Approximate GCDs of Polynomials and SOS Relaxation

Bin Li
Key Laboratory of Mathematics Mechanization
Chinese Academy of Sciences
Beijing 100080, China
bli@mmrc.iss.ac.cn

Jiawang Nie
Institute for Mathematics and Applications
University of Minnesota
Minneapolis, MN 55455, USA
njw@ima.umn.edu

Lihong Zhi
Key Laboratory of Mathematics Mechanization
Chinese Academy of Sciences
Beijing 100080, China
lzhi@mmrc.iss.ac.cn

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The problem of computing the approximate GCD of several polynomials $f_1, \ldots, f_s \in F[x_1, \ldots, x_n]$, where F is either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$, can be written as

$$\min_{p,u_1,\ldots,u_s} \|f_1 - p\cdot u_1\|_2^2 + \|f_2 - p\cdot u_2\|_2^2 + \cdots + \|f_s - p\cdot u_s\|_2^2 \quad (1)$$

where $p, u_1, \ldots, u_s$ are polynomials such that $\deg(p) \le k$, $\deg(p\cdot u_i) \le d_i = \deg f_i$ for $1 \le i \le s$. Here $\|f\|_2$ denotes the norm of the coefficient vector of polynomial f.

The minimization problem has many different formulations and various numeric optimization strategies have been proposed, see [1, 3] and references therein. In particular, Karmarkar and Lakshman [4] proposed an algorithm based on the global minimization of a rational function to compute the approximate GCD of univariate polynomials. The most expensive part of their algorithm is to find all the real solutions of two bivariate polynomials with high degrees. It has been shown in [8] that sum of squares (SOS) relaxation[5, 9] can be used to find the global minimum of the rational function arising from the approximate GCD computation. The SOS relaxation can be solved by reformulating them as semidefinite programs (SDPs), which in turn are solved efficiently by using interior point methods [7]. Motivated by the interesting results obtained in [8], we show how to apply SOS relaxation to solve different optimization problems formulated in [1, 3, 4].

Let $\mathbf{f_i}, \mathbf{u_i}, \mathbf{p}$ be coefficient vectors corresponding to polynomials f_i, u_i, p respectively, and $A_i = A_i(\mathbf{p})$ be convolution matrices such that $A_i\mathbf{u_i}$ produce the coefficient vectors of $p\cdot u_i$. Then the straight-forward formulation of the minimization problem (1) can be written as:

$$\min_{\mathbf{p},\mathbf{u_1},\ldots,\mathbf{u_s}} \|\mathbf{f_1} - A_1\mathbf{u_1}\|_2^2 + \cdots + \|\mathbf{f_s} - A_s\mathbf{u_s}\|_2^2 \quad (2)$$

If we fix the coefficients of p, the minimum is achieved at $\mathbf{u_i} := (A_i^* A_i)^{-1} A_i^* \mathbf{f_i}$, $1 \le i \le s$, and the minimization problem becomes

$$\min_{\mathbf{p}} \sum_{i=1}^{s} \mathbf{f_i}^* \mathbf{f_i} - \mathbf{f_i}^* A_i (A_i^* A_i)^{-1} A_i^* \mathbf{f_i}. \quad (3)$$

Here A_i^* and $\mathbf{f_i}^*$ denote the conjugate transpose of A_i and $\mathbf{f_i}$ respectively. This is an unconstrained minimization problem of rational function with positive denominator. It generalizes the formulations presented in [1, 2, 4, 12] for computing the approximate GCD of univariate polynomials and in [2] for computing the nearest bivariate polynomials with a linear (or fixed degree) factor. As shown in [1], if a good initial guess is taken, then Newton-like optimization method or Levenberg-Marquardt method can converge very fast to the global minimum of the nonlinear least squares problems (2,3). However, if we start with poor initial guess, then these methods may converge to local minimum after taking a large number of iterations. Suppose we have no information about the approximate GCD, then the following method based on SOS relaxation can be used to compute the GCD and global minimal residue.

Suppose we separate the real and imaginary parts of the unknown vectors, the minimization problems (2,3) can be expressed as $\min_{z\in\mathbb{R}^t} \frac{f(z)}{g(z)}$, where $f(z), g(z) \in \mathbb{R}[z_1, \ldots, z_t]$ and $g(z)$ is a real positive definite polynomial. Here z denotes the vector of coefficients of polynomials p and u_i. This problem can be solved by the following SOS relaxation

$$r_{sos}^* := \max_{r\in\mathbb{R}} \quad r \quad \text{s.t.} \quad f(z) - rg(z) \quad \text{is SOS in } \mathbb{R}^t[z]. \quad (4)$$

The degrees and number of variables for the formulations (2), (3) are: $\deg(f) = 4$, $g(z) = 1$, $t = 2\binom{n+k}{n} + 2\sum_{i=1}^{s}\binom{n+d_i-k}{n}$, and $\deg(f), \deg(g) \le 2\sum_{i=1}^{s}\binom{n+d_i-k}{n}$, $t = 2\binom{n+k}{n}$ respectively. There is a trade off between the number of variables and the degrees of polynomials. The method described above has been implemented by the first author in Matlab based on algorithms in SOSTOOLS [10], YALMIP [6] and SeDuMi [11].

Example 0.1 Consider two polynomials $f_1(x) = x(x+1)^2$, $f_2(x) = (x-1)(x+1)^2 + 1/10$ and $k = 2$, $F = \mathbb{R}$. Applying SOS relaxation to the optimization problem (4) with

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$\deg(f) = \deg(g) = 4$, we get the minimal value $r^*_{sos} \approx 9.388 \cdot 10^{-4}$. Solving the dual problem of the SOS relaxation, we get $z^*_0 \approx 0.9336, z^*_1 \approx 1.9781$. Evaluating the rational function at $z^* = (z^*_0, z^*_1)$ shows that $\frac{f(z^*)}{g(z^*)} \approx 9.388 \cdot 10^{-4} \approx r^*_{sos}$, which implies that z^* is the global minimizer for the problem (4). It corresponds to the monic approximate GCD $0.9336 + 1.9781x + x^2$.

In the following table, we compare the minimal residues achieved by different methods for examples in [1]. The second and fourth columns are the minimal residues computed by SOS relaxation of the problems (2,3) respectively. The third and fifth columns are the best results of Direct Solution and Reduced System in [1]. The last column consists of the minimal residues computed by STLN in [3]. We notice

k	polynomial	direct	rational	reduced	STLN
2	7.204e-8	1.6e-8	1.573e-8	1.6e-8	1.560e-8
3	1.574e-2	1.6e-2	1.562e-2	1.6e-2	1.562e-2
2	1.702e-2	1.7e-2	1.702e-2	1.7e-2	1.702e-2
4	7.427e-5	7.1e-5	7.086e-5	7.1e-5	7.086e-5
2	1.731e-5	1.7e-5	1.729e-5	1.7e-5	1.729e-5

Table 1: Comparison of optimization strategies.

that the SOS relaxation method can find the global minimum efficiently for small GCD problems. There is no need for a good initial guess. However, the global method is very expensive since the size of the matrix involved in the SOS relaxations and their dual problems increase exponentially in the degrees and number of variables of polynomials. For example, the size of the matrix involved in example 0.1 for computing the global minimum is 39, however, the size of the matrix for extracting the approximate GCD is already 4462. This forces us to look for the most efficient formulation.

As in [3], the problem of finding the approximate GCDs of several polynomials can also be formulated as

$$\min \quad \|\mathbf{z}\|_2^2, \quad \text{s.t.} \quad S_k(\mathbf{c}+\mathbf{z})\mathbf{x} = 0, \ \|\mathbf{x}\|_2 = 1, \qquad (5)$$

where $\mathbf{c}$ is the coefficient vector of $f_1, \ldots, f_s$, and the perturbations to the polynomials are parameterized via the vector $\mathbf{z}$, and $S_k(\mathbf{c}+\mathbf{z})$ is the multi-polynomial generalized Sylvester matrix [3].

Similar to the method used in [3], we can also choose one column of S_k and reformulate the problem as

$$\min_{\mathbf{z},\mathbf{x}} \quad \|\mathbf{z}\|_2^2 + \rho\|\mathbf{x}\|_2^2, \quad \text{s.t.} \quad A(\mathbf{c}+\mathbf{z})\mathbf{x} = \mathbf{b}(\mathbf{c}+\mathbf{z}), \qquad (6)$$

where ρ is a small number and $\mathbf{v}$ is a random vector, $\dim(\mathbf{z}) = \sum_{i=1}^{s}\binom{n+d_i}{n}$ and $\dim(\mathbf{x}) = \sum_{i=1}^{s}\binom{n+d_i-k}{n}$. We have tried to solve the minimization problems (5,6) by Lasserre SDP relaxations [5] in Matlab.

Example 0.2 Consider two polynomials $f_1(x) = x^3 - 1$, $f_2(x) = x^2 - 1.01$ and $k = 1, F = \mathbb{R}, \rho = 10^{-6}$. For minimization problem (5), the lower bounds given by the first and second order SOS relaxations are $r_1 = 0$, and $r_2 \approx 2.0852 \cdot 10^{-5}$ respectively. The size of the matrix involved in the second-order SOS relaxation is 8829. Here we notice that one feasible solution corresponding to the first-order relaxation in the homogenous model (5) is $\mathbf{z} = \mathbf{0}$, $\mathbf{x} = [\mathbf{0}, 1]^T$ with objective value zero. For minimization problem (6), we choose the first column of S_1 to be $\mathbf{b}$ and the remaining columns to form matrix A, the minimal perturbation computed by the first-order and the second-order SOS relaxations are $r_1 \approx 9.9673 \cdot 10^{-6}$, and $r_2 \approx 2.0871 \cdot 10^{-5}$. The dimensions of the matrices involved in the SOS relaxations are 152 and 6476 respectively. The minimizer can be extracted by the second-order SOS relaxation.

Although we may compute the global minimizer from high-order SOS relaxations, the size of the matrix increases quickly. The experiments show that the first-order SOS relaxation can give us some useful information on the minimal perturbations. However, as pointed out by Erich Kaltofen, if we want to compute the lower bound for the minimization problem (5), we have to try all the possible selection of $\mathbf{b}$ in (6). In the future, we are going to investigate how to reduce the size of the SOS problem by exploring the special structures of the minimization problems formulated in the approximate GCD computation.

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