Computing Multiple Zeros of Polynomial Systems: Case of Breadth One (*Invited Talk*)

Lihong Zhi[1,2(✉)]

[1] Key Laboratory of Mathematics Mechanization, Academy of Mathematics and System Sciences, Beijing, China
[2] School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100190, China
lzhi@mmrc.iss.ac.cn

Abstract. Given a polynomial system f with a multiple zero x whose Jacobian matrix at x has corank one, we show how to compute the multiplicity structure of x and the lower bound on the minimal distance between the multiple zero x and other zeros of f. If x is only given with limited accuracy, we give a numerical criterion to guarantee that f has μ zeros (counting multiplicities) in a small ball around x. Moreover, we also show how to compute verified and narrow error bounds such that a slightly perturbed system is guaranteed to possess an isolated breadth-one singular solution within computed error bounds. Finally, we present modified Newton iterations and show that they converge quadratically if x is close to an isolated exact singular solution of f. This is joint work with Zhiwei Hao, Wenrong Jiang, Nan Li.

1 Introduction

Let I_f be an ideal generated by polynomials $f = \{f_1, \ldots, f_n\}$, where $f_i \in \mathbb{C}[X_1, \ldots, X_n]$. An isolated zero of multiplicity μ for f is a point $x \in \mathbb{C}^n$ such that

1. $f(x) = 0$,
2. there exists a ball $B(x, r)$ of radius $r > 0$ such that $B(x, r) \cap f^{-1}(0) = \{x\}$,
3. $\mu = \dim(\mathbb{C}[X]/Q_{f,x})$,

where

$$B(x, r) := \{y \in \mathbb{C}^n : \|y - x\| < r\},$$

and $Q_{f,x}$ is a primary component of the ideal I_f whose associate prime is

$$m_x = (X_1 - x_1, \ldots, X_n - x_n).$$

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Let $\mathbf{d}_x^{\boldsymbol{\alpha}} : \mathbb{C}[X] \to \mathbb{C}$ denote the differential functional defined by

$$\mathbf{d}_x^{\boldsymbol{\alpha}}(g) = \frac{1}{\alpha_1! \cdots \alpha_n!} \cdot \frac{\partial^{|\boldsymbol{\alpha}|} g}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}(x), \quad \forall g \in \mathbb{C}[X], \tag{1}$$

where $x \in \mathbb{C}^n$ and $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_n] \in \mathbb{N}^n$. We have

$$\mathbf{d}_x^{\boldsymbol{\alpha}}\left((X - x)^{\boldsymbol{\beta}}\right) = \begin{cases} 1, \text{ if } \ \boldsymbol{\alpha} = \boldsymbol{\beta}, \\ 0, \text{ otherwise.} \end{cases} \tag{2}$$

The local dual space of I_f at a given isolated singular solution x is a subspace $\mathcal{D}_{f,x}$ of $\mathfrak{D}_x = \mathrm{span}_{\mathbb{C}}\{\mathbf{d}_x^{\alpha}\}$ such that

$$\mathcal{D}_{f,x} = \{\Lambda \in \mathfrak{D}_x \ \mid \ \Lambda(g) = 0, \ \forall g \in I_f\}. \tag{3}$$

When the evaluation point x is clear from the context, we write $d_1^{\alpha_1} \cdots d_n^{\alpha_n}$ instead of $\mathbf{d}_x^{\boldsymbol{\alpha}}$ for simplicity.

Let $\mathcal{D}_{f,x}^{(k)}$ be the subspace of $\mathcal{D}_{f,x}$ with differential functionals of orders bounded by k, we define

1. breadth $\kappa = \dim\left(\mathcal{D}_{f,x}^{(1)} \setminus \mathcal{D}_{f,x}^{(0)}\right)$,
2. depth $\rho = \min\left(\left\{k \ \mid \ \dim\left(\mathcal{D}_{f,x}^{(k+1)} \setminus \mathcal{D}_{f,x}^{(k)}\right) = 0\right\}\right)$,
3. multiplicity $\mu = \dim\left(\mathcal{D}_{f,x}^{(\rho)}\right)$.

If x is an isolated singular solution of f, then $1 \leq \kappa \leq n$ and $\rho < \mu < \infty$.

We recall α-theory below according to [1] and refer to [16,37–41,43] for more details.

Let $Df(x)$ denote the Jacobian matrix of f at x. Suppose $Df(x)$ is invertible, x is called a simple (regular) zero of f. The Newton's iteration is defined by

$$N_f(x) = x - Df(x)^{-1} f(x). \tag{4}$$

Shub and Smale [37] defined

$$\gamma(f, x) = \sup_{k \geq 2} \left\| Df(x)^{-1} \cdot \frac{D^k f(x)}{k!} \right\|^{\frac{1}{k-1}}, \tag{5}$$

where $D^k f$ denotes the k-th derivative of f which is a symmetric tensor whose components are the partial derivatives of f of order k, $\|\cdot\|$ denotes the classical operator norm.

According to [1, Theorem 1], if

$$\|z - x\| \leq \frac{3 - \sqrt{7}}{2\gamma(f, x)}, \tag{6}$$

then Newton's iterations starting at z will converge quadratically to the simple zero x.

If y is another zero of f, according to [1, Corollary 1], we have

$$\|y - x\| \geq \frac{5 - \sqrt{17}}{4\gamma(f, x)}, \tag{7}$$

which separates the simple zero x from other zeros of f.

Furthermore, according to [1, Theorem 2], if only a system f and a point x are given such that

$$\alpha(f,x) \leq \frac{13-3\sqrt{17}}{4} \approx 0.157671, \tag{8}$$

where $\alpha(f,x) = \beta(f,x)\gamma(f,x)$ and

$$\beta(f,x) = \|x - N_f(x)\| = \|Df(x)^{-1}f(x)\|,$$

then Newton's iterations starting at x will converge quadratically to a simple zero ξ of f and

$$\|x - \xi\| \leq 2\beta(f,x).$$

It is a challenge to extend α-theory for polynomial systems with singular solutions. When $Df(x)$ is not invertible, many modifications of Newton's iteration to restore the quadratic convergence for singular solutions have been proposed in [2,6–8,12–14,29–33,36,46]. Recently, some symbolic-numeric methods based on deflated systems have also been proposed for refining approximate isolated singular solutions to high accuracy [3–5,10,11,18–20,25]. For example, as shown in [19], let $r = \operatorname{rank}(Df(x))$, with probability one, there exists a unique vector $\lambda = (\lambda_1, \lambda_2 \ldots, \lambda_{r+1})^T$ such that (x, λ) is an isolated solution of a deflated polynomial system, i.e.,

$$\begin{cases} f(x) = 0, \\ Df(x)B\lambda = 0, \\ h^T\lambda = 1, \end{cases} \tag{9}$$

where $B \in \mathbb{C}^{n\times(r+1)}$ is a random matrix, $h \in \mathbb{C}^{r+1}$ is a random vector. If (x, λ) is still a singular solution of (9), the deflation is repeated. Furthermore, they proved that the number of deflations needed to derive a regular solution of an augmented system is strictly less than the multiplicity of x. Dayton and Zeng showed that the depth of $\mathcal{D}_{f,x}$ is a tighter bound for the number of deflations [5].

In [44,45], we present a method based on the reduction to geometric involutive form to compute the primary component and a basis of the local dual space of a polynomial system at an isolated singular solution. We also present an algorithm based on correctly computed multiplicity structure such as index and multiplicity at an approximate singular solution to restore the quadratic convergence of Newton's iterations.

In this paper, we introduce some recent contributions related to extending α-theory for polynomial systems with singular zeros satisfying $f(x) = 0$, $\dim\ker Df(x) = 1$. It is also called breadth-one singular zero in [5] as

$$\dim(\mathcal{D}^{(k)}_{f,x} \setminus \mathcal{D}^{(k-1)}_{f,x}) = 1,\ k = 1\ldots,\rho,\ \rho = \mu - 1. \tag{10}$$

Therefore, the local dual space of I_f at x is

$$\mathcal{D}_{f,x} = \operatorname{span}_{\mathbb{C}}\{\Lambda_0, \Lambda_1, \ldots, \Lambda_{\mu-1}\},$$

where $\deg(\Lambda_k) = k$ and $\Lambda_0 = 1$.

As pointed out in [11], the breath one case is the least degenerate one and therefore most likely to be of practical significance. Moreover, it is also the worst case for the deflation method [5,19,29,30] since the deflation always terminates at step $\mu - 1$. Hence the size of the matrices grows extremely fast with the multiplicity.

2 Local Dual Space

Let us introduce a morphism $\Phi_\sigma : \mathfrak{D}_x \to \mathfrak{D}_x$ which is an anti-differentiation operator defined by

$$\Phi_\sigma(d_1^{\alpha_1}\cdots d_n^{\alpha_n}) = \begin{cases} d_1^{\alpha_1}\cdots d_\sigma^{\alpha_\sigma - 1}\cdots d_n^{\alpha_n}, & \text{if } \alpha_\sigma > 0,\\ 0, & \text{otherwise.}\end{cases}$$

Computing a closed basis of the local dual space is done essentially by matrix-kernel computations based on the stability property of $\mathcal{D}_{f,x}$ [26,28,42]:

$$\forall \Lambda \in \mathcal{D}_{f,x}^{(k)},\ \Phi_\sigma(\Lambda) \in \mathcal{D}_{f,x}^{(k-1)},\quad \sigma = 1,\ldots,n. \tag{11}$$

Let $\mathcal{D}_{f,x}^{(k)}$ be the subspace of $\mathcal{D}_{f,x}$ with differential functionals of orders bounded by k. Let $\Psi_\sigma : \mathfrak{D}_x \to \mathfrak{D}_x$ be a differential operator defined by

$$\Psi_\sigma(d_1^{\alpha_1}\cdots d_n^{\alpha_n}) = \begin{cases} d_\sigma^{\alpha_\sigma + 1}\cdots d_n^{\alpha_n}, & \text{if } \alpha_1 = \cdots = \alpha_{\sigma-1} = 0,\\ 0, & \text{otherwise.}\end{cases}$$

We deal with multiple zeros satisfying $f(x) = 0$, $\dim\ker Df(x) = 1$. The local dual space of I_f at a given isolated singular solution x is

$$\mathcal{D}_{f,x} = \mathrm{span}_{\mathbb{C}}\{\Lambda_0, \Lambda_1, \ldots, \Lambda_{\mu-1}\},$$

where $\deg(\Lambda_k) = k$ and $\Lambda_0 = 1$.

As shown in [23, Theorem 3.4], suppose $\Lambda_1 = a_{1,1}d_1 + \cdots + a_{1,n}d_n$, without loss of generality, we assume $a_{1,1} = 1$, $a_{k,1} = 0$, $k = 2,\ldots,n$. Then for $k = 2,\ldots,\mu-1$, we have

$$\Lambda_k = \Delta_k + a_{k,2}d_2 + \cdots + a_{k,n}d_n, \tag{12}$$

where

$$\Delta_k = \sum_{\sigma=1}^{n} \Psi_\sigma(a_{1,\sigma}\Lambda_{k-1} + \cdots + a_{k-1,\sigma}\Lambda_1), \tag{13}$$

and $a_{k,2},\ldots,a_{k,n}$ are determined by solving the linear system obtained from setting $\Lambda_k(f_i) = 0, i = 1,\ldots,n$:

$$\begin{pmatrix} d_2(f_1) & \cdots & d_n(f_1)\\ \vdots & \ddots & \vdots\\ d_2(f_n) & \cdots & d_n(f_n)\end{pmatrix}\begin{pmatrix} a_{k,2}\\ \vdots\\ a_{k,n}\end{pmatrix} = -\begin{pmatrix} \Delta_k(f_1)\\ \vdots\\ \Delta_k(f_n)\end{pmatrix}. \tag{14}$$

Definition 1 [15]. *For a polynomial function $f : \mathbb{C}^n \to \mathbb{C}^n$, suppose $f(x) = 0, \dim \ker Df(x) = 1$. Then $Df(x)$ has a normalized form if*

$$Df(x) = \begin{pmatrix} 0 & D\hat{f}(x) \\ 0 & 0 \end{pmatrix}, \tag{15}$$

$D\hat{f}(x)$ is the nonsingular Jacobian matrix of polynomials $\hat{f} = \{f_1, \ldots, f_{n-1}\}$ with respect to variables $X_2, \ldots, X_n$.

If x is a multiple zero of multiplicity μ for f and $Df(x)$ has the normalized form (15), which is always possible to obtain by performing unitary transformations when $\dim \ker Df(x) = 1$, see [15, Sect. 2.3], then we have $\Delta_k(f_n) = 0$, for $k = 2, \ldots, \mu - 1$, $\Delta_\mu(f_n) \neq 0$, and the linear system (14) for getting the values of $a_{k,2}, \ldots, a_{k,n}$ can be simplified to:

$$\begin{pmatrix} d_2(f_1) & \cdots & d_n(f_1) \\ \vdots & \ddots & \vdots \\ d_2(f_{n-1}) & \cdots & d_n(f_{n-1}) \end{pmatrix} \begin{pmatrix} a_{k,2} \\ \vdots \\ a_{k,n} \end{pmatrix} = - \begin{pmatrix} \Delta_k(f_1) \\ \vdots \\ \Delta_k(f_{n-1}) \end{pmatrix}. \tag{16}$$

3 Local Separation Bound and Cluster Location

In [9], Dedieu and Shub gave quantitative results for simple double zeros satisfying $f(x) = 0$ and

(A) $\dim \ker Df(x) = 1$,
(B) $D^2 f(x)(v, v) \notin \mathrm{im} Df(x)$,

where $\ker Df(x)$ is spanned by a unit vector $v \in \mathbb{C}^n$. They generalized the definition of γ in (5) to

$$\gamma_2(f, x) = \max\left(1, \sup_{k \geq 2} \left\| A(f, x, v)^{-1} \cdot \frac{D^k f(x)}{k!} \right\|^{\frac{1}{k-1}}\right), \tag{17}$$

where

$$A(f, x, v) = Df(x). + \frac{1}{2} D^2 f(x)(v, \Pi_v), \tag{18}$$

is a linear operator which is invertible at the simple double zero x, and Π_v denotes the Hermitian projection onto the subspace $[v] \subset \mathbb{C}^n$.

In [9, Theorem 1], Dedieu and Shub also presented a lower bound for separating simple double zeros x from the other zeros y of f,

$$\|y - x\| \geq \frac{d}{2\gamma_2(f, x)^2}, \tag{19}$$

where $d \approx 0.2976$ is a positive real root of

$$\sqrt{1 - d^2} - 2d\sqrt{1 - d^2} - d^2 - d = 0. \tag{20}$$

In [9, Theorem 4], Dedieu and Shub showed that if the following criterion is satisfied at a given point x and a given vector v

$$\|f(x)\| + \|Df(x)v\| \frac{d}{4\gamma_2(f,x,v)^2} < \frac{d^3}{32\gamma_2^4 \|B(f,x,v)^{-1}\|}, \tag{21}$$

then f has two zeros in the ball of radius

$$\frac{d}{4\gamma_2(f,x)^2}, \tag{22}$$

around x. Let us set

$$B(f,x,v) = A(f,x,v) - L,$$

where $L(v) = Df(x)v$, $L(w) = 0$ for $w \in v^{\perp}$, and

$$\gamma_2(f,x) = \max\left(1, \sup_{k\geq 2} \left\| B(f,x,v)^{-1} \cdot \frac{D^k f(x)}{k!} \right\|^{\frac{1}{k-1}}\right). \tag{23}$$

Based on the multiplicity structure of the singular zero x of f computed in the last section, we generalize Dedieu and Shub's results to multiple zeros with arbitrary large multiplicity.

Let $f : \mathbb{C}^n \to \mathbb{C}^n$, and x be a singular zero of f of multiplicity μ, where $Df(x)$ has the normalized form $Df(x) = \begin{pmatrix} 0 & D\hat{f}(x) \\ 0 & 0 \end{pmatrix}$, $D\hat{f}(x)$ is invertible and

$$\Delta_k(f_n) = 0, \text{ for } k = 2, \ldots, \mu - 1, \quad \Delta_\mu(f_n) \neq 0. \tag{24}$$

Let y be another vector in $\mathbb{C}^n$ and $y \neq x$. Recall that $\varphi = d_P(v, y - x)$, $v = (1, 0, \ldots, 0)^T$ and $w = x - y = (\zeta, \eta_2, \ldots, \eta_n)^T$, $\eta = (\eta_2, \ldots, \eta_n)^T$, then we have $|\zeta| = \|w\| \sin\varphi$, $\|\eta\| = \|w\| \cos\varphi$. Let

$$\mathcal{A} = \begin{pmatrix} \sqrt{2} D\hat{f}(x) & 0 \\ 0 & \frac{1}{\sqrt{2}} \Delta_\mu(f_n) \end{pmatrix},$$

and $\gamma_\mu = \max(\hat{\gamma}_\mu,\ \gamma_{\mu,n})$, where

$$\hat{\gamma}_\mu = \hat{\gamma}_\mu(f,x) = \max\left(1,\ \sup_{k\geq 2} \left\| D\hat{f}(x)^{-1} \frac{D^k \hat{f}(x)}{k!} \right\|^{\frac{1}{k-1}}\right), \tag{25}$$

where $D^k \hat{f}(x)$ for $k \geq 2$ denote the partial derivatives of $\hat{f}$ of order k with respect to $X_1, X_2, \ldots, X_n$ evaluated at x, and

$$\gamma_{\mu,n} = \gamma_{\mu,n}(f,x) = \left(1,\ \sup_{k\geq 2} \left\| \frac{1}{\Delta_\mu(f_n)} \cdot \frac{D^k f_n(x)}{k!} \right\|^{\frac{1}{k-1}}\right), \tag{26}$$

Definition 2 [15, Defintion 3]**.** *We define* $d = \min(d_1, d_2, d_3)$*, where*

$$d_1 = \sqrt{\frac{1}{c_{\mu-1,1}^2 + 1}},\ d_2 = \sqrt{\frac{1}{\mu - 1}},$$

and d_3 *is the smallest positive real root of the polynomial*

$$\begin{aligned} p(d) = (1-d^2)^{\frac{\mu}{2}} - &\sum_{i+j=\mu, j>0} c_{i,j} d(1-d^2)^{\frac{i}{2}} d^{j-1} \\ - d &\left(\sum_{1 \le i \le \mu-2} t_{i,0} + \sum_{1 \le i+j \le \mu-2, j>0} t_{i,j} (1-d^2)^{\frac{i}{2}} d^j + 1 \right), \end{aligned} \tag{27}$$

where $c_{i,j}$ *and* $t_{i,j}$ *can be obtained by the method given in [15, Case 2].*

Theorem 1 [15, Theorem 5]**.** *Let* x *be a multiple zero of* f *of multiplicity* μ*,* $\dim \ker Df(x) = 1$*, and* y *be another zero of* f*, then*

$$\|y - x\| \ge \frac{d}{2\gamma_\mu^\mu}.$$

Remark 1. For $\mu = 2$, we have [15, Sect. 3.3]

$$p(d) = 1 - 2d^2 - 2d\sqrt{1-d^2} - d. \tag{28}$$

The smallest positive real root of $p(d)$ is

$$d \approx 0.2865.$$

For $\mu = 3$, we have [15, Lemma 3]

$$p(d) = (1 - 2d - 8d^2)\sqrt{1-d^2} - 9d - d^2 + 6d^3. \tag{29}$$

The smallest positive root of $p(d)$ is

$$d \approx 0.08507.$$

Theorem 2 [15, Theorem 8]**.** *Given* $f : \mathbb{C}^n \to \mathbb{C}^n,\ x \in \mathbb{C}^n$*, such that* $D\hat{f}(x)$ *is invertible, and* $\Delta_\mu(f_n) \ne 0$*. Let*

$$H_1 = \begin{pmatrix} \frac{\partial \hat{f}(x)}{\partial X_1} & 0 \\ \frac{\partial f_n(x)}{\partial X_1} & \frac{\partial f_n(x)}{\partial \hat{X}} \end{pmatrix},$$

$$H_k = \left(\begin{pmatrix} 0 & 0 \\ \Delta_k(f_n) & 0 \end{pmatrix} \mathbf{0}_{\underbrace{n \times \cdots \times n}_{k} \times (n-1)} \right),\ 2 \le k \le \mu - 1,$$

and polynomials

$$g(X) = f(X) - f(x) - \sum_{1 \le k \le \mu-1} H_k (X - x)^k.$$

Let $\gamma_\mu = \gamma_\mu(g, x)$*, if*

$$\|f(x)\| + \sum_{1 \le k \le \mu-1} \|H_k\| \left(\frac{d}{4\gamma_\mu^\mu}\right)^k < \frac{d^{\mu+1}}{2\,(4\gamma_\mu^\mu)^\mu \,\|\mathcal{A}^{-1}\|}, \tag{30}$$

then f *has* μ *zeros (counting multiplicities) in the ball of radius* $\frac{d}{4\gamma_\mu^\mu}$ *around* x*.*

4 Verified Error Bound

Let $\mathbb{IR}$ be the set of real intervals, and let $\mathbb{IR}^n$ and $\mathbb{IR}^{n\times n}$ be the set of real interval vectors and real interval matrices, respectively. Standard verification methods for nonlinear systems are based on the following theorem.

Theorem 1 [17,27,34]**.** *Let* $f : \mathbb{R}^n \to \mathbb{R}^n$ *be a system of nonlinear equations. Suppose* $x \in \mathbb{R}^n$*,* $\mathbf{X} \in \mathbb{IR}^n$ *with* $0 \in \mathbf{X}$ *and* $R \in \mathbb{R}^{n\times n}$ *are given. Let* $\mathbf{M} \in \mathbb{IR}^{n\times n}$ *be given such that*

$$\{Df_i(y) : y \in x + \mathbf{X}\} \subseteq \mathbf{M}_{i,:}, i = 1, \ldots, n. \tag{31}$$

Denote by I_n *the* $n \times n$ *identity matrix and assume*

$$-Rf(x) + (I_n - R\mathbf{M})\mathbf{X} \subseteq \mathrm{int}(\mathbf{X}). \tag{32}$$

Then there is a unique $\tilde{x} \in x + \mathbf{X}$ *satisfying* $f(\tilde{x}) = 0$*. Moreover, every matrix* $\tilde{M} \in \mathbf{M}$ *is nonsingular. In particular, the Jacobian matrix* $Df(\tilde{x})$ *is nonsingular.*

Theorem 1 is restricted to verifying the existence of a simple solution of a square and regular system. Notice that Theorem 1 is valid over complex numbers with the necessary modifications. In [35], by introducing a smoothing parameter, Rump and Graillat developed a verification method for computing verified and narrow error bounds, such that a slightly perturbed system is proved to possess a double root within computed error bounds.

In [23], by adding a univariate polynomial in one selected variable with some smoothing parameters to one selected equation of the original system, we generalized the algorithm in [35] to compute guaranteed error bounds such that a slightly perturbed system is proved to have a breadth-one isolated singular solution within computed error bounds.

For a polynomial function $f : \mathbb{C}^n \to \mathbb{C}^n$, where $f_i \in \mathbb{C}[X_1, \ldots, X_n]$, and suppose x is a zero of f of multiplicity μ and satisfying $\dim\ker Df(x) = 1$. Suppose the i-th column of $Df(x)$ can be written as a linear combination of the other $n-1$ columns, then we choose x_i as the variable. Similarly, suppose the j-th row of $Df(x)$ can be written as a linear combination of the other $n-1$

linearly independent rows, then we add the perturbed univariate polynomial in x_i to f_j. Finally, we permute

$$x_1 \leftrightarrow x_i \text{ and } f_1 \leftrightarrow f_j$$

to construct a deflated system below.

We introduce $\mu - 1$ smoothing parameters $b_0, b_1, \ldots, b_{\mu-2}$ and construct a deflated system $G(X, b, a)$ with μn variables and μn equations:

$$G(X,b,a) = \begin{pmatrix} F_1(X,b) = f(X) - \left(\sum_{\nu=0}^{\mu-2} \frac{b_\nu x_1^\nu}{\nu!}\right) e_1 \\ F_2(X,b,a_1) \\ F_3(X,b,a_1,a_2) \\ \vdots \\ F_\mu(X,b,a_1,\ldots,a_{\mu-1}) \end{pmatrix}, \tag{33}$$

where $e_1 = (1, 0, \ldots, 0)^T$, $b = (b_0, b_1, \ldots, b_{\mu-2})$, $a = (a_1, a_2, \ldots, a_{\mu-1})$, $a_1 = (1, a_{1,2}, \ldots, a_{1,n})^T$, $a_k = (0, a_{k,2}, \ldots, a_{k,n})^T$ for $1 < k \le \mu$, and

$$F_k(X, b, a_1, \ldots, a_{k-1}) = L_{k-1}(F_1), \tag{34}$$

where L_k are differentiation operators corresponding to Λ_k defined by (12).

Theorem 2 [23, Theorem 4.3]**.** *Suppose $G(x, \tilde{b}, \tilde{a}) = 0$. If the Jacobian matrix of the deflated polynomial system $G(X, b, a)$ at $(x, \tilde{b}, \tilde{a})$ is nonsingular, then x is an isolated root of the perturbed polynomial system $F(X) = F_1(X, \tilde{b})$ with multiplicity μ and the corank of $DF(x)$ is one.*

Theorem 3 [23, Theorem 4.5]**.** *Suppose Theorem 1 is applicable to $G(X, b, a)$ in (33) and yields inclusions for x, $\tilde{b}$ and $\tilde{a}$ such that $G(x, \tilde{b}, \tilde{a}) = 0$. Then x is an isolated breadth-one root of $F(X) = F_1(X, \tilde{b})$ with multiplicity μ.*

5 Modified Newton Iterations

In [22], we presented a symbolic-numeric method to refine an approximate isolated singular solution $\tilde{x} = (x_1, \ldots, x_n)$ of a polynomial system $f = \{f_1, \ldots, f_n\}$ when the Jacobian matrix of f evaluated at $\tilde{x}$ has corank one approximately. Our approach is based on the regularized Newton iteration and the computation of differential conditions satisfied at the approximate singular solution. The size of matrices involved in our algorithm is bounded by $n \times n$. The algorithm converges quadratically if $\tilde{x}$ is close to the isolated exact singular solution of f.

Theorem 4 [22, Theorem 3.16]**.** *If the Jacobian matrix of f evaluated at x has corank one and the approximate singular solution $\tilde{x}$ of f satisfying*

$$\|\tilde{x} - x\| = \varepsilon \ll 1,$$

where the positive number ϵ is small enough such that there are no other solutions of f nearby, then the refined singular solution $\tilde{x}$ returned by Algorithm 1 satisfies

$$\|N_f(\tilde{x}) - x\| = O(\varepsilon^2).$$

Algorithm 1. Modified Newton's Iterations for Breadth-one Multiple Zero

Input:

f: a polynomial system;
$\tilde{x}$: an approximate singular zero of f;
μ: the multiplicity

Output:

$N_f(\tilde{x})$: a refined solution after one iteration;

1: solve the regularized least squares problem

$$(Df(\tilde{x})^* Df(\tilde{x}) + \sigma_n I_n)\tilde{y} = Df(\tilde{x})^* b,$$

where $b = -f(\tilde{x})$, I_n is the $n \times n$ identity matrix and σ_n is the smallest singular value of $Df(\tilde{x})$;

2: compute the singular value decomposition of $Df(\tilde{x} + \tilde{y}) = U \cdot \Sigma \cdot V^*$, let

$$g(X) = f(W \cdot X), \quad W = (v_n, v_1, \ldots, v_{n-1}),$$

and set $\tilde{z} \leftarrow W^*(\tilde{x} + \tilde{y})$;

3: construct Δ_μ and a closed approximate basis of the local dual space

$$\mathcal{D}_{g,\tilde{z}} = \mathrm{Span}(\Lambda_0, \Lambda_1, \ldots, \Lambda_{\mu-1}),$$

by Algorithm MultiplicityStructureBreadthOneNumeric in [21];

4: solve the linear system

$$\left[\Delta_\mu(g), \frac{\partial g(\tilde{z})}{\partial z_2}, \ldots, \frac{\partial g(\tilde{z})}{\partial z_n}\right] \delta \;= -\Lambda_{\mu-1}(g)$$

5: update the zero of g

$$\tilde{z}_1 \leftarrow \tilde{z}_1 + \frac{\delta_1}{\mu}, \quad \tilde{z}_i \leftarrow \tilde{z}_i, \; 2 \leq i \leq n$$

and

$$N_f(\tilde{x}) \leftarrow W \cdot \tilde{z}.$$

The proof of Theorem 4 in [22] is based on studying zeros of deflated systems. It is difficult to quantify the quadratical convergence of Algorithm 1. In [15], we present a new algorithm for refining an approximate singular zero whose Jacobian matrix has corank one. The main idea is to perform the unitary transformations to both variables and equations defined at the approximate singular solutions, then define the modified Newton's iteration which are very similar to Step 4 in Algorithm 1.

Theorem 3. *Given an approximate zero z of a polynomial system f associated to a multiple zero ξ of multiplicity μ and satisfying $f(\xi) = 0$, $\dim\ker Df(\xi) = 1$. Suppose*

$$\hat{\gamma}_\mu(f, z)\|z - \xi\| < \frac{1}{2},$$

Algorithm 2. Modified Newton's Iteration for Breadth-one Multiple Zeros

Input:

f: a polynomial system;

z: an approximate singular zero of f;

μ: the multiplicity;

Output:

$N_f(z)$: a refined solution after one iteration;

1: compute the singular value decomposition

$$Df(z) = U \cdot \begin{pmatrix} \Sigma_{n-1} & 0 \\ 0 & \sigma_n \end{pmatrix} \cdot V^*,\ W_\dagger = (v_n, v_1, \ldots, v_{n-1});$$

2: perform the unitary transformations to equations and variables

$$f(X) \leftarrow U^* \cdot f(W_\dagger \cdot X), \quad z \leftarrow W_\dagger^* z;$$

3: update the last $n-1$ elements of the approximate zero

$$N_1(\hat{f}, \hat{z}) \leftarrow \hat{z} - D\hat{f}(z)^{-1}\hat{f}(z), \quad y = (y_1, \hat{y}) \leftarrow (z_1, N_1(\hat{f}, \hat{z}));$$

4: compute the singular value decomposition

$$Df(y) = U \cdot \begin{pmatrix} \Sigma_{n-1} & 0 \\ 0 & \sigma_n \end{pmatrix} \cdot V^*, W_\ddagger = (v_n, v_1, \ldots, v_{n-1});$$

5: perform the unitary transformations to equations and variables:

$$g(X) \leftarrow U^* \cdot f(W_\ddagger \cdot X), \quad w = (w_1, \hat{w}) \leftarrow W_\ddagger^* y;$$

6: update the first element of the approximate zero

$$N_2(g_n, w) \leftarrow w_1 - \frac{1}{\mu} \Delta_\mu(g_n)^{-1} \Delta_{\mu-1}(g_n), \quad x = (x_1, \hat{x}) \leftarrow (N_2(g_n, w), \hat{w});$$

7: update the zero of f

$$N_f(z) \leftarrow W_\dagger \cdot W_\ddagger \cdot x.$$

where $\hat{\gamma}_\mu(f, z)$ is defined by (25), then the refined singular solution $N_f(z)$ returned by Algorithm 2 satisfies

$$\|N_f(z) - \xi\| = O(\|z - \xi\|^2). \tag{35}$$

In [15, Theorem 12], we give a quantified quadratic convergence proof of the Algorithm 2 for simple triple zeros. There is no significant obstacle to extend the proof to multiple zeros of higher multiplicities. However, the computation will become more complicated.

Theorem 4 [15, Theorem 12]**.** *Given an approximate zero z of a system f associated to a simple triple zero ξ of multiplicity 3 and satisfying $f(\xi) = 0$, $\dim\ker Df(\xi) = 1$. Let $u = \max\{\gamma_3(f, \xi)^3 \|\xi - z\|, L\gamma_3(f, \xi)^2 \|\xi - z\|\}$, where L is the Lipschitz constant of the function $Df(X)$.*

(1) If $u < u_3 \approx 0.0137$,
then the output of Algorithm 2 satisfies:

$$\|N_f(z) - \xi\| < \|z - \xi\| .$$

(2) If $u < u_3' \approx 0.0098$ *then after* k *times of iteration we have*

$$\left\|N_f^k(z) - \xi\right\| < \left(\frac{1}{2}\right)^{2^k-1} \|z - \xi\| .$$

6 Conclusion

The Maple code of algorithms mentioned in the paper and test results are available http://www.mmrc.iss.ac.cn/~lzhi/Research/hybrid.

Although the algorithms and proofs of quadratic convergence given in the paper are for polynomial systems with exact multiple zeros, examples are given to demonstrate that our algorithms are also applicable to analytic systems and polynomial systems with a cluster of simple roots.

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