* is a closed convex cone in V^* such that

$$C^* = \overline{\text{conv}}(\text{Er}(C^*)), \tag{22}$$

with $\text{Er}(\cdot)$ the set of all the extreme rays of a cone and $\overline{\text{conv}}(\cdot)$ the closure of the convex hull of a set. In the following, we assume that (b) does not hold and deduce the negation of condition (a).

According to (21), "(b) does not hold" means $\overline{x} - \overline{y} \notin \overline{C}$. By the geometric Hahn-Banach theorem (Rudin, 1991), there is an $\ell \in C^*$ such that $\ell(\overline{x} - \overline{y}) < 0$. Combining with the property in (22), one further claims that

$$\exists f \in \text{Er}(C^*) \subset C^*, \ f(\overline{x} - \overline{y}) < 0. \tag{23}$$

Then we have $f(\overline{1}_S - \overline{0}_S) > 0$ and the function

$$\hat{f} : S \to \mathbb{R}_+, c \mapsto \frac{1}{f(\overline{1}_S - \overline{0}_S)} f(\overline{c} - \overline{0}_S) \tag{24}$$

is a monotone semialgebra homomorphism such that $\hat{f}(x) < \hat{f}(y)$, contradicting to the condition (a).

When F is not order-cancellative, we apply the reduction to the order-cancellative case described in Fritz (2021b): equipping F with the new preorder relation in which g is less than or equal to h if and only if

$$\exists w \in F,\ g + w \preccurlyeq h + w.$$

Then F is order-cancellative by construction. Therefore by what we have already shown, condition (a) implies that for every $\epsilon > 0$ there are $m \in \mathbb{N}$ and $w \in F$ with

$$\overline{x} + \sum_{j=0}^{m} \epsilon^{j+1} \cdot \overline{u}^j + w \succcurlyeq \overline{y} + w.$$

We can now use the following standard argument to get rid of w. By induction and chaining inequalities – see the proof of (Fritz, 2017, Theorem 6.18) and references thereafter – we can prove that for every $n \in \mathbb{N}_{>0}$,

$$n \cdot (\overline{x} + \sum_{j=0}^{m} \epsilon^{j+1} \cdot \overline{u}^j) + w \succcurlyeq n \cdot \overline{y} + w.$$

Thus choosing $k > m$ with $\overline{u}^k \succcurlyeq w$ and taking $n \geq \epsilon^{-(k+1)}$, we get

$$\overline{x} + \sum_{j=0}^{k} \epsilon^{j+1} \cdot \overline{u}^j \succcurlyeq \overline{x} + \sum_{j=0}^{m} \epsilon^{j+1} \cdot \overline{u}^j + \epsilon^{k+1} \cdot \overline{u}^k \succcurlyeq \overline{y} + \frac{1}{n} \cdot w \succcurlyeq \overline{y},$$

which again proves condition (b). □

6.2. Some observations needed in the proof

This subsection provides some technical details that are skipped in the last subsection. Skipping this subsection does not harm to understand the entire paper.

In Observations 6.1–6.5, we verify the condition given in (21), the equation (22), the claim given in (23) and the claim that the function $\hat{f}$ in (24) is a monotone homomorphism. These claims have already been given by Fritz in Fritz (2021b), and part of the proofs of them are also contained therein. However, we rewrite these proofs for completeness. Note that the proof of the equation (22) below (Observation 6.3) is provided by Fritz through private communications.

Observation 6.1. $\overline{x} - \overline{y} \in \overline{C} \ \Leftrightarrow$ *the condition* (b) *holds.*

Proof. The condition $\overline{x} - \overline{y} \in \overline{C}$ implies that

$$\forall \epsilon > 0,\ \exists v \in C \text{ s.t. } v \in (\overline{x} - \overline{y}) + (\mathcal{N}_\epsilon - \mathcal{N}_\epsilon).$$

There are $c, d \in \mathcal{N}_\epsilon$ such that $(\overline{x} - \overline{y}) + (c - d) = v \trianglerighteq 0_V$. By the assumption that F is order-cancellative, this implies $\overline{x} + c \succcurlyeq \overline{y} + d \succcurlyeq \overline{y}$. Since $c \in \mathcal{N}_\epsilon$, then (18) holds.

Since the inequality (18) holds, we have

$$\left(\overline{x} + \sum_{j=0}^{m} \epsilon^{j+1} \cdot \overline{u}^j\right) - \overline{0}_S \trianglerighteq \overline{y} - \overline{0}_S.$$

Adding $\overline{0}_S - \overline{y}$ to the both sides, one obtains

$$\big(\overline{x}+\sum_{j=0}^{m}\epsilon^{j+1}\cdot\overline{u}^{j}\big)-\overline{y}\trianglerighteq\overline{y}-\overline{y}=0_{V},$$

which is equivalent to

$$(\overline{x}-\overline{y})+\big(\sum_{j=0}^{m}\epsilon^{j+1}\cdot\overline{u}^{j}-\overline{0}_{S}\big)\in C, \tag{25}$$

with $\sum_{j=0}^{m}\epsilon^{j+1}\cdot\overline{u}^{j}-\overline{0}_{S}\in\mathcal{N}_{\epsilon}-\mathcal{N}_{\epsilon}$. By inequality (25), we know that there is a point in C that is also in the neighborhood $(\overline{x}-\overline{y})+(\mathcal{N}_{\epsilon}-\mathcal{N}_{\epsilon})$ for any $\epsilon>0$. Thus $\overline{x}-\overline{y}\in\overline{C}$. □

In the proof of the next observation, we need the following definition:

Definition 6.2. Asimow (1969) Suppose P is a cone in some locally convex topological vector space. A set M is said to be a *cap* of the cone P if M is a compact convex subset of P for which $P\backslash M$ is also convex. If each point of P is contained in a cap, then P is said to be *well-capped*.

Observation 6.3. $C^{*}=\overline{\mathrm{conv}}(\mathrm{Er}(C^{*}))$.

Proof. Proposition 21 in Choquet (1962) indicates that any well-capped closed convex cone in a locally convex topological vector space is the closed convex hull of its extreme rays. To prove (22), it is sufficient to show that C^{*} is well-capped (since V^{*} is already locally convex). In Fritz (2021b), Fritz defined the sets

$$C_{\epsilon}^{*}=\big\{\ell\in C^{*}\,|\,\ell(\mathcal{N}_{\epsilon}-\overline{0}_{S})\text{ is bounded}\big\}$$

and

$$D_{\epsilon}=\big\{\ell\in C_{\epsilon}^{*}\,|\,\ell(\mathcal{N}_{\epsilon}-\overline{0}_{S})\subset[0,1]\big\},$$

which satisfy the equation

$$C^{*}=\bigcup_{\epsilon>0}C_{\epsilon}^{*}=\bigcup_{\substack{\epsilon>0\\ n\in\mathbb{N}}}nD_{\epsilon} \tag{26}$$

and claimed that D_{ϵ} was a cap of C^{*} (hence C^{*} is well-capped by equation (26)). While the other conditions in the definition of a cap are verified in Fritz (2021b) for the set D_{ϵ}, the condition "the set $C^{*}\backslash D_{\epsilon}$ is convex" is somehow left out. Hence, Fritz provides the following proof of the condition through personal communications:

Note that $\sum_{j=0}^{m}\epsilon^{j+1}\cdot\overline{u}^{j}\in\mathcal{N}_{\epsilon}$ for any $m\in\mathbb{N}$, we have

$$D_{\epsilon}=\Big\{\ell\in C_{\epsilon}^{*}\,\Big|\,\forall m\in\mathbb{N},\ell\big(\sum_{j=0}^{m}\epsilon^{j+1}\cdot\overline{u}^{j}-\overline{0}_{S}\big)\leq 1\Big\}.$$

Then

$$C_{\epsilon}^{*}\backslash D_{\epsilon}=\Big\{\ell\in C_{\epsilon}^{*}\,\Big|\,\exists m\in\mathbb{N},\ell\big(\sum_{j=0}^{m}\epsilon^{j+1}\cdot\overline{u}^{j}-\overline{0}_{S}\big)>1\Big\}$$

is convex: Set $f_{i}\in C_{\epsilon}^{*}\backslash D_{\epsilon}$ with m_{i} such that $f_{i}\big(\sum_{j=0}^{m_{i}}\epsilon^{j+1}\cdot\overline{u}^{j}-\overline{0}_{S}\big)>1$, $i=1,2$. For any $\mu_{1},\mu_{2}=1-\mu_{1}\in(0,1)$ and $\hat{m}=\max\{m_{1},m_{2}\}$, we have

$$\begin{aligned} & (\mu_1 f_1 + \mu_2 f_2)\big(\sum_{j=0}^{\hat{m}} \epsilon^{j+1} \cdot \overline{u}^j - \overline{0}_S\big) \\ = \;& \mu_1 f_1\big(\sum_{j=0}^{\hat{m}} \epsilon^{j+1} \cdot \overline{u}^j - \overline{0}_S\big) + \mu_2 f_2\big(\sum_{j=0}^{\hat{m}} \epsilon^{j+1} \cdot \overline{u}^j - \overline{0}_S\big) \\ \geq \;& \mu_1 f_1\big(\sum_{j=0}^{m_1} \epsilon^{j+1} \cdot \overline{u}^j - \overline{0}_S\big) + \mu_2 f_2\big(\sum_{j=0}^{m_2} \epsilon^{j+1} \cdot \overline{u}^j - \overline{0}_S\big) \\ > \;& 1. \end{aligned} \tag{27}$$

Thus $\mu_1 f_1 + \mu_2 f_2 \in C_\epsilon^* \backslash D_\epsilon$.

The set $C^* \backslash D_\epsilon$ is also convex. Set $f_1, f_2 \in C^* \backslash D_\epsilon$ and $\mu_1, \mu_2 = 1 - \mu_1 \in (0,1)$. If one of f_1 and f_2 is not in C_ϵ^* (*i.e.*, is not bounded over the set $\mathcal{N}_\epsilon - \overline{0}_S$), then neither is the functional $\mu_1 f_1 + \mu_2 f_2$. Thus, $\mu_1 f_1 + \mu_2 f_2 \in C^* \backslash C_\epsilon^* \subset C^* \backslash D_\epsilon$. On the contrary, if both f_1 and f_2 are in C_ϵ^*, they are in $C_\epsilon^* \backslash D_\epsilon$. Now (27) implies $\mu_1 f_1 + \mu_2 f_2 \in C_\epsilon^* \backslash D_\epsilon \subset C^* \backslash D_\epsilon$. □

Observation 6.4. $\exists f \in \mathrm{Er}(C^*)$ s.t. $f(\overline{x} - \overline{y}) < 0$.

Proof. This part was also omitted in Fritz (2021b). Hence we give a proof here. Since there is an $\ell \in C^*$ so that $\ell(\overline{x} - \overline{y}) < 0$, the intersection $\mathcal{H} \cap C^*$ is not empty, where

$$\mathcal{H} = \{\phi \in V^* \mid \phi(\overline{x} - \overline{y}) < 0\}$$

is an open half-space in V^*. It is sufficient to show that there is $f \in \mathrm{Er}(C^*)$ so that $f \in \mathcal{H}$. Assume that none of the extreme rays of C^* is in $\mathcal{H}$. By the equation (22), for any $\phi \in C^*$ and any neighborhood $O \ni \phi$, there is $\phi' \in O \cap \mathrm{conv}(\mathrm{Er}(C^*))$. Thus $\phi' = \sum_{i=1}^{s} \mu_i \phi_i$ for some $s \in \mathbb{N}$, some $\phi_i \in \mathrm{Er}(C^*)$ and some real numbers $\mu_i \in [0,1]$ with $\sum_{i=1}^{s} \mu_i = 1$. Since $\phi_i \notin \mathcal{H}$ by the assumption, $\phi_i(\overline{x} - \overline{y}) \geq 0$ for any $1 \leq i \leq s$. Hence $\phi'(\overline{x} - \overline{y}) \geq 0$, i.e., $\phi' \notin \mathcal{H}$. This means that $O \cap (V^* \backslash \mathcal{H}) \neq \emptyset$ for any neighborhood O of ϕ. Since $V^* \backslash \mathcal{H}$ is clearly closed, $\phi \in V^* \backslash \mathcal{H}$. Noting that ϕ is an arbitrary element in C^*, we conclude that $C^* \cap \mathcal{H} = \emptyset$, which is a contradiction. □

Observation 6.5. *Let $f \in \mathrm{Er}(C^*)$ be any extreme ray of C^*, then $f(\overline{1}_S - \overline{0}_S) > 0$ and $\hat{f}$ defined in (24) is a monotone semialgebra homomorphism.*

Proof. Without loss of generality, we may assume that for any nonzero $a \in F$, $\overline{1}_S + a \neq \overline{0}_S$. Otherwise, there would be a nonzero $a \in F$ so that $\overline{0}_S = \overline{1}_S + a \succcurlyeq \overline{1}_S$. It is also true that $\overline{1}_S \succcurlyeq \overline{0}_S$, hence one concludes that the preorder $\preccurlyeq$ is trivial. Now that conditions (a)–(d) are all trivially true, and so is the theorem's conclusion.

Set $f_F(w) = f(w - \overline{0}_S)$ for any $w \in F$, then we have

$$f_F\big((\overline{1}_S + a)^{-1} * w\big) + f_F\big(a * (\overline{1}_S + a)^{-1} * w\big) = f_F(w)$$

by the linearity of f. Let

$$\begin{cases} g(w) &= f_F\big((\overline{1}_S + a)^{-1} * w\big) \\ h(w) &= f_F\big(a * (\overline{1}_S + a)^{-1} * w\big) \end{cases}$$

then g, h are both linear and monotone on F since f_F is. Set $g^V(b-d) = g(b) - g(d)$ and $h^V(b-d) = h(b) - h(d)$ for any $b, d \in F$ (or equivalently, for any $b - d \in V$), then both g^V and h^V are well-defined on V. Moreover, a routine check indicates that they are monotone and linear maps from V to $\mathbb{R}$ and

$$g^V + h^V = f. \tag{28}$$

For any nonzero $t \in F$, the operator

$$t_* : V \to V,\ b - d \mapsto t * b - t * d \tag{29}$$

is well-defined and continuous (Lemma 6.6). Noting that $g^V = f \circ \left((\overline{1}_S + a)^{-1}\right)_*$ and $h^V = f \circ \left(a * (\overline{1}_S + a)^{-1}\right)_*$, one observes that both g^V and h^V are continuous since f is. To conclude, we have $g^V, h^V \in C^*$.

By the equation (28) and the fact that f is an extreme ray, we conclude that there are nonnegative real numbers $s_1(a)$ and $s_2(a)$ such that $g^V = s_1(a) f$ and $h^V = s_2(a) f$. As a ray of C^*, $f \neq 0$. There is some $b_0 - d_0 \in V$ such that $f(b_0 - d_0) \neq 0$. Thus

$$g^V\left((\overline{1}_S + a) * b_0 - (\overline{1}_S + a) * d_0\right) = f(b_0 - d_0) \neq 0$$

and

$$h^V\left((\overline{1}_S + a) * a^{-1} * b_0 - (\overline{1}_S + a) * a^{-1} * d_0\right) = f(b_0 - d_0) \neq 0,$$

which means $g^V \neq 0$ and $h^V \neq 0$. Hence $s_1(a), s_2(a) \in \mathbb{R}_{>0}$. Therefore, we have $h^V = \gamma(a) g^V$ for the positive number $\gamma(a) = s_2(a)/s_1(a)$ and $h^V(w - \overline{0}_S) = \gamma(a) g^V(w - \overline{0}_S)$ for any $w \in F$, *i.e.*,

$$f_F\left(a * (\overline{1}_S + a)^{-1} * w\right) = \gamma(a) f_F\left((\overline{1}_S + a)^{-1} * w\right). \tag{30}$$

Replacing w in (30) by $(\overline{1}_S + a) * w$ one obtains

$$f_F(a * w) = \gamma(a) f_F(w) \tag{31}$$

for any $w \in F$. Taking $w = \overline{1}_S$ in (31), we have

$$f_F(a) = \gamma(a) f_F(\overline{1}_S). \tag{32}$$

Now $0 \neq f(b_0 - d_0) = f_F(b_0) - f_F(d_0)$ indicates that $f_F(b_0) \neq 0$ or $f_F(d_0) \neq 0$. We may assume $f_F(b_0) \neq 0$, then clearly $b_0 \neq \overline{0}_S$. Taking $a = b_0$ in (32), we have $f_F(b_0) = \gamma(b_0) f_F(\overline{1}_S)$. Since $f_F(b_0) \neq 0$, we obtain $f_F(\overline{1}_S) \neq 0$. In fact, $f_F(\overline{1}_S) > 0$ since $f \in C^*$. From (32) we see that $\gamma(a) = f_F(a)/f_F(\overline{1}_S)$. Substituting that back to (31), we have

$$f_F(a * w) = \frac{f_F(a)}{f_F(\overline{1}_S)} f_F(w) \tag{33}$$

for any $w \in F$ and any nonzero $a \in F$. Indeed, when $a = \overline{0}_S$, the identity (33) also holds since both of its two sides equal 0. Define $\hat{f}(c) = \frac{1}{f_F(\overline{1}_S)} f_F(\overline{c})$ for any $c \in S$, then the identity (33) indicates that $\hat{f} : S \to \mathbb{R}_+$ preserves the multiplications. The fact that it also preserves the additions and scalar multiplications and is monotone is straightforward. Thus, $\hat{f}$ is a monotone semialgebra homomorphism. □

The following lemma (also claimed in Fritz (2021b)) is needed in the proof above.

Lemma 6.6. *For any nonzero $t \in F$, the map given in (29) is well-defined and continuous* w.r.t. *the topology in V.*

Proof. Suppose $b - d = b' - d'$ in V for some b, d, b', d' in F. Then there is $e \in F$ such that $b + d' + e = b' + d + e$ in F. Hence

$$t * b + t * d' + t * e = t * b' + t * d + t * e,$$

which means $t * b - t * d = t * b' - t * d'$ in V. Therefore, the map t_* is well-defined.

The map t_* is linear. To prove the continuity, it is sufficient to show that it is continuous at the point $0_V = \overline{0}_S - \overline{0}_S$: Let $k \in \mathbb{N}$ such that $\overline{u}^k \succcurlyeq t$. For any $\epsilon > 0$, we set $\delta = (\min\{\epsilon, 1\})^{k+1}$. Then, for any $b \in \mathcal{N}_\delta$, $b \preccurlyeq \sum_{j=0}^{m} \delta^{j+1} \cdot \overline{u}^j$ for some $m \in \mathbb{N}$. On the other hand,

$$\begin{aligned} t * b &\preccurlyeq \overline{u}^k * (\sum_{j=0}^{m} \delta^{j+1} \cdot \overline{u}^j) \\ &= \sum_{j=0}^{m} \delta^{j+1} \cdot \overline{u}^{j+k} \\ &\preccurlyeq \sum_{j=0}^{m} (\min\{\epsilon, 1\})^{k+j+1} \cdot \overline{u}^{j+k} \\ &= \sum_{j=k}^{m+k} (\min\{\epsilon, 1\})^{j+1} \cdot \overline{u}^{j} \\ &\preccurlyeq \sum_{j=k}^{m+k} (\min\{\epsilon, 1\})^{j+1} \cdot \overline{u}^{j} + \sum_{j=0}^{k-1} (\min\{\epsilon, 1\})^{j+1} \cdot \overline{u}^{j} \\ &= \sum_{j=0}^{m+k} (\min\{\epsilon, 1\})^{j+1} \cdot \overline{u}^{j}, \end{aligned}$$

which means $t * b \in \mathcal{N}_{\min\{\epsilon,1\}} \subset \mathcal{N}_\epsilon$. Therefore, for any $b, d \in \mathcal{N}_\delta$, we have $t * b - t * d \in \mathcal{N}_\epsilon - \mathcal{N}_\epsilon$. That is, t_* is continuous at 0_V. □

6.3. A variation of the main theorem and some examples

Theorem 1.6 is a noncommutative counterpart of Theorem 2.12 in Fritz (2021b), although in the noncommutative case, an equivalent characterization for the preorder relation $\preccurlyeq$ via the original relation $\leq$ similar to the one in (13) is presently not available. However, via Proposition 3.2, we obtain another version of Theorem 1.6 involving only the preorder "$\leq$" and the semialgebra of fractions, rather than the derived preorder "$\preccurlyeq$."

Theorem 6.7. *Let $(S, \leq)$ be a preordered semialgebra satisfying Assumptions 1.4 and 1.5, with a power universal element $u \in S$. Suppose that F is the semialgebra of the fractions of S defined in Subsection 2.3 and $\overline{\ } : S \to F, x \mapsto \overline{x}$ is the canonical map in Definition 2.10. Then, for every nonzero $x, y \in S$, the following are equivalent:*

(a) $f(x) \geq f(y)$ for every monotone semialgebra homomorphism $f : S \to \mathbb{R}_+$.

(b) For every real number $\epsilon > 0$, there are $m \in \mathbb{N}$, $k \in \mathbb{N}_{\geq 1}$, $g_1, \ldots, g_k$, $h_1, \ldots, h_k, w \in F$ and $A_1, \ldots, A_k, B_1, \ldots, B_k \in S$ such that $A_i \geq B_i$ for all i, and that

$$\begin{cases} \overline{x} + \sum_{j=0}^{m} \epsilon^{j+1} \cdot \overline{u}^j + w = \sum_{i=1}^{k} g_i \overline{A}_i h_i, \\ \overline{y} + w = \sum_{i=1}^{k} g_i \overline{B}_i h_i. \end{cases} \tag{34}$$

(c) For every $r \in \mathbb{R}_+$ and every real number $\epsilon > 0$, there are $p \in \mathbb{Q}_+[X]$, $k \in \mathbb{N}_{\geq 1}$, $g_1, \ldots, g_k, h_1, \ldots, h_k, w \in F$ and $A_1, \ldots, A_k, B_1, \ldots, B_k \in S$ such that $p(r) \leq \epsilon$, that $A_i \geq B_i$ for all i, and that

$$\begin{cases} \overline{x} + p(\overline{u}) + w = \sum_{i=1}^{k} g_i \overline{A}_i h_i, \\ \overline{y} + w = \sum_{i=1}^{k} g_i \overline{B}_i h_i. \end{cases} \tag{35}$$

(d) For every $r \in \mathbb{R}_+$ and every real number $\epsilon > 0$, there are $p \in \mathbb{Q}_+[X]$, $k \in \mathbb{N}_{\geq 1}$, $g_1, \ldots, g_k, h_1, \ldots, h_k, w \in F$ and $A_1, \ldots, A_k, B_1, \ldots, B_k \in S$ such that $p(r) \leq 1 + \epsilon$, that $A_i \geq B_i$ for all i, and that

$$\begin{cases} p(\overline{u}) * \overline{x} + w &= \sum_{i=1}^{k} g_i \overline{A}_i h_i, \\ \overline{y} + w &= \sum_{i=1}^{k} g_i \overline{B}_i h_i. \end{cases}$$

Proof. From Corollary 3.3, one observes that conditions (b) – (d) in Theorem 1.6 imply, respectively, conditions (b) – (d) in this theorem. Now it suffices to prove that (b) $\Rightarrow$ (a), (c) $\Rightarrow$ (a), and (d) $\Rightarrow$ (a) in this theorem.

(b) $\Rightarrow$ (a): Suppose $f : S \to \mathbb{R}_+$ is any monotone homomorphism. Take

$$\epsilon = \begin{cases} \delta / f(u), & \text{if} \quad f(u) \neq 0, \\ \delta, & \text{if} \quad f(u) = 0, \end{cases}$$

where $0 < \delta < 1$ is any real number, then equations (34) hold. Evaluating f^F at both sides of each equation in (34), we will get two equations involving real numbers. Subtracting the second one from the first one, we obtain

$$f(x) + \sum_{j=0}^{m} \epsilon^{j+1} f(u)^j - f(y) \geq 0.$$

If $f(u) = 0$ then $f(x) \geq f(y)$. Otherwise, we have

$$\sum_{j=0}^{m} \epsilon^{j+1} f(u)^j = \sum_{j=0}^{m} \delta^{j+1} / f(u) < \frac{\delta}{f(u)(1-\delta)}.$$

Therefore, $f(x) + \delta/(f(u)(1-\delta)) > f(y)$. Letting δ tend to zero, we have $f(x) \geq f(y)$.

(c) $\Rightarrow$ (a): Suppose $f : S \to \mathbb{R}_+$ is any monotone homomorphism. Take $r = f(u)$ and ϵ to be any positive number. Applying f^F to equations (35) as before, we obtain $f(x) + p(r) - f(y) \geq 0$. Since $p(r) \leq \epsilon$ and ϵ is arbitrary, $f(x) \geq f(y)$.

The proof of the implication (d) $\Rightarrow$ (a) is similar. □

In Fritz (2021b) Example 4.1, the commutative version of the above theorem (Fritz (2021b) Theorem 1.12) is applied to the commutative semiring $S = \mathbb{R}_+[x_1, \ldots, x_n]'$, which is the semiring of commutative real polynomials with nonnegative coefficients and strictly positive constant coefficient together with zero. It is indicated that the most difficult implication in Fritz (2021b) Theorem 1.12, which is a commutative version for implication (a) $\Rightarrow$ (b) in the above theorem, follows from Pólya's Positivstellensatz (Pólya, 1928), which moreover gives a concrete form for the consequent of that implication in Fritz (2021b) Theorem 1.12.

Positivstellensätze are at the core of real algebraic geometry since they have both theoretical value and significant importance for scientific computing (such as polynomial optimization), while Pólya's Positivstellensatz is well-known as one of the classic Positivstellensätze. It claims that if a homogeneous polynomial f in the commutative ring $\mathbb{R}[x_1, \ldots, x_n]$ is strictly positive on the standard codimension-1 simplex

$$\Delta_n = \{(x_1, \ldots, x_n) \in \mathbb{R}_+^n \mid x_1 + \cdots + x_n = 1\},$$

then $(x_1 + \cdots + x_n)^m f$ strictly has positive coefficients for some nonnegative integer $m \in \mathbb{N}$.

Now that the most difficult implication in Fritz (2021b) Theorem 1.12 can be derived from Pólya's Positivstellensatz, it can be regarded as a weaker version of Pólya's Positivstellensatz. Note that Pólya's Positivstellensatz holds for the commutative polynomial ring, one may conjecture that there are noncommutative versions of Pólya's Positivstellensatz. As was done in Fritz (2021b) Example 4.1, we can apply Theorem 6.7 to the noncommutative semialgebra $\mathbb{R}_+\langle x_1, \ldots, x_n\rangle'$, obtaining the implication from (a) to (b) and regarding it as a weaker version of certain unknown noncommutative Pólya's Positivstellensatz. This can be of significance in noncommutative algebraic geometry, where noncommutative Positivstellensätze play important roles.

The following example illustrates implication from (a) to (b) in Theorem 6.7 in the bivariate case with the semialgebra being $\mathbb{R}_+\langle a,b\rangle'$.

Example 6.8. Set $S=\mathbb{R}_+\langle a,b\rangle'$ to be the preordered semialgebra defined as in Example 1.2 with variables a,b and the coefficient-wise preorder. Then $u=2\cdot(1+a+b)$ is a power universal element. Moreover, every monotone semialgebra homomorphism $f:S\to\mathbb{R}_+$ is an evaluation at some point in $\mathbb{R}_+^2$. Take $x=a^2+b^2$ and $y=a*b+b*a$, then condition (a) in Theorem 6.7 holds since $x(P)\geq y(P)$ at any point $P=(a,b)^T\in\mathbb{R}_+^2$. Therefore, condition (b) in Theorem 6.7 holds.

In the following, we illustrate condition (b) in Theorem 6.7 as the value of ϵ varies. A useful fact is that when $\epsilon<1/2$ and m tends to infinity, the limit of the coefficient of the monomial $a*b$ (and $b*a$) in the polynomial $\sum_{j=0}^m\epsilon^{j+1}\cdot u^j$ is $C_\epsilon=4\epsilon^3/(1-2\epsilon)^3$.

When $\epsilon=0.3$, $C_\epsilon>1$, we have $x+\sum_{j=0}^7\epsilon^{j+1}\cdot u^j\geq y$ coefficient-wise. Therefore, we can take $m=7$, $k=1$, $g_1=h_1=\overline{1}$, $w=\overline{0}$, $A_1=x+\sum_{j=0}^7\epsilon^{j+1}\cdot u^j$ and $B_1=y$ in condition (b). This is rather trivial.

If $\epsilon=0.3$, and we take $m=6$. Since the polynomial $x+\sum_{j=0}^6\epsilon^{j+1}\cdot u^j-y$ has negative coefficients, $x+\sum_{j=0}^6\epsilon^{j+1}\cdot u^j\geq y$ does not hold. However, the polynomial

$$(1+a+b)*(x+\sum_{j=0}^6\epsilon^{j+1}\cdot u^j-y)$$

has no negative coefficients, which leads to

$$(1+a+b)*(x+\sum_{j=0}^6\epsilon^{j+1}\cdot u^j)\geq(1+a+b)*y \tag{36}$$

coefficient-wise. Therefore, in condition (b), we can take $m=6$, $k=1$, $g_1=\overline{1+a+b}^{-1}$, $h_1=\overline{1}$, $w=\overline{0}$, $A_1=(1+a+b)*(x+\sum_{j=0}^6\epsilon^{j+1}\cdot u^j)$ and $B_1=(1+a+b)*y$. This is nontrivial compared with the case $m=7$.

When $\epsilon=0.278$, $C_\epsilon<1$. The polynomial $x+\sum_{j=0}^m\epsilon^{j+1}\cdot u^j-y$ has negative coefficients for any natural number m. This is different from the case $\epsilon=0.3$. However, if we take $m=6$, then inequality (36) still holds. If $m=5$, then we have

$$(1+a+b)^2*(x+\sum_{j=0}^5\epsilon^{j+1}\cdot u^j)\geq(1+a+b)^2*y \tag{37}$$

and

$$(1+a+b)*(x+\sum_{j=0}^5\epsilon^{j+1}\cdot u^j)*(1+a+b)\geq(1+a+b)*y*(1+a+b) \tag{38}$$

coefficient-wise. Therefore, in condition (b) of Theorem 6.7, we may take $m=5$, $k=1$, $g_1=\overline{1+a+b}^{-2}$, $h_1=\overline{1}$, $w=\overline{0}$, $A_1=(1+a+b)^2*(x+\sum_{j=0}^5\epsilon^{j+1}\cdot u^j)$ and $B_1=(1+a+b)^2*y$. Among them, we may also take $g_1=h_1=\overline{1+a+b}^{-1}$,

$$A_1=(1+a+b)*(x+\sum_{j=0}^5\epsilon^{j+1}\cdot u^j)*(1+a+b)$$

and $B_1=(1+a+b)*y*(1+a+b)$ instead.

When $\epsilon=0.25$, we also have $C_\epsilon<1$, and the polynomial $x+\sum_{j=0}^m\epsilon^{j+1}\cdot u^j-y$ always has negative coefficients. Now that ϵ is smaller, we need larger m. For instance, if we take $m=10$, then

$$(1+a*b+b*a)*(x+\sum_{j=0}^{10}\epsilon^{j+1}\cdot u^j)*(1+a+b)\geq(1+a*b+b*a)*y*(1+a+b)$$

coefficient-wise. Therefore, in condition (b) of Theorem 6.7, we may take $m=10$, $k=1$, $g_1=\overline{1+a*b+b*a}^{-1}$, $h_1=\overline{1+a+b}^{-1}$, $w=\overline{0}$,

$$A_1=(1+a*b+b*a)*(x+\sum_{j=0}^{10}\epsilon^{j+1}\cdot u^j)*(1+a+b)$$

and $B_1=(1+a*b+b*a)*y*(1+a+b)$. If we take $m=6$, then we have $(1+a+b)^6*(x+\sum_{j=0}^{6}\epsilon^{j+1}\cdot u^j)\geq(1+a+b)^6*y$. And those quantities in condition (b) of Theorem 6.7 can be assigned as before.

For all the cases mentioned above, we have $w=\overline{0}$ in equations (34), which implies the inequality $\overline{x}+\sum_{j=0}^{m}\epsilon^{j+1}\cdot\overline{u}^j\succcurlyeq\overline{y}$ in $F(S)$. Therefore, they also illustrate condition (b) in Theorem 1.6.

Roughly speaking, the smaller the number ϵ is, the larger the number m, or the higher the degree of A_i and B_i, shall be taken in condition (b) of Theorem 6.7.

Here is another example of noncommutative Laurent polynomials:

Example 6.9. Set $S=\mathbb{R}_+\langle a,b,a^{-1},b^{-1}\rangle'$ to be the preordered semialgebra consisting of the zero polynomial together with other nonzero noncommutative Laurent polynomials in variables a,b with nonnegative coefficients and positive constant terms. S is equipped with the coefficient-wise preorder $\leq$ (see Example 1.2). Set $D=a^{-1}+b^{-1}+1+a+b$, then $u=2\cdot D$ is a power universal element. Moreover, the monotone semialgebra homomorphisms mapping from S to $\mathbb{R}_+$ are exactly the evaluations of noncommutative Laurent polynomials at the points in $\mathbb{R}^2_{>0}$.

Note that the commutative polynomial $1+s^4t^2+s^2t^6-1/2s^2t^2\in\mathbb{R}[s,t]$ is a nonnegative circuit polynomial (Dressler et al., 2017). This implies that, for any $s,t\in\mathbb{R}_{>0}$, we have $1+s^2t+st^3-1/2st\geq 0$, and thus $1/(st)+s+t^2-1/2\geq 0$. Therefore, we may take $x=a^{-1}*b^{-1}+b^{-1}*a*b+2\cdot b^2+1$ and $y=b^2+3/2$ in S such that condition (a) in Theorem 6.7 holds. As in Example 6.8, we illustrate condition (b) of Theorem 6.7 as follows:

When $\epsilon=1/4$, we may take $m=1$, since $D*(x+\sum_{j=0}^{1}\epsilon^{j+1}\cdot u^j)\geq D*y$ (also $(x+\sum_{j=0}^{1}\epsilon^{j+1}\cdot u^j)*D\geq y*D$) in S. While $m\geq 2$, we always have $x+\sum_{j=0}^{m}\epsilon^{j+1}\cdot u^j\geq y$ in S, which is not interesting.

When $\epsilon=1/8$, if we take $m=2$, then we have

$$D^2*(x+\sum_{j=0}^{2}\epsilon^{j+1}\cdot u^j)*D^2\geq D^2*y*D^2,$$

$$D^3*(x+\sum_{j=0}^{2}\epsilon^{j+1}\cdot u^j)*D\geq D^3*y*D$$

and

$$D*(x+\sum_{j=0}^{2}\epsilon^{j+1}\cdot u^j)*D^3\geq D*y*D^3.$$

However, one observes that

$$D^4*(x+\sum_{j=0}^{2}\epsilon^{j+1}\cdot u^j)\not\geq D^4*y$$

and

$$(x+\sum_{j=0}^{2}\epsilon^{j+1}\cdot u^j)*D^4 \not\geq y*D^4.$$

This indicates that in condition (b) of Theorem 6.7, sometimes the "left" and the "right" certificates g_i and h_i need to be both nontrivial (i.e., not equal to one). This is the essential difference between the noncommutative case and the commutative case.

The following theorem illustrates the first application of our result to noncommutative ordered $\mathbb{R}$-algebras. It is similar to the Positivstellensatz of Krivine–Kadison–Dubois (Krivine, 1964, Proposition 3), but replaces its Archimedeanicity assumption by the power universal condition. As is known, the Archimedeanicity assumption is a common condition for many Positivstellensätze in real algebraic geometry. Therefore, Positivstellensätze without that assumption are of particular interest.

Theorem 6.10. *Let R be an $\mathbb{R}$-algebra and $P\subset R$ a sub-semialgebra satisfying Assumption 1.5. Suppose there is an element $v\in P$ such that for every $a\in R$ there is a polynomial $\Phi\in\mathbb{R}_+[X]$ satisfying $\Phi(v)-a\in P$. Define the sub-semialgebra*

$$\begin{aligned} S &= \{0\}\cup\{a\in P\mid \exists n\in\mathbb{N}_{>0}, na-1\in P\}\\ &= \{0\}\cup\bigcup_{n\in\mathbb{N}_{>0}}(P+1)/n, \end{aligned}$$

then $R=S-S$. For any $x,y\in S$, the following statements are equivalent:

(a) $f(x-y)\geq 0$ for every algebra homomorphism $f: R\to\mathbb{R}$ satisfying $f(P)\subset\mathbb{R}_+$.

(b) For every $r\in\mathbb{R}_+$, and every real number $\epsilon>0$, there are $q\in\mathbb{Q}_+[X]$, $k\in\mathbb{N}_{\geq 1}$, $g_1,\ldots,g_k,h_1,\ldots,h_k$, $w,w'\in W(S)\subset W(P)$ and $p_1,\ldots,p_k\in P$ such that $q(r)\leq\epsilon$, and

$$\begin{cases} \overline{x}+q(\overline{v})+\overline{w} &= \overline{w'}+\sum_{i=1}^{k}\overline{g}_i\overline{p}_i\overline{h}_i,\\ \overline{y}+\overline{w} &= \overline{w'}, \end{cases} \tag{39}$$

where $x\to\overline{x}$ is the generalized canonical map from $W(P)$ to $F(P)$.

Proof. As in the proof of (Fritz, 2021b, Theorem 4.4): The existence of v clearly implies that $R=P-P$. The set S is indeed a semialgebra because it is closed under addition, multiplication and nonnegative scalar multiplication: Taking $\frac{p+1}{m},\frac{p'+1}{n}\in S$ and $r\in\mathbb{R}_{>0}$ with $p,p'\in P$ and $m,n\in\mathbb{N}_{>0}$, we have

$$\begin{cases} \frac{p+1}{m}+\frac{p'+1}{n} &= \frac{(np+mp'+m+n-1)+1}{mn}\\ \frac{p+1}{m}\frac{p'+1}{n} &= \frac{(pp'+p+p')+1}{mn}\\ r\frac{p+1}{m} &= \frac{(krp+kr-1)+1}{mk} \end{cases},$$

where $k>0$ is an integer such that $kr>1$. Clearly, $np+mp'+m+n-1$, $pp'+p+p'$ and $krp+kr-1$ belong to P. Thus the term on the right-hand side of every equation above belongs to S.

The sub-semialgebra S can also be equipped with the preorder $\leq$ such that $s\leq t$ in S if and only if $t-s\in P$. Moreover, $\lambda(1+v)$ is a power universal element of S for any rational number $\lambda>1$: For any nonzero $t\in S$, there is $\Phi\in\mathbb{R}_+[X]$ such that $\Phi(v)-t\in P$ (i.e., $\Phi(v)\geq t$). Taking $k\in\mathbb{N}$ so that $k\geq\deg(\Phi)$ and $\lambda^k\geq\Phi(1)$, we have

$$(\lambda(1+v))^k\geq\Phi(1)(1+v)^{\deg(\Phi)}\geq\Phi(1+v)\geq\Phi(v)\geq t.$$

On the other hand, since $t\in S$ is nonzero, $t=\frac{p+1}{n}$ for some $p\in P$ and $n\in\mathbb{N}_{>0}$. Taking $k\in\mathbb{N}_{\geq 1}$ so that $\lambda^k\geq n$, we have

$$(\lambda(1+v))^k t \geq n(1+v)\frac{p+1}{n} \geq 1,$$

which proves that $\lambda(1+v)$ is power universal in S.

For any $a \in R$, there are $p, p' \in P$ such that $a = p - p' = (p+1) - (p'+1)$. By the definition of S, we have $p+1, p'+1 \in S$. Thus, we have $R = S - S$. These imply that restriction from R to S implements a bijection between the monotone homomorphisms $f : S \to \mathbb{R}_+$ and the $\mathbb{R}$-algebra homomorphisms $f : R \to \mathbb{R}$ with $f(P) \subset \mathbb{R}_+$.

Applying Theorem 6.7 to the semialgebra S, we see condition (a) implies that for every $r \in \mathbb{R}_+$ and every real number $\epsilon > 0$, there are $H \in \mathbb{Q}_+[X]$, $k \in \mathbb{N}_{\geq 1}$, $g_1, \ldots, g_k, h_1, \ldots, h_k, w \in W(S)$, $A_1, \ldots, A_k$, $B_1, \ldots, B_k \in S$ such that $H(\lambda(1+r)) \leq \epsilon$, $A_i \geq B_i$ for all i, and

$$\begin{cases} \overline{x}^S + H(\lambda(\overline{1+v}^S)) + \overline{w}^S &= \sum_{i=1}^k \overline{g_i}^S \overline{A_i}^S \overline{h_i}^S, \\ \overline{y}^S + \overline{w}^S &= \sum_{i=1}^k \overline{g_i}^S \overline{B_i}^S \overline{h_i}^S, \end{cases} \tag{40}$$

where the map $x \to \overline{x}^S$ is the generalized canonical map from $W(S)$ to $F(S)$. By Proposition 2.16, there is a natural semialgebra homomorphism from $F(S)$ to $F(P)$, which means the superscript "S" in equations (40) can be dropped. According to the equations obtained by dropping "S", we can take $q(X) = H(\lambda(1+X))$, $p_i = A_i - B_i \in P$ and $w' \in W(S)$ to be any expression such that $\overline{w'}^S = \sum_{i=1}^k \overline{g_i}^S \overline{B_i}^S \overline{h_i}^S$, so that equations (39) hold.

To prove (b) $\Rightarrow$ (a), we fix any algebra homomorphism $f : R \to \mathbb{R}$ with $f(P) \subset \mathbb{R}_+$. Then $g = f|_P : P \to \mathbb{R}_+$ is a semialgebra homomorphism. If we equip P with the preorder "$\leq$" such that $p \leq p'$ if and only if $p' - p \in P$ for every $p, p' \in P$, then P becomes a preordered semialgebra and g becomes a monotone semialgebra homomorphism. By Proposition 4.2, g can be extended to a semialgebra homomorphism $g^F : F(P) \to \mathbb{R}_+$ such that $g^F(\overline{p}) = g(p) = f(p)$ for any $p \in P$. Take $r = f(v) \in \mathbb{R}_+$ and fix an arbitrary $\epsilon > 0$, then equations (39) hold. Applying g^F to both sides of each equation in (39), we have

$$\begin{cases} f(x) + q(f(v)) + g^F(\overline{w}) &= g^F(\overline{w'}) + \sum_{i=1}^k g^F(\overline{g_i}) f(p_i) g^F(\overline{h_i}), \\ f(y) + g^F(\overline{w}) &= g^F(\overline{w'}). \end{cases} \tag{41}$$

Subtracting the second equation from the first one, we obtain $f(x-y) + q(f(v)) = \sum_{i=1}^k g^F(\overline{g_i}) f(p_i) \times g^F(\overline{h_i}) \geq 0$, with $q(f(v)) = q(r) \leq \epsilon$. Since ϵ is arbitrary, we have $f(x-y) \geq 0$. □

7. Conclusion

We provide a noncommutative version of the Vergleichsstellensatz proved by Fritz in Fritz (2021b). This is the first Vergleichsstellensatz for noncommutative semialgebras in the literature. It characterizes the relaxed preorder induced by monotone homomorphisms to $\mathbb{R}_+$ by "asymptotic" inequalities in the semialgebra of the fractions. Finding out how it can be applied to other areas, such as probability theory and quantum information theory, or how it is related to classical noncommutative real algebraic geometry, would be interesting topics in the future.

CRediT authorship contribution statement

Tao Zheng: Formal analysis, Methodology, Validation, Writing – original draft, Writing – review & editing. **Lihong Zhi:** Conceptualization, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Supervision, Validation, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgement

We thank the editor and the reviewers for their careful reading and constructive suggestions. We also thank Tobias Fritz for his/her valuable discussions and input. These contributions have greatly helped improve the paper.

Lihong Zhi is supported by the National Key R&D Program of China 2023YFA1009401 and the National Basic Research Program of China 2018YFA0306702, while Tao Zheng is supported by the Postdoctoral Fellowship Program of CPSF GZC20232045 and the National Natural Science Foundation of China 12501701, 12471387.

Data availability

No data was used for the research described in the article.

References

Asimow, L., 1969. Extremal structure of well-capped convex sets. Trans. Am. Math. Soc. 138, 363–375.
Bunth, G., Vrana, P., 2021. Asymptotic relative submajorization of multiple-state boxes. Lett. Math. Phys. 111 (4), 1–23.
Choquet, G., 1962. Les cônes convexes faiblement complets dans l'analyse. In: Proc. Intern. Congr. Mathematicians, Stockholm, pp. 317–330.
Cohn, P.M., 1995. Skew Fields: Theory of General Division Rings. Cambridge University Press.
Dressler, M., Iliman, S., De Wolff, T., 2017. A Positivstellensatz for sums of nonnegative circuit polynomials. SIAM J. Appl. Algebra Geom. 1 (1), 536–555.
Fritz, T., 2017. Resource convertibility and ordered commutative monoids. Math. Struct. Comput. Sci. 27 (6), 850–938.
Fritz, T., 2021a. Abstract Vergleichsstellensätze for preordered semifields and semirings II. arXiv preprint. arXiv:2112.05949v2.
Fritz, T., 2021b. A generalization of Strassen's Positivstellensatz. Commun. Algebra 49 (2), 482–499.
Fritz, T., 2021c. Abstract Vergleichsstellensätze for preordered semifields and semirings I. arXiv preprint. arXiv:2003.13835v3.
Fritz, T., 2021d. The asymptotic comparison of random walks on topological abelian groups. arXiv preprint. arXiv:2004.13655v2.
Golan, J.S., 1999. Semirings and Their Applications. Springer Science & Business Media.
Karoubi, M., 1978. K-Theory: An Introduction. Springer, Berlin, Heidelberg.
Krivine, J.-L., 1964. Quelques propriétés des préordres dans les anneaux commutatifs unitaires. C. R. Acad. Sci. Paris 258 (3417–3418), 4.
Li, Y., Zuiddam, J., 2020. Quantum asymptotic spectra of graphs and non-commutative graphs, and quantum Shannon capacities. IEEE Trans. Inf. Theory 67 (1), 416–432.
Perry, C., Vrana, P., Werner, A.H., 2021. The semiring of dichotomies and asymptotic relative submajorization. IEEE Trans. Inf. Theory 68 (1), 311–321.
Pólya, G., 1928. Uber positive Darstellung von Polynomen. Vierteljahrsschr. Nat.forsch. Ges. Zür. 73, 141–145. see Collected Papers, vol. 2 (1974).
Robere, R., Zuiddam, J., 2022. Amortized circuit complexity, formal complexity measures, and catalytic algorithms. In: 2021 IEEE 62nd Annual Symposium on Foundations of Computer Science (FOCS). IEEE, pp. 759–769.
Rudin, W., 1991. Functional Analysis, 2nd ed. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York.
Strassen, V., 1988. The asymptotic spectrum of tensors. J. reine angew. Math. 384, 102–152.
Vrana, P., 2021. A generalization of Strassen's theorem on preordered semirings. Order, 1–20.
Zheng, T., Zhi, L., 2023. A Vergleichsstellensatz of Strassen's type for a noncommutative preordered semialgebra through the semialgebra of its fractions. arXiv preprint. arXiv:2204.02577v3.
Zuiddam, J., 2019. The asymptotic spectrum of graphs and the Shannon capacity. Combinatorica 39 (5), 1173–1184.