Determining Singular Solutions of Polynomial Systems via Symbolic-numeric Reduction to Geometric Involutive Form

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Abstract

We present a method based on symbolic-numeric reduction to geometric involutive form to compute the primary component and a basis of Max Noether space of a polynomial system at an isolated singular solution. The singular solution can be known exactly or approximately. If the singular solution is known with limited accuracy, then we propose a generalized quadratic Newton iteration to refine it to high accuracy.

1 Introduction

Consider an ideal I generated by a polynomial system $F = \{f_1, \ldots, f_t\}$, where $f_i \in \mathbb{C}[x_1, \ldots, x_s]$, $i = 1, \ldots, t$. For a given isolated singular solution $\hat{\mathbf{x}} = (\hat{x}_1, \ldots, \hat{x}_s)$ of F, suppose Q is the isolated primary component whose associate prime is $P = (x_1 - \hat{x}_1, \ldots, x_s - \hat{x}_s)$, in [Wu and Zhi 2008], we use symbolic-numeric method based on the geometric jet theory of partial differential equations introduced in [Reid et al. 2003; Zhi and Reid 2004] to compute the index ρ and multiplicity μ, such that $Q = (I, P^\rho)$ and $\mu = \dim(\mathbb{C}[\mathbf{x}]/Q)$. We also derive a simple involutive criterion based on the special structure of the ideal (I, P^k) and apply it to the truncated coefficient matrices formulated from the Taylor series expansions of polynomials in prolonged systems of F at $\hat{\mathbf{x}}$ to order k. The number of columns of these coefficient matrices is fixed

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by $\binom{k+s-1}{s}$. A basis for the Max Noether space of I at $\hat{\mathbf{x}}$ is obtained from the null space of the truncated coefficient matrix of the involutive system.

If a singular solution is only known with limited accuracy, by choosing a tolerance, we can compute the index, multiplicity and a basis of Max Noether space for this approximate singular solution. It is well known that numeric computations deeply depend on the choice of tolerance. In order to obtain accurate information about the multiplicity structure, we present in [Wu and Zhi 2008] a method to improve the accuracy of the singular root. Suppose $\hat{\mathbf{x}} = \hat{\mathbf{x}}_{\text{exact}} + \hat{\mathbf{x}}_{\text{error}}$, where $\hat{\mathbf{x}}_{\text{exact}}$ denotes the exact singular solution of F and $\hat{\mathbf{x}}_{\text{error}}$ denotes the error in the solution. In [Wu and Zhi 2008], we observe that a good approximation $\hat{\mathbf{y}}$ of $-\hat{\mathbf{x}}_{\text{error}}$ is computed from the null vectors of the truncated coefficient matrix of the involutive system. The singular solution $\hat{\mathbf{x}}+\hat{\mathbf{y}}$ has higher accuracy compared with $\hat{\mathbf{x}}$. For a set of benchmark problems, we show that singular solutions accurately to the full machine precision are obtained in a few iterations after applying the procedure iteratively to $\hat{\mathbf{x}} + \hat{\mathbf{y}}$ with smaller tolerances. However, we did not provide the convergence and complexity analysis of our refining algorithm.

In this paper, we prove the quadratic convergence of the refining algorithm under the condition that the index and multiplicity of the singular solution are computed correctly. Newton's original thought about calculating corrector for an approximate solution is to compute the Taylor series expansion of the system at the approximate zero to the first order and solve the linear equations to obtain the Newton iterator. If the solution is simple, then the Newton iteration is well-defined and converges quadratically. We generalize the Newton iteration for handling approximate multiple roots. We compute the truncated coefficient matrix of the involutive system by shifting the coefficient matrix formulated from the truncated multivariate Taylor series expansions of the polynomials $f_1, \ldots, f_t$ at $\hat{\mathbf{x}}$ to order ρ, then generate multiplication matrices from its null vectors. Let $\hat{\mathbf{y}}$ be the averages of the traces of the multiplication matrices. We prove that if the given singular solution satisfies $\|\hat{\mathbf{x}} - \hat{\mathbf{x}}_{\text{exact}}\| = \mathcal{O}(\varepsilon)$, then the refined solution obtained by adding $\hat{\mathbf{y}}$ to $\hat{\mathbf{x}}$ will satisfy $\|\hat{\mathbf{x}}+\hat{\mathbf{y}} - \hat{\mathbf{x}}_{\text{exact}}\| = \mathcal{O}(\varepsilon^2)$. That's the reason we call our algorithm generalized quadratic Newton iteration. If we underestimate or overestimate the index due to poorly chosen tolerance, then we can rediscover the correct index after the accuracy of the approximate singular solution improved after one or two iterations.

Since the size of the coefficient matrices we used for computing the primary component, and refining a singular solution is bounded by $t\binom{\rho+s}{s} \times \binom{\rho+s}{s}$, the complexities of Algorithm 1 and Algorithm 3 are $\mathcal{O}\left(t\binom{\rho+s}{s}^3\right)$. The complexity of Algorithm 2 is $\mathcal{O}\left(t\binom{\rho+s-1}{s}^3\right)$ because the size of the matrix for computing a basis of the Max Noether space of I at $\hat{\mathbf{x}}$ is bounded by $t\binom{\rho+s-1}{s} \times \binom{\rho+s-1}{s}$.

The algorithms have been implemented in Maple 11 and Matlab. We give an example to illustrate our method along the paper. Test results are presented in Maple 11 with Digits := 14 for a set of benchmark problems. We show that for most of examples, we can refine a singular solution with only two correct digits to high precision by only two or three generalized Newton iterations.

2 Isolated Primary Component

The following paragraphs give a brief outline of the notations and tools we use throughout this paper. We refer to [Cox et al. 1992; van der Waerden B. L. 1970] for detailed introduction.

Definition 1 *If P and Q are ideals and have the property that (1) $fg \in Q$ and $f \notin Q$ implies $g \in P$, (2) $Q \subseteq P$, (3) $g \in P$ implies $g^\rho \in Q$ for some positive integer ρ, then Q is* primary *and P the* prime ideal belonging to Q.

If Q is a primary ideal then $P = \sqrt{Q}$ is the prime ideal belonging to Q and Q is called P-primary.

Definition 2 *Every polynomial ideal has an irredundant primary decomposition, i.e. $I = \cap_{i=1}^{r} Q_i$, where Q_i are primary, $Q_i \subsetneq \cap_{j \neq i} Q_j$. We call Q_i a* primary component (ideal) *of I. Q_i is said to be isolated if no prime ideal belonging to Q_j, $j \neq i$, is divisible by a prime ideal belonging to Q_i.*

Definition 3 *ρ is called the index of a primary ideal Q if ρ is the minimal nonnegative integer such that $\sqrt{Q}^\rho \subseteq Q$.*

Theorem 4 *[van der Waerden B. L. 1970] Suppose the polynomial ideal I has an isolated primary component Q whose associated prime P is maximal, and ρ is the index of Q.*
If $\sigma < \rho$, then

$$\dim(\mathbb{C}[\mathbf{x}]/(I, P^{\sigma-1})) < \dim(\mathbb{C}[\mathbf{x}]/(I, P^\sigma)). \tag{1}$$

If $\sigma \geq \rho$, then

$$Q = (I, P^\rho) = (I, P^\sigma). \tag{2}$$

Corollary 5 *[Wu and Zhi 2008] If a polynomial ideal I has an isolated primary component Q whose associated prime P is maximal, then the index ρ of Q is less than or equal to the multiplicity μ of Q.*

3 Involutiveness of Polynomial Systems

Consider a polynomial system $F = \{f_1, \ldots, f_t\}$, where $f_i \in \mathbb{C}[x_1, \ldots, x_s]$ is of degree d, $i = 1, \ldots, t$ and $s \leq t$. We study the variety

$$V(F) = \left\{ [\mathbf{x}^d, \ldots, 1] \in \mathbb{C}^{N_d} \mid M_d^{(0)} \cdot [\mathbf{x}^d, \ldots, 1]^T = \mathbf{0} \right\},$$

where $N_d = \binom{d+s}{s}$, $\mathbf{x}^j$ denotes all monomials of total degree equal to j. All distinct monomials are regarded as independent variables and $V(F)$ is simply the null space of $M_d^{(0)}$.

A single prolongation of the system F is to multiply the polynomials in F by variables, so that the resulting augmented system has degree $d+1$. Successive prolongations of the system yield $F = F^{(0)}, F^{(1)}, F^{(2)}, \ldots$, and a sequence of corresponding linear constant matrix systems:

$$M_d^{(0)} \cdot \mathbf{v_d} = \mathbf{0}, M_d^{(1)} \cdot \mathbf{v_{d+1}} = \mathbf{0}, M_d^{(2)} \cdot \mathbf{v_{d+2}} = \mathbf{0}, \cdots$$

where $\mathbf{v_i} = [\mathbf{x}^i, \mathbf{x}^{i-1}, \ldots, \mathbf{x}, 1]^T$.

A single geometric projection is defined as

$$\boldsymbol{\pi}(F) = \left\{ [\mathbf{x}^{d-1}, \ldots, 1] \in \mathbb{C}^{N_{d-1}} \mid \exists \mathbf{x}^d, M_d^{(0)} \cdot [\mathbf{x}^d, \ldots, 1]^T = \mathbf{0} \right\}.$$

The projection operator $\boldsymbol{\pi}$ maps a point in $\mathbb{C}^{N_d}$ to one in $\mathbb{C}^{N_{d-1}}$ by eliminating the monomials of the highest degree d. A numeric projection operator $\hat{\boldsymbol{\pi}}$ based on singular value decomposition (SVD) was proposed in [Bonasia et al. 2004; Reid et al. 2003; Wittkopf and Reid 2001]. We first find the singular value decomposition $M_d^{(0)} = U \cdot \Sigma \cdot V$. The approximate rank r of $M_d^{(0)}$ is the number of singular values bigger than a fixed tolerance. The tolerance is chosen according to the number of correct digits for the coefficients of the input polynomials. The dimension of F is defined as the dimension of the null space of $M_d^{(0)}$, so we have $\dim F = \dim \text{Nullspace}(M_d^{(0)}) = N_d - r$. Deleting the first r rows of V yields an approximate basis for the null space of $M_d^{(0)}$. To estimate $\dim \hat{\boldsymbol{\pi}}(F)$, the components of the approximate basis for the null space of $M_d^{(0)}$ corresponding to the monomials of the highest degree d are deleted. This projected basis yields an approximate spanning set for $\hat{\boldsymbol{\pi}}(F)$. Application of the SVD to each of these approximate spanning sets yields the approximate dimensions of $\hat{\boldsymbol{\pi}}(F)$, $\hat{\boldsymbol{\pi}}^2(F)$, $\hat{\boldsymbol{\pi}}^3(F)$, ..., which are required for the approximate involutive form test.

The symbol matrix of polynomials of degree d is simply the submatrix of the coefficient matrix $M_d^{(0)}$ corresponding to the monomials of the highest degree d. One of the most important requirements of involutive systems is

that their symbols are involutive. The following criterion of involution for zero dimensional polynomial systems is given in [Zhi and Reid 2004].

Theorem 6 *[Zhi and Reid 2004] A zero dimensional polynomial system F is involutive at prolongation order m and projected order ℓ if and only if $\boldsymbol{\pi}^{\ell}(F^{(m)})$ satisfies the projected elimination test:*

$$\dim \boldsymbol{\pi}^{\ell}\left(F^{(m)}\right) = \dim \boldsymbol{\pi}^{\ell+1}\left(F^{(m+1)}\right), \tag{3}$$

and the symbol involutive test:

$$\dim \boldsymbol{\pi}^{\ell}\left(F^{(m)}\right) = \dim \boldsymbol{\pi}^{\ell+1}\left(F^{(m)}\right). \tag{4}$$

Suppose $\hat{\mathbf{x}} = (\hat{x}_1, \ldots, \hat{x}_s)$ is an isolated singular solution of $F = \{f_1, \ldots, f_t\}$. Let $P = (x_1 - \hat{x}_1, \ldots, x_s - \hat{x}_s)$ and I be an ideal having P-primary isolated component. Let

$$\mathrm{T}_k(F) = \{\mathrm{T}_k(f_1), \ldots, \mathrm{T}_k(f_t)\},$$

where $\mathrm{T}_k(f_i) = \sum_{|\alpha|<k} f_{i,\alpha}(\mathbf{x}-\hat{\mathbf{x}})^{\alpha}$ denotes truncated Taylor series expansions of the polynomial f_i at $\hat{\mathbf{x}}$ to order k.

The singular solution can be moved to the origin by changing of variables. For simplicity, we suppose $\hat{\mathbf{x}}$ is the origin and still use $M_k^{(0)}$ and $M_k^{(j)}$ to denote the coefficient matrices of the truncated polynomial system $\mathrm{T}_k(F)$ and truncated prolonged polynomial systems $\mathrm{T}_k(F^{(j)})$. Let $d_k^{(j)} = \dim \text{Nullspace}(M_k^{(j)})$ for $j \geq 0$. The coefficient matrices $M_k^{(0)}$ and $M_k^{(j)}$ are much smaller than the coefficient matrices of $F \cup P^k$ and their prolongations.

Theorem 7 *[Wu and Zhi 2008] The polynomial system $F_k = \mathrm{T}_k(F) \cup P^k$ is involutive at prolongation order m if and only if*

$$\dim \text{Nullspace}(M_k^{(m)}) = \dim \text{Nullspace}(M_k^{(m+1)}). \tag{5}$$

Proof: The lower submatrices of the symbol matrices of F_k and prolonged system $F_k^{(j)}$ are identity matrices corresponding to monomials in P^k and P^{k+j}. Therefore the symbol matrices are of full column rank, i.e., the symbols of $F_k^{(j)}$ are involutive according to (4). Furthermore, the dimensions of null spaces of coefficient matrices of the systems F_k and $F_k^{(j)}$ can be computed from the coefficient matrices $M_k^{(0)}$ and $M_k^{(j)}$ generated by the truncated systems $\mathrm{T}_k(F)$ and $\mathrm{T}_k(F^{(j)})$. □

4 Algorithm for Computing Isolated Primary Component

For a given isolated solution $\hat{\mathbf{x}}$ of the ideal $I = (f_1, \ldots, f_t)$, without loss of generality, suppose $\hat{\mathbf{x}}$ is the origin. Let Q be the isolated primary component whose associate prime is $P = (x_1 - \hat{x}_1, \ldots, x_s - \hat{x}_s)$, we compute the index ρ, such that $Q = (I, P^\rho)$ and the multiplication matrices of the quotient ring $\mathbb{C}[\mathbf{x}]/Q$.

Algorithm 1 *IsolatedPrimaryComponent*

Input: *An isolated multiple solution $\hat{\mathbf{x}}$ of an ideal $I = (f_1, \ldots, f_t)$ and a tolerance τ.*

Output: *Multiplicity μ, index ρ, and multiplication matrices $M_{x_1}, \ldots, M_{x_s}$ of the quotient ring $\mathbb{C}[\mathbf{x}]/Q$ where $Q = (I, P^\rho)$, $P = (x_1 - \hat{x}_1, \ldots, x_s - \hat{x}_s)$.*

- *Form the coefficient matrix $M_k^{(0)}$. The prolonged matrix $M_k^{(j)}$ is computed by shifting $M_k^{(0)}$ accordingly.*
- *Compute $d_k^{(j)} = \dim \text{Nullspace}(M_k^{(j)})$ for the given tolerance τ. The prolongation is stopped until $d_k^{(m)} = d_k^{(m+1)} = d_k$.*
- *If $d_k = d_{k-1}$, then set $\rho = k - 1$ and $\mu = d_\rho$ which are the index and multiplicity of the singular solution $\hat{\mathbf{x}}$.*
- *Compute the multiplication matrices $M_{x_1}, \ldots, M_{x_s}$ of $\mathbb{C}[\mathbf{x}]/Q$ from the null vectors of the matrix $M_{\rho+1}^{(m)}$, where $Q = (I, P^\rho)$.*

Remark 8 *If the solution is not at the origin, we can compute the matrix $M_k^{(0)}$ by changing of variables $y_i = x_i - \hat{x}_i$ for $i = 1, \ldots, s$, and compute the coefficients of polynomials $f_1(y_1 + \hat{x}_1, \ldots, y_s + \hat{x}_s), \ldots, f_t(y_1 + \hat{x}_1, \ldots, y_s + \hat{x}_s)$ with respect to the variables $y_1, \ldots, y_s$. However, since we only need the coefficients of $y_1^{\alpha_1} \cdots y_s^{\alpha_s}$ with total degree less than k to formulate the matrices $M_k^{(0)}$. Hence it is more efficient and numerically stable to obtain the coefficient matrix $M_k^{(0)}$ by computing the truncated multivariate Taylor series expansions of $f_1, \ldots, f_t$ at $\hat{\mathbf{x}}$ to order k. The prolonged matrix $M_k^{(j)}$ is computed by shifting $M_k^{(0)}$ accordingly.*

Remark 9 *Since all polynomials are truncated by degree k, the coefficient matrix $M_k^{(j)}$ has only $\binom{k+s-1}{s}$ columns. Furthermore, the number of prolongations m has an upper bound: $m \leq \max\{1, k - 1 - \min(\text{ldeg}(f_1), \ldots, \text{ldeg}(f_t))\}$, where $\text{ldeg}(f)$ denotes the lowest degree of f. The dimensions of the matrices appeared in the Algorithm 1 are bounded by $t\binom{\rho+s}{s} \times \binom{\rho+s}{s}$.*

Symbolic methods based on the uniqueness of the reduced Gröbner basis are given in [Gianni et al. 1988; Lakshman 1994] to determine the index of Q. However, when the singular solution is only known with finite precision, their

methods are subject to numerical stability problem.

Example 4.1 *[Ojika 1987] Consider the polynomials*

$$\{f_1 = x_1^2 + x_2 - 3,\ f_2 = x_1 + 0.125x_2^2 - 1.5\}.$$

The system has $(1, 2)$ *as a* 3*-fold solution.*

By changing of variables, we get a new system

$$\{g_1 = x_1^2 + 2x_1 + x_2,\ g_2 = x_1 + 0.125x_2^2 + 0.5x_2\},$$

which has a 3-fold solution $\hat{\mathbf{x}} = (0, 0)$. Let $I = (g_1, g_2)$ and $P = (x_1, x_2)$.

- $k = 2$. We have:

$$[\mathrm{T}_2(g_1), \mathrm{T}_2(g_2)]^T = M_2^{(0)} \cdot [x_1, x_2, 1]^T,$$

where

$$M_2^{(0)} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0.5 & 0 \end{bmatrix}$$

and $d_2^{(0)} = 2$. The prolonged matrix $M_2^{(1)}$ is obtained by adding zero elements to $M_2^{(0)}$. Hence $d_2^{(1)} = 2$ and

$$d_2 = \dim(\mathbb{C}[\mathbf{x}]/(I, P^2)) = 2.$$

- $k = 3$. We have:

$$[\mathrm{T}_3(g_1), \mathrm{T}_3(g_2)]^T = M_3^{(0)} \cdot \left[x_1^2, x_1x_2, x_2^2, x_1, x_2, 1\right]^T,$$

where

$$M_3^{(0)} = \begin{bmatrix} 1 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0.125 & 1 & 0.5 & 0 \end{bmatrix}$$

and $d_3^{(0)} = 4$. After the first prolongation, we have:

$$[\mathrm{T}_3(x_1g_1), \ldots, \mathrm{T}_3(x_2g_2), \mathrm{T}_3(g_1), \mathrm{T}_3(g_2)]^T = M_3^{(1)} \cdot \left[x_1^2, x_1x_2, x_2^2, x_1, x_2, 1\right]^T,$$

where

$$M_3^{(1)} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0.5 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0.125 & 1 & 0.5 & 0 \end{bmatrix}.$$

We have $d_3^{(1)} = 3$. The prolonged matrix $M_3^{(2)}$ is obtained by adding zeros to $M_3^{(1)}$, hence $d_3^{(2)} = 3$, and

$$d_3 = \dim(\mathbb{C}[\mathbf{x}]/(I, P^3)) = 3.$$

- $k = 4$. We compute $d_4^{(0)} = 4$, and $d_4^{(1)} = d_4^{(2)} = 3$. Hence,

$$d_4 = \dim(\mathbb{C}[\mathbf{x}]/(I, P^4)) = 3.$$

Since $d_3 = d_4 = 3$, the multiplicity of $\hat{\mathbf{x}} = (0, 0)$ is $\mu = 3$ and the index of the primary component is $\rho = 3$. Last but not least, multiplication matrices computed from null vectors of $M_4^{(1)}$ with respect to the normal set $\{x_1, x_2, 1\}$ are:

$$M_{x_1} = \begin{bmatrix} -2 & -1 & 0 \\ 4 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \; M_{x_2} = \begin{bmatrix} 4 & 2 & 0 \\ -8 & -4 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The primary component of I at $(0, 0)$ is

$$(x_1^2 + 2x_1 + x_2,\; x_2^2 + 8x_1 + 4x_2,\; x_1x_2 - 4x_1 - 2x_2).$$

Obviously, we can get the primary component of $(f_1,\ f_2)$ at $(1, 2)$ by changing the variables back:

$$(x_1^2 + x_2 - 3,\; x_2^2 + 8x_1 - 12,\; x_1x_2 - 6x_1 - 3x_2 + 10).$$

5 Algorithm for Computing a Basis of Max Noether Space

Let $D(\alpha) = D(\alpha_1, \ldots, \alpha_s) : \mathbb{C}[\mathbf{x}] \to \mathbb{C}[\mathbf{x}]$ denote the differential operator defined by:

$$D(\alpha_1, \ldots, \alpha_s) = \frac{1}{\alpha_1! \cdots \alpha_s!} \partial x_1^{\alpha_1} \cdots \partial x_s^{\alpha_s},$$

for non-negative integer array $\alpha = [\alpha_1, \ldots, \alpha_s]$. We write $\mathfrak{D} = \{D(\alpha),\ |\alpha| \geq 0\}$ and denote by $Span_{\mathbb{C}}(\mathfrak{D})$ the $\mathbb{C}$-vector space generated by $\mathfrak{D}$ and introduce a morphism on $\mathfrak{D}$ that acts as "integral":

$$\sigma_{x_j}(D(\alpha)) = \begin{cases} D(\alpha_1, \ldots, \alpha_j - 1, \ldots, \alpha_s), \text{ if } \alpha_j > 0, \\ 0, \qquad\qquad\qquad\qquad \text{otherwise.} \end{cases}$$

Definition 10 *A subspace L of $Span_{\mathbb{C}}(\mathfrak{D})$ is said to be closed if*

$$\sigma_{x_j}(L) \subseteq L,\ j = 1, \ldots, s.$$

Definition 11 *[Lasker 1905; Möller and Tenberg 2001] Given a zero* $\hat{\mathbf{x}} = (\hat{x}_1, \ldots, \hat{x}_s)$ *of an ideal* I*, we define the Max Noether space of* I *at* $\hat{\mathbf{x}}$ *as*

$$\triangle_{\hat{\mathbf{x}}} := \{L \in Span_{\mathbb{C}}(\mathfrak{D}) | \, L(f)|_{\mathbf{x}=\hat{\mathbf{x}}} = 0, \ \forall f \in I\}. \tag{6}$$

Theorem 12 *[Damiano et al. 2007] Let* M *be the maximal ideal* $(x_1, \ldots, x_s)$ *of* $\mathbb{C}[\mathbf{x}]$*. There is a bijective correspondence between* M*-primary ideals of* $\mathbb{C}[\mathbf{x}]$ *and closed subspaces of* $Span_{\mathbb{C}}(\mathfrak{D})$*:*

$$\{M\text{-primary ideals in } \mathbb{C}[\mathbf{x}]\}$$
$$\uparrow\downarrow$$
$$\{\text{closed subspaces of } \mathrm{Span}_{\mathbb{C}}(\mathfrak{D})\}.$$

Moreover, for a zero dimensional M*-primary ideal of* $\mathbb{C}[\mathbf{x}]$ *whose multiplicity is* μ*, we have that* $\dim_{\mathbb{C}}(\triangle_{\hat{\mathbf{x}}}) = \mu$*.*

Dayton and Zeng [Dayton and Zeng 2005] compute a set of basis from the dual space. We obtain a basis from studying the primal space directly.

Theorem 13 *[Wu and Zhi 2008] Let* $Q = (I, P^\rho)$ *be the isolated primary component of an ideal* $I = (f_1, \ldots, f_t)$ *at the multiple solution* $\hat{\mathbf{x}}$ *and* μ *be the multiplicity of* $\hat{\mathbf{x}}$*. Suppose the system* $F_\rho = \mathrm{T}_\rho(F) \cup P^\rho$ *is involutive at prolongation order* m*, the null space of the matrix* $M_\rho^{(m)}$ *is generated by vectors* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_\mu$*. Let*

$$\mathbf{L} = [D(\rho - 1, 0, \ldots, 0), D(\rho - 2, 1, 0, \ldots, 0), \ldots, D(0, \ldots, 0)]$$

denote the vector consists of all differential operators of order less than ρ*. Then a basis for the Max Noether space* $\triangle_{\hat{\mathbf{x}}}$ *can be computed as*

$$L_j = \mathbf{L} \cdot \mathbf{v}_j, \quad \text{for } 1 \leq j \leq \mu.$$

Algorithm 2 *MaxNoetherSpace*

Input: *An isolated multiple solution* $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_s)$ *of an ideal* $I = (f_1, \ldots, f_t)$ *and the index* ρ *of the primary component* $Q = (I, P^\rho)$*, a tolerance* τ*.*

Output: *A basis* $L = \{L_1, \ldots, L_\mu\}$ *of Max Noether space of* I *at* $\hat{\mathbf{x}}$*, where* μ *is the multiplicity.*

- *Compute* $d_\rho^{(j)} = \dim \mathrm{Nullspace}(M_\rho^{(j)})$ *for the given* τ*. The prolongation is stopped until* $d_\rho^{(m)} = d_\rho^{(m+1)}$ *and* $\mu = d_\rho^{(m)}$*.*
- *Compute the null vectors of* $M_\rho^{(m)}$ *denoted by* $\mathbf{v}_1, \ldots, \mathbf{v}_\mu$*. The basis is computed as* $L_j = \mathbf{L} \cdot \mathbf{v}_j$*, for* j *from* 1 *to* μ*.*

Example 4.1(continued) The null space of the matrix $M_3^{(1)}$ has dimension 3 and can be written as

$$N_3^{(1)} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -4 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ -8 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

A basis can be obtained by multiplying the differential operators of order less than 3 with the null vectors in $N_3^{(1)}$:

$$\begin{cases} L_1 = D(0,0), \\ L_2 = D(0,1) - D(2,0) + 2D(1,1) - 4D(0,2), \\ L_3 = D(1,0) - 2D(2,0) + 4D(1,1) - 8D(0,2). \end{cases}$$

6 Algorithm for Refining Approximate Singular Solution

Suppose we are given an approximate solution

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}_{\text{exact}} + \hat{\mathbf{x}}_{\text{error}},$$

where $\hat{\mathbf{x}}_{\text{error}}$ denotes the error in the solution and $\hat{\mathbf{x}}_{\text{exact}}$ denotes the exact solution of the polynomial system $F = \{f_1, \ldots, f_t\}$ with multiplicity μ and index ρ. By changing of variables $y_i = x_i - \hat{x}_i$, $i = 1, \ldots, s$, we obtain a new polynomial system $G = \{g_1, \ldots, g_t\}$, where

$$g_j(y_1, \ldots, y_s) = f_j(y_1 + \hat{x}_{1,\text{exact}} + \hat{x}_{1,\text{error}}, \ldots, y_s + \hat{x}_{s,\text{exact}} + \hat{x}_{s,\text{error}}), \quad j = 1, \ldots, t.$$

Next lemma may be clear at a glance, but it is important for our proof of convergence.

Lemma 14 *The polynomial system G has an exact solution*

$$\hat{\mathbf{y}} = -\hat{\mathbf{x}}_{\text{error}} = (-\hat{x}_{1,\text{error}}, \ldots, -\hat{x}_{s,\text{error}}) \tag{7}$$

with the same index ρ and multiplicity μ as $\hat{\mathbf{x}}_{\text{exact}}$ to F.

Proof: It is clear that $g_i(-\hat{\mathbf{x}}_{\text{error}}) = f_i(\hat{\mathbf{x}}_{\text{exact}}) = 0$ for $i = 1, \ldots, t$. Moreover, given a set of basis $\{L_1, \ldots, L_\mu\}$ of Max Noether space of F at $\hat{\mathbf{x}}_{\text{exact}}$, for $j = 1, \ldots, \mu$, and $i = 1, \ldots, t$,

$$L_j(g_i(\mathbf{y})) \mid_{\mathbf{y}=-\hat{\mathbf{x}}_{\text{error}}} = L_j(f_i(\mathbf{y} + \hat{\mathbf{x}})) \mid_{\mathbf{y}=-\hat{\mathbf{x}}_{\text{error}}} = L_j(f_i(\mathbf{x})) \mid_{\mathbf{x}=\hat{\mathbf{x}}_{\text{exact}}} = 0.$$

Hence $\{L_1, \ldots, L_\mu\}$ is also a set of basis of Max Noether space of G at $\hat{\mathbf{y}}$. □

Consider the polynomial system

$$\bar{G} = \{g_1, \ldots, g_t, \quad (\mathbf{y} + \hat{\mathbf{x}}_{\text{error}})^\alpha, |\alpha| = \rho + 1\},$$

where ρ is the index of $-\hat{\mathbf{x}}_{\text{error}}$. By Lemma 14, we know that this system has only one exact multiple solution $-\hat{\mathbf{x}}_{\text{error}}$ with index ρ. The polynomials in $\bar{G}$ generate an isolated primary component denoted by $\bar{Q}$. Suppose the system $\bar{G}$ is involutive after m prolongations. We denote its coefficient matrix by

$$M = \begin{bmatrix} M'_h & M'_l \\ M_h & M_l \end{bmatrix}, \tag{8}$$

where $[M'_h\, M'_l]$ and $[M_h\, M_l]$ are coefficient matrices of polynomial systems $\{(\mathbf{y}+\hat{\mathbf{x}}_{\text{error}})^\alpha, |\alpha| = \rho+1\}$, $\{g_1, \ldots, g_t\}$ and their prolongations with respective to the high order monomials

$$[\mathbf{y}^{\rho+m}, \ldots, \mathbf{y}^{\rho+1}]$$

and low order monomials

$$[\mathbf{y}^\rho, \ldots, \mathbf{y}, 1].$$

Notice that

$$M_l = M^{(m)}_{\rho+1}, \tag{9}$$

where $M^{(m)}_{\rho+1}$ is the coefficient matrix of the truncated system

$$G_{\rho+1} = \{\mathrm{T}_k(g_1), \ldots, \mathrm{T}_k(g_t)\}$$

prolonged to order m, and

$$M'_h = \begin{bmatrix} I_{\rho+m} & \cdots & \tilde{M}_h \\ & \ddots & \vdots \\ & & I_{\rho+1} \end{bmatrix}, \tag{10}$$

where $I_{\rho+i}$ is an identity matrix with dimension $\binom{s+\rho+i-1}{\rho+i}$.

Theorem 15 *Suppose $\{L_1, \ldots, L_\mu\}$ is a set of basis of Max Noether space of G at $\hat{\mathbf{y}}$ and M is the coefficient matrix of the involutive form of $\bar{G}$ after m*

prolongations, we have

$$\{L_1(\mathbf{v}(\mathbf{y})_{\rho+m}) \mid_{\mathbf{y}=\hat{\mathbf{y}}}, \ldots, L_\mu(\mathbf{v}(\mathbf{y})_{\rho+m}) \mid_{\mathbf{y}=\hat{\mathbf{y}}}\} \tag{11}$$

is a basis of null space of the matrix M, where $\hat{\mathbf{y}} = -\hat{\mathbf{x}}_{\text{error}}$ and

$$\mathbf{v}(\mathbf{y})^T_{\rho+m} = [y_1^{\rho+m}, \ldots, y_s^{\rho+1}, y_1^{\rho}, \ldots, y_s, 1]^T. \tag{12}$$

Proof: Since $\{L_1, \ldots, L_\mu\}$ is a set of basis of Max Noether space of G at $\hat{\mathbf{y}}$, for any $L_i \in L$, $i = 1, \ldots, \mu$, we have

$$L_i(M \cdot [y_1^{\rho+m}, \ldots, y_s^{\rho+1}, y_1^{\rho}, \ldots, \ldots, y_s, 1]^T) \mid_{\mathbf{y}=\hat{\mathbf{y}}} = \mathbf{0}.$$

It is clear that

$$M[L_i(y_1^{\rho+m}), \ldots, L_i(y_s^{\rho+1}), L_i(y_1^{\rho}), \ldots, L_i(y_s), L_i(1)]^T \mid_{\mathbf{y}=\hat{\mathbf{y}}} = \mathbf{0}.$$

Hence $L_i(\mathbf{v}(\mathbf{y})_{\rho+m}) \mid_{\mathbf{y}=\hat{\mathbf{y}}}$, $i = 1, \ldots, \mu$ are null vectors of M.

By the definition of index, we can select μ monomials from the monomial vector $\mathbf{v}(\mathbf{y})_{\rho-1}$, which consists of all monomials of order at most $\rho - 1$, to constitute a basis of $\mathbb{C}[\mathbf{y}]/\bar{Q}$. Since $\{L_1 \mid_{\mathbf{y}=\hat{\mathbf{y}}}, \ldots, L_\mu \mid_{\mathbf{y}=\hat{\mathbf{y}}}\}$ is a basis of dual space of $\mathbb{C}[\mathbf{y}]/\bar{Q}$, we have that

$$L_1(\mathbf{v}(\mathbf{y})_{\rho-1}) \mid_{\mathbf{y}=\hat{\mathbf{y}}}, \ldots, L_\mu(\mathbf{v}(\mathbf{y})_{\rho-1}) \mid_{\mathbf{y}=\hat{\mathbf{y}}} \tag{13}$$

are linearly independent. Therefore, $L_1(\mathbf{v}(\mathbf{y})_{\rho+m}) \mid_{\mathbf{y}=\hat{\mathbf{y}}}, \ldots, L_\mu(\mathbf{v}(\mathbf{y})_{\rho+m}) \mid_{\mathbf{y}=\hat{\mathbf{y}}}$ are linearly independent. □

If we form multiplication matrices $\{M_{y_1}, \ldots, M_{y_s}\}$ using null vectors of M, then $\frac{1}{\mu} Tr(M_{y_i}) = -\hat{x}_{i,\text{error}}$ since the system $\bar{G}$ has only one solution $\hat{\mathbf{y}} = -\hat{\mathbf{x}}_{\text{error}}$ with multiplicity μ.

Suppose the given approximate solution $\hat{\mathbf{x}}$ of F is not far away from the exact solution $\hat{\mathbf{x}}_{\text{exact}}$, i.e.,

$$\|\hat{\mathbf{y}}\| = \| - \hat{\mathbf{x}}_{\text{error}}\| = \mathcal{O}(\varepsilon) \ll 1. \tag{14}$$

Here and hereafter, $\| \cdot \|$ is denoted as the infinity norm. We prove that an approximate solution of $\hat{\mathbf{y}}$ of double precision can be computed from the null vectors of the matrix $M^{(m)}_{\rho+1}$. Therefore, there is no need to construct and work with the full size big matrix M (8).

Corollary 16 *Suppose $\mathbf{v} = [\mathbf{v}_h^T, \mathbf{v}_l^T]^T$ is a normalized null vector of the big matrix M defined in (8), where $\mathbf{v}_h$ and $\mathbf{v}_l$ are column vectors of length $\binom{\rho+m+s}{\rho+m} - \binom{\rho+s}{\rho}$ and $\binom{\rho+s}{\rho}$ respectively, then $\|\mathbf{v}_h\| = \mathcal{O}(\varepsilon^2)$.*

Proof: By Theorem 15, we know that

$$\{L_1(\mathbf{v}(\mathbf{y})_{\rho+m}) \mid_{\mathbf{y}=\hat{\mathbf{y}}}, \ldots, L_\mu(\mathbf{v}(\mathbf{y})_{\rho+m}) \mid_{\mathbf{y}=\hat{\mathbf{y}}}\}$$

is a basis of null space of the matrix M.

Pick up a differential operator of the highest order, without loss of generality, denoted it by L_μ, its order is $\rho - 1$. Applying it to the monomials of order α, we obtain

$$L_\mu(\mathbf{y}^\alpha) = \sum\nolimits_{|\beta|=|\alpha|-\rho+1} c_\beta \mathbf{y}^\beta, \quad c_\beta \in \mathbb{C}. \tag{15}$$

If $|\alpha| \geq \rho + 1$, then $|\beta| \geq 2$, hence $\| L_\mu(\mathbf{y}^\alpha) \mid_{\mathbf{y}=\hat{\mathbf{y}}} \| = \mathcal{O}(\varepsilon^2)$ as $\|\hat{\mathbf{y}}\| = \mathcal{O}(\varepsilon)$.

For i from 1 to μ, we have

$$\| (L_i([\mathbf{y}^{\rho+m}, \ldots, \mathbf{y}^{\rho+1}]^T) \mid_{\mathbf{y}=\hat{\mathbf{y}}} \| \leq \| (L_\mu([\mathbf{y}^{\rho+m}, \ldots, \mathbf{y}^{\rho+1}]^T) \mid_{\mathbf{y}=\hat{\mathbf{y}}} \| = \mathcal{O}(\varepsilon^2). \tag{16}$$

Moreover, by duality, we have

$$\| L_i(\mathbf{v}(\mathbf{y})_{\rho-1}) \mid_{\mathbf{y}=\hat{\mathbf{y}}} \| = \mathcal{O}(1). \tag{17}$$

For any normalized null vector $\mathbf{v} = [\mathbf{v}_h^T, \mathbf{v}_l^T]^T$ of M, $\mathbf{v}$ can be written as linear combination of $L_i(\mathbf{v}(\mathbf{y})_{\rho+m}) \mid_{\mathbf{y}=\hat{\mathbf{y}}}$ and $\mathbf{v}_h$ corresponds to the linear combination of terms $\| L_i(\mathbf{y}^\alpha) \mid_{\mathbf{y}=\hat{\mathbf{y}}} \| = \mathcal{O}(\varepsilon^2)$ for $|\alpha| \geq \rho + 1$ and $i = 1, \ldots, \mu$. We have $\|\mathbf{v}_h\| = \mathcal{O}(\varepsilon^2)\|\mathbf{v}_l\| = \mathcal{O}(\varepsilon^2)$ because of (16),(17) and $\|\mathbf{v}\| = 1$. □

For the matrix M defined in (8), due to the special structure of the matrix M_h' displayed in (10), there exists an invertible matrix P_1, such that $P_1 M_h' = I$.

$$\tilde{M} = \begin{bmatrix} I & \mathbf{0} \\ -M_h & I \end{bmatrix} \begin{bmatrix} P_1 & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} M_h' & M_l' \\ M_h & M_l \end{bmatrix} = \begin{bmatrix} I & \tilde{M}_l \\ \mathbf{0} & M_l - M_h \tilde{M}_l \end{bmatrix},$$

where, $\tilde{M}_l = P_1 M_l'$. Thence, computing null vectors of M is equivalent to computing null vectors of $M_l - M_h \tilde{M}_l$.

Suppose $\mathbf{v} = [\mathbf{v}_h^T, \mathbf{v}_l^T]^T$ is a normalized null vector of M,

$$\begin{bmatrix} I & \tilde{M}_l \\ \mathbf{0} & M_l - M_h \tilde{M}_l \end{bmatrix} \begin{bmatrix} \mathbf{v}_h \\ \mathbf{v}_l \end{bmatrix} = \mathbf{0}.$$

Since $\mathbf{v}_h + \tilde{M}_l \mathbf{v}_l = \mathbf{0}$, according to Corollary 16,

$$\|\tilde{M}_l \mathbf{v}_l\| = \|\mathbf{v}_h\| = \mathcal{O}(\varepsilon^2).$$

Furthermore, since $M_l\mathbf{v}_l - M_h\tilde{M}_l\mathbf{v}_l = \mathbf{0}$, we have

$$\|M_l\mathbf{v}_l\| = \|M_h\tilde{M}_l\mathbf{v}_l\| \leq \|M_h\|\|\tilde{M}_l\mathbf{v}_l\| = \mathcal{O}(\varepsilon^2).$$

We have the following theorem:

Theorem 17 *Suppose* $\{L_1, \ldots, L_\mu\}$ *is a set of basis of Max Noether space of* G *at* $\hat{\mathbf{y}}$ *and the truncated system*

$$G_{\rho+1} = \{\mathrm{T}_k(g_1), \ldots, \mathrm{T}_k(g_t)\}$$

is involutive after m *prolongations, whose coefficient matrix is* $M_l = M_{\rho+1}^{(m)}$. *For the threshold* $\mathcal{O}(\varepsilon^2)$, *we have that*

$$\{L_1(\mathbf{v}(\mathbf{y})_\rho) \mid_{\mathbf{y}=\hat{\mathbf{y}}}, \ldots, L_\mu(\mathbf{v}(\mathbf{y})_\rho) \mid_{\mathbf{y}=\hat{\mathbf{y}}}\} \tag{18}$$

is a basis of null space of the matrix $M_{\rho+1}^{(m)}$, *where* $\hat{\mathbf{y}} = -\hat{\mathbf{x}}_{\text{error}}$ *and*

$$\mathbf{v}(\mathbf{y})_\rho^T = [y_1^\rho, y_1^{\rho-1}y_2, \ldots, y_1, \ldots, y_s, 1]^T.$$

Proof: By Theorem 15,

$$\{L_1(\mathbf{v}(\mathbf{y})_{\rho+m}) \mid_{\mathbf{y}=\hat{\mathbf{y}}}, \ldots, L_\mu(\mathbf{v}(\mathbf{y})_{\rho+m}) \mid_{\mathbf{y}=\hat{\mathbf{y}}}\}$$

is a basis of null space of the matrix M. According to Corollary 16 and analysis given above, we have

$$\|M_{\rho+1}^{(m)}L_i(\mathbf{v}(\mathbf{y})_\rho) \mid_{\mathbf{y}=\hat{\mathbf{y}}} \| = \mathcal{O}(\varepsilon^2), \quad 1 \leq i \leq \mu.$$

By virtue of (13), these null vectors (18) are linearly independent. □

Remark 18 *According to Theorem 17, if we choose a threshold* $\mathcal{O}(\varepsilon^2)$ *to compute the rank of* M_l, *then we will root out*

$$\text{dim Nullspace}(M_{\rho+1}^{(m)}) = \text{dim Nullspace}(M).$$

Furthermore, the multiplication matrices $\{\tilde{M}_{y_1}, \ldots, \tilde{M}_{y_s}\}$ *are formed from the null vectors of* $M_{\rho+1}^{(m)}$ *by linear system solving, see [Reid and Zhi 2008] for details. Hence, by Theorem 15 and Theorem 17, we have*

$$\frac{1}{\mu}Tr(\tilde{M}_{y_i}) = \frac{1}{\mu}Tr(M_{y_i}) + \mathcal{O}(\varepsilon^2) = -\hat{x}_{i,\text{error}} + \mathcal{O}(\varepsilon^2). \tag{19}$$

Therefore, we can get the error part of the singular solution with double correct digits.

Based on the above discussions, we are ready to give an algorithm to refine an approximate singular solution.

Algorithm 3 *MultipleRootRefiner*

Input: *An isolated approximate singular solution* $\hat{\mathbf{x}}$ *of an ideal* $I = (f_1, \ldots, f_t)$ *and a tolerance* τ.

Output: *Refined solution* $\hat{\mathbf{x}}$, *the multiplicity* μ *and index* ρ *of the primary component* $Q = (I, P^\rho)$ *and a basis* $L = \{L_1, \ldots, L_\mu\}$ *of Max Noether space of* I *at the refined solution.*

- *For given approximate root* $\hat{\mathbf{x}}$ *and tolerance* τ, *applying Algorithm 1 to estimate the multiplicity* μ *and index* ρ.
- *Suppose the truncated system* $G_{\rho+1}$ *is involutive at prolongation order* m, *then form multiplication matrices* $M_{x_1}, \ldots, M_{x_s}$ *from null vectors of the matrix* $M_{\rho+1}^{(m)}$. *An approximate solution* $\hat{\mathbf{y}}$ *is obtained by averaging the trace of each multiplication matrix.*
- *Set* $\hat{\mathbf{x}} = \hat{\mathbf{x}} + \hat{\mathbf{y}}$ *and run the first two steps for the refined solution. The tolerance is decreased according to the solution* $\hat{\mathbf{y}}$.
- *If* $\hat{\mathbf{y}}$ *converges to the origin, then we get the refined solution* $\hat{\mathbf{x}}$ *with high accuracy. We apply Algorithm 1 to compute the primary component and Algorithm 2 to compute a basis of Max Noether space of* I *at the refined solution. Otherwise, we decrease the tolerance and run the above steps again.*

Theorem 19 *Algorithm 3 provides a stable quadratically convergent method to refine a multiple solution.*

Proof: The quadratical convergency follows from Theorem 17 and Remark 18. Our algorithm is stable since we use singular value decomposition to compute the dimensions and null vectors of the coefficient matrices and formulate the multiplication matrices. Moreover, we form the coefficient matrices by computing the Taylor expansions up to desired order instead of changing variables and expanding the polynomials. □

We apply symbolic and numeric perturbations to Example 4.1 to illustrate the quadratical convergence of Algorithm 3.

Example 4.1 (continued) Suppose we are given an approximate singular solution $\hat{\mathbf{x}} = (1 + \varepsilon,\ 2 + \varepsilon)$ and set $\rho = 3$. Applying Gaussian elimination to

$M_4^{(1)}$, we get an upper triangular matrix of 12×10:

$$\begin{bmatrix}
I_{5\times5} & \star & \star & \star & \star & \star \\
\mathbf{0}_{1\times5} & 1+\frac{1}{2}\varepsilon & \star & \star & \star & \star \\
\mathbf{0}_{1\times5} & 0 & 1 & \star & \star & \star \\
\mathbf{0}_{1\times5} & 0 & 0 & -2\varepsilon^2+\mathcal{O}(\varepsilon^3) & -\frac{1}{4}\varepsilon^2+\mathcal{O}(\varepsilon^3) & -\frac{3}{16}\varepsilon^3+\mathcal{O}(\varepsilon^4) \\
\mathbf{0}_{1\times5} & 0 & 0 & 0 & -\frac{1}{144}\varepsilon^3+\mathcal{O}(\varepsilon^4) & -\frac{1}{192}\varepsilon^4+\mathcal{O}(\varepsilon^5) \\
\mathbf{0}_{1\times5} & 0 & 0 & 0 & 0 & -\frac{3}{32}\varepsilon^4+\mathcal{O}(\varepsilon^5) \\
\mathbf{0}_{1\times5} & 0 & 0 & 0 & 0 & 0 \\
\mathbf{0}_{1\times5} & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

where $\star$ represents rational function of ε.

It is clear that elements of the last five rows of the above matrix are $\mathcal{O}(\varepsilon^2)$. Let us take off the last five rows and compute null vectors and form multiplication matrices with respect to the normal set $\{x_1, x_2, 1\}$:

$$M_{x_1} = \begin{bmatrix}
-2-2\varepsilon & -1 & -3\varepsilon-\varepsilon^2 \\
4-\varepsilon+\mathcal{O}(\varepsilon^2) & 2-\varepsilon+\mathcal{O}(\varepsilon^2) & 6\varepsilon+\mathcal{O}(\varepsilon^2) \\
1 & 0 & 0
\end{bmatrix},$$

$$M_{x_2} = \begin{bmatrix}
4-\varepsilon+\mathcal{O}(\varepsilon^2) & 2-\varepsilon+\mathcal{O}(\varepsilon^2) & 6\varepsilon+\mathcal{O}(\varepsilon^2) \\
-8 & -4-2\varepsilon & -12\varepsilon-\varepsilon^2 \\
0 & 1 & 0
\end{bmatrix}.$$

The average of the trace of M_{x_1} is the same as M_{x_2}, which is $-\varepsilon + \mathcal{O}(\varepsilon^2)$. Adding it to $\hat{\mathbf{x}}$, we get a refined solution $(1 + \mathcal{O}(\varepsilon^2), 2 + \mathcal{O}(\varepsilon^2))$.

Example 4.1 (continued) Suppose we are given an approximate solution $\hat{\mathbf{x}} = (1.001, 1.998)$. Applying Algorithm 1 to estimate multiplicity and index for a given tolerance $\tau = 10^{-3}$.

- The singular values of $M_3^{(1)}$ are:

  $$\{3.1234, \cdots, 1.8285, 2.8400 \times 10^{-4}, 4.4717 \times 10^{-10}, 1.3509 \times 10^{-20}\}$$

  and $d_3^{(1)} = 3$.
- The singular values of $M_3^{(2)}$ are:

  $$\{3.1234, \cdots, 1.8285, 2.8400 \times 10^{-4}, 2.3352 \times 10^{-9}, 3.2703 \times 10^{-13}\}$$

and $d_3^{(2)} = 3$.

- The singular values of $M_4^{(1)}$ are:

$$\{3.5118, \cdots, 4.6516 \times 10^{-2}, 1.2633 \times 10^{-6}, 6.1389 \times 10^{-13}, 6.4274 \times 10^{-20}\}$$

and $d_4^{(1)} = 3$.

So that for the given tolerance $\tau = 10^{-3}$, the multiplicity is $\mu = 3$ and the index is $\rho = 3$. The multiplication matrices are formed from the null space of $M_4^{(1)}$. The solution computed by averaging the traces of the multiplication matrices is

$$\hat{\mathbf{y}} = (-0.001000696, 0.002003323).$$

Adding $\hat{\mathbf{y}}$ to $\hat{\mathbf{x}}$, we obtain the refined solution $\hat{\mathbf{x}}$. We apply Algorithm 3 twice to the new singular solution $\hat{\mathbf{x}}$ for tolerance 10^{-5} and 10^{-8} respectively, and get the refined solution:

$$\hat{\mathbf{x}} = (1 - 3.5470 \times 10^{-16}, 2 - 2.3068 \times 10^{-15}).$$

7 Complexity and Experiments

There are quantity of masterpieces on designing algorithms for computing involutive bases of polynomial systems [Apel 1995; Gerdt and Blinkov 1998; Seiler 2002]. However, they seldom talk about complexity of algorithms. In [Chistov and Gigorier 2007], they work out a double-exponential complexity bound of constructing a Janet basis of a D-module. In [Gerdt and Zinin 2008], they give some bound estimations for cardinality of Boolean Gröbner bases. As to Boolean Pommaret bases, they give an example of single-exponential cardinality and they have a conjecture that their exact cardinality is single-exponential. Since we are working with the special polynomial system $F_k = \mathrm{T}_k(F) \cup P^k$. It has been shown in the above sections that the truncated coefficient matrices $M_k^{(j)}$ have provided us enough information to compute the multiplicity structure of the singular solution and refine an approximate multiple solution. Hence we have the following combinatorial complexity for our algorithms.

Theorem 20 *The complexity of Algorithm 2 is:*

$$\mathcal{O}\left(t\binom{\rho+s-1}{s}^3\right).$$

Proof: Algorithm 2 computes null vectors of a matrix of size at most $t\binom{\rho+s-1}{s} \times \binom{\rho+s-1}{s}$. □

Remark 21 *In [Mourrain 1996], the complexity of the algorithm to compute differential operators is* $\mathcal{O}((s^2+t)\mu^3)$. *Generally speaking,* $\mu \leq \binom{\rho+s-1}{s}$. *We need more operations because we do prolongations in all directions. In some sense, we sacrifice efficiency to ensure numeric stability.*

Theorem 22 *The complexities of Algorithm 1 and Algorithm 3 are:*

$$\mathcal{O}\left(t\binom{\rho+s}{s}^3\right)$$

Proof: The size of the matrices used in Algorithm 1 and Algorithm 3 are bounded by $t\binom{\rho+s}{s} \times \binom{\rho+s}{s}$.

Remark 23 *The complexity of the deflation algorithm for refining a singular solution [Lecerf 2002] is bounded by the multiplicity which could be much smaller than ours.*

The following experiments are done for Digits := 14 in Maple 11 under Windows. The systems DZ1 and DZ2 are quoted from [Dayton and Zeng 2005]. The system D2 [Dayton 2007] is positive dimensional, but we can compute its isolated zero dimensional primary component at the origin. The other examples are cited from the PHCpack demos http://www.math.uic.edu/~jan/. We use s, ρ and μ to represent the number of variables, the index and the multiplicity respectively. The fifth column lists the increase in the number of correct digits from the approximate solutions obtained by Algorithm MultipleRootRefiner. The Maple code of three algorithms and test results are available http://www.mmrc.iss.ac.cn/~lzhi/Research/hybrid/polysolver.

8 Conclusion

The multiplicity structure of a singular solution has been studied extensively in [Bates et al. 2006; Damiano et al. 2007; Dayton 2007; Dayton and Zeng 2005; Kobayashi et al. 1998; Marinari et al. 1995, 1996; Möller and Stetter 1995; Möller and Tenberg 2001; Mourrain 1996]. Various methods have been proposed for computing the singular solutions to high accuracy [Corless et al. 1997; Lecerf 2002; Leykin et al. 2006, 2008; Ojika 1987; Ojika et al. 1983]. In this paper, we describe algorithms based on the geometric involutive form to completely describe the multiplicity structure of an isolated singular solution.

If the polynomial system and the singular solutions are known exactly, the tolerance is set to be zero. We compute the multiplicity, index and a basis of Max Noether space by exact linear algebra computation. If we are given an approximate isolated singular solution of an exact polynomial system, then we

System	s	ρ	μ	# Digits
cmbs1	3	5	11	$2 \to 7 \to 14$
cmbs2	3	4	8	$2 \to 5 \to 14$
mth191	3	3	4	$2 \to 6 \to 13 \to 15$
LVZ	3	8	18	$4 \to 7 \to 14$
KSS	5	5	16	$3 \to 7 \to 13 \to 14$
Caprasse	4	3	4	$3 \to 9 \to 12 \to 13$
DZ1	4	11	131	$2 \to 8 \to 15$
DZ2	3	8	16	$3 \to 7 \to 14$
tangents1	6	4	4	$2 \to 6 \to 12 \to 13$
D2	3	5	5	$2 \to 4 \to 7 \to 14$
Ojika1	2	3	3	$2 \to 4 \to 8 \to 14$
Ojika2	3	2	2	$2 \to 4 \to 9 \to 13$
Ojika3	3	3	4	$2 \to 4 \to 9 \to 13$
Ojika4	3	3	3	$2 \to 6 \to 13$
Cyclic9	9	3	4	$3 \to 5 \to 11 \to 13$

Table 1. Algorithm Performance

apply generalized quadratic Newton iterations to refine the singular solution to have high accuracy and obtain accurate multiplicity structure with respect to the refined solution.

Our refinement procedure is different from deflation methods in [Leykin et al. 2006, 2008; Ojika 1987; Ojika et al. 1983]. They restore quadratic convergence of a modified Newton's method by producing a new polynomial system which has the original multiple solution as a simple one. Since the column dimension of the matrix we used to refine an approximate singular solution is $\binom{\rho+s}{s}$. Our algorithm for refining an approximate singular solution is not efficient when the index ρ is big. For the example in [Kobayashi et al. 1998], the number of variables is 10, and the index can be as large as 11, we still have trouble to deal with this one. We are going to explore more algebraic structure to reduce the size of the matrices in our methods as Zeng did in his latest work [Zeng 2009], which aims at developing a new method to improve the computational efficiency for large systems.

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References

Apel, Joachim. A Gröbner approach to involutive bases. *J. Symbolic Comput.*, 19:441–457, 1995.

Bates, D., Peterson, C., and Sommese, A. A numerical-symbolic algorithm for computing the multiplicity of a component of an algebraic set. *Journal of Complexity*, 22:475–489, 2006.

Bonasia, J., Lemaire, F., Reid, G., Scott, R., and Zhi, L. Determination of approximate symmetries of differential equations. In *CRM Proceedings and Lecture Notes*, pages 233–249, 2004.

Chistov, A. and Gigorier, D. Complexity of a Janet basis of a D-module. Preprints in MPIM, 2007.

Corless, R., Gianni, P., and Trager, B. A reordered Schur factorization method for zero-dimensional polynomial systems with multiple roots. In Küchlin, editor, *Proc. 1997 Internat. Symp. Symbolic Algebraic Comput. ISSAC'97*, pages 133–140, New York, 1997. ACM Press. ISBN 0-89791-875-4.

Cox, D., Little, J., and O'Shea, D. *Ideals, Varieties, and Algorithms*. Springer-Verlag, New York, 1992. Undergraduate Texts in Mathematics.

Damiano, A., Sabadini, I., and Struppa, D. Computational methods for the construction of a class of Noetherian operators. *Experiment. Math.*, 16: 41–55, 2007.

Dayton, B. Numerical local rings and local solutions of nonlinear systems. In Verschelde, Jan and Watt, Stephen M., editors, *SNC'07 Proc. 2007 Internat. Workshop on Symbolic-Numeric Comput.*, pages 79–86, New York, N. Y., 2007. ACM Press. ISBN 978-1-59593-744-5.

Dayton, B. and Zeng, Z. Computing the multiplicity structure in solving polynomial systems. In Kauers, Manuel, editor, *ISSAC '05 Proc. 2005 Internat. Symp. Symbolic Algebraic Comput.*, pages 116–123, New York, N. Y., 2005. ACM Press. ISBN 1-59593-095-7.

Gerdt, V. and Blinkov, Y. Involutive bases of polynomial ideals. *Mathematics and Computers in Simulation*, 45:519–541, 1998.

Gerdt, V. and Zinin, M. A Pommaret division algorithm for computing Gröbner bases on Boolean rings. In [Jeffrey 2008], pages 95–102, 2008.

Gianni, P., Trager, B., and Zacharias, G. Gröbner bases and primary decomposition of polynomial ideals. *J. Symbolic Computation*, 6:149–167, 1988.

Jeffrey, David, editor. *ISSAC 2008 Proc. 2008 Internat. Symp. Symbolic Algebraic Comput.*, 2008. ACM Press. ISBN 978-1-59593-904-3.

Kobayashi, H., Suzuki, H., and Sakai, Y. Numerical calculation of the multiplicity of a solution to algebraic equations. *Math. Comp.*, 67(221):257–270, 1998.

Lakshman, Y. A single exponential bound on the complexity of computing Gröbner bases of zero dimensional ideals. *Effective Methods in Algebraic Geometry, Progress in Mathematics*, pages 227–234, 1994.

Lasker, E. Zur theorie der moduln und ideale. *Math. Ann.*, 60:20–116, 1905.

Lecerf, G. Quadratic Newton iteration for systems with multiplicity. *Foundations of Computational Mathematics*, 2(3):247–293, 2002.

Leykin, A., Verschelde, J., and Zhao, A. Newton's method with deflation for isolated singularities of polynomial systems. *Theoretical Computer Science*, 359:111–122, 2006.

Leykin, A., Verschelde, J., and Zhao, A. Newton's method with deflation for isolated singularities of polynomial systems. *IMA Volume 146: Algorithms in Algebraic Geometry, edited by Alicia Dickenstein, Frank-Olaf Schreyer, and Andrew J. Sommese*, pages 79–97, 2008.

Marinari, M., Mora, T., and Möller, H. Gröbner duality and multiplicities in polynomial solving. In Levelt, A. H. M., editor, *Proc. 1995 Internat. Symp. Symbolic Algebraic Comput. (ISSAC'95)*, pages 167–179, New York, N. Y., 1995. ACM Press.

Marinari, M., Mora, T., and Möller, H. On multiplicities in polynomial system solving. *Trans. Amer. Math. Soc.*, 348:3283–3321, 1996.

Möller, H. and Stetter, H. Multivariate polynomial equations with multiple zeros solved by matrix eigenproblems. *Numer. Math.*, 70:311–329, 1995.

Möller, H. and Tenberg, R. Multivariate polynomial system solving using intersections of eigenspaces. *J. symbolic Computation*, 30:1–19, 2001.

Mourrain, B. Isolated points, duality and residues. *J. of Pure and Applied Algebra*, 117 & 118:469–493, 1996.

Ojika, T. Modified deflation algorithm for the solution of singular problems. *J. Math. Anal. Appl.*, 123:199–221, 1987.

Ojika, T., Watanabe, S., and Mitsui, T. Deflation algorithm for the multiple roots of a system of nonlinear equations. *J. Math. Anal. Appl.*, 96:463–479, 1983.

Reid, G., Tang, J., and Zhi, L. A complete symbolic-numeric linear method for camera pose determination. In Sendra, J., editor, *Proc. 2003 Internat. Symp. Symbolic Algebraic Comput. ISSAC'03*, pages 215–223, New York, 2003. ACM Press. ISBN 1-58113-641-2.

Reid, G. and Zhi, L. Solving polynomial systems via symbolic-numeric reduction to geometric involutive form. *J. symbolic Computation*, 2008.

Seiler, W. *Involution - The formal theory of differential equations and its applications in computer algebra and numerical analysis*. Habilitation thesis, Univ. of Mannheim, Germany, 2002.

van der Waerden B. L. *Algebra*. Frederick Ungar Pub. Co., 1970.

Wittkopf, A. and Reid, G. Fast differential elimination in C: The CDiffelim Environment. *Comp. Phys. Comm.*, 139(2):192–217, 2001.

Wu, X. and Zhi, L. Computing the multiplicity structure from geometric involutive form. In [Jeffrey 2008], pages 325–332, 2008.

Zeng, Z. The closedness subspace method for computing the multiplicity structure of a polynomial system. To appear, 2009.

Zhi, L. and Reid, G. Solving nonlinear polynomial system via symbolic-numeric elimination method. In Faugére, J. and Rouillier, F., editors, *Proc. International conference on polynomial system solving*, pages 50–53, 2004.