J. Symbolic Computation (1998) **26**, 667–675
Article No. sy980233

Nearest Singular Polynomials[†]

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The nearest singular polynomial with a multiple zero of multiplicity $k \geq 2$ is considered based on the minimization of a quadratic form. Some recursive relations between the polynomials determining the multiple zeros for consecutive k's are presented.

1. Introduction

Some problems have been studied based on the minimization of a quadratic form (Corless *et al.*, 1995; Karmarkar and Lakshman, 1996a; Karmakar and Lakshman, 1996b; Zhi Lihong and Wu Wenda, 1997). One of them is to find the nearest singular polynomial with a double zero to a given polynomial. In this paper, we will consider the case of the nearest singular polynomial with a multiple zero of multiplicity k for any positive integer $k \geq 2$. This is not only a natural generalization of Zhi Lihong and Wu Wenda (1997), but we will also derive some recursive relations related to the determination of nearest singular polynomials for consecutive k's.

In Section 2, we will give a suitable expression for the quadratic form in terms of the undetermined multiple zero c and its conjugate which leads to an easy factorization of its derivatives with respect to c and $\overline{c}$. The factored form of its first derivative is given in Section 3. In Section 4, some recursive relations for the factors of the first derivative are derived for consecutive k's. In Section 5, we will determine which factor is useful for our purpose. Numerical examples are given in Section 6.

2. Expressions for the Quadratic Form

The problem considered is the following: given a monic polynomial f

$$f = x^m + \sum_{j=1}^{m} f_j x^{m-j}, \qquad f_j \in \mathcal{C},$$

[†]Work supported in part by the national climbing project No. 8505, and in part by China–Austria Scientific-Technological Cooperation Project No. IV B 15.
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0747–7171/98/120667 + 09 $30.00/0

find a monic polynomial $h(x)$ of the form

$$h = (x-c)^k \left(x^{m-k} + \sum_{j=1}^{m-k} \phi_j x^{m-k-j} \right), \qquad c, \phi_j \in \mathcal{C},$$

such that

$$\mathcal{N} := \| f - h \|^2$$

is minimized, where

$$\| p(x) \|^2 = \sum_{j=0}^{k} |p_j|^2$$

for any

$$p(x) = \sum_{j=0}^{k} p_j x^{k-j}, \ p_j \in \mathcal{C}.$$

Since

$$f(x) - h(x) = \sum_{j=1}^{m} \left(f_j - \sum_{i=0}^{k} \binom{k}{i} (-1)^i c^i \phi_{j-i} \right) x^{m-j},$$

where $\phi_{-k+1} = \phi_{-k+2} = \cdots = \phi_{-1} = 0$, $\phi_0 = 1$, $\phi_{m-k+1} = \phi_{m-k+2} = \cdots = \phi_m = 0$, we have

$$\mathcal{N} = \sum_{j=1}^{m} \left| f_j - \sum_{i=0}^{k} \binom{k}{i} (-1)^i c^i \phi_{j-i} \right|^2. \tag{2.1}$$

Let $r := f - g - A\phi$, where

$$\begin{aligned} f &= (f_1, \cdots, f_m)^T, \\ g &= (a_{21}, \cdots, a_{k1}, 0, \cdots, 0)^T, \\ \phi &= (\phi_1, \cdots, \phi_{m-k})^T, \\ A &= (a_{ij})_{m \times (m-k)}, \\ a_{ij} &= \binom{k}{i-j} (-1)^{i-j} c^{i-j}. \end{aligned}$$

Here, as usual

$$\binom{p}{q} = 0 \qquad \text{if } q > p \text{ or } q < 0.$$

It is easy to see that the jth component of r is the coefficient of x^{m-j} in $f(x) - h(x)$. Thus

$$\mathcal{N} = r^* r = (f - g - A\phi)^* (f - g - A\phi). \tag{2.2}$$

For any $c, \mathcal{N}$ attains its minimum $\mathcal{N}_m$ when ϕ is the least squares solution of the equation

$$f - g - A\phi = 0,$$

i.e.

$$\phi = A^+ (f - g), \tag{2.3}$$

where A^+ is the Penrose inverse of A.

In the following, we assume $\phi = A^+(f-g)$. Consequently

$$
\begin{aligned}
r &= (I - AA^+)(f-g), \\
\mathcal{N}_m &= (f-g)^*(I-AA^+)(f-g).
\end{aligned} \tag{2.4}
$$

Note that $\mathcal{N}_m$ depends on c only. Our problem becomes finding c such that $\mathcal{N}_m$ is minimized.

Since the columns of A are linearly independent, we have

$$A^+ = (A^*A)^{-1}A^*.$$

Let L be an $m \times m$ bi-diagonal Toeplitz matrix

$$
L = \begin{bmatrix} 1 & & & & \\ -c & 1 & & & \\ & -c & 1 & & \\ & & & \ddots & \\ & & & -c & 1 \end{bmatrix}. \tag{2.5}
$$

It is easy to check that for any positive integer k, we have

$$
\begin{aligned}
L^k &= (l_{ij}^{(k)}), \qquad & l_{ij}^{(k)} &= \binom{k}{i-j}(-1)^{i-j}c^{i-j}. \\
L^{-k} &= (l_{ij}^{(-k)}), \qquad & l_{ij}^{(-k)} &= \binom{i-j+k-1}{k-1}c^{i-j}.
\end{aligned}
$$

The submatrix composed of the first $m-k$ columns of L^k is A. We partition L^k and L^{-k} correspondingly:

$$L^k = (A \quad B), \qquad L^{-k} = \begin{pmatrix} U \\ V \end{pmatrix},$$

and let $W := U(I - V^*(VV^*)^{-1}V)$; then

$$A^*AW = A^*(I-BV)(I-V^*(VV^*)^{-1}V) = A^*(I-V^*(VV^*)^{-1}V) = A^*$$

and

$$W = (A^*A)^{-1}A^* = A^+.$$

Now r and $\mathcal{N}_m$ become

$$
\begin{aligned}
r &= (I-AA^+)(f-g) = V^*(VV^*)^{-1}V(f-g), \\
\mathcal{N}_m &= (f-g)^*V^*(VV^*)^{-1}V(f-g).
\end{aligned} \tag{2.6}
$$

It is easy to see that

$$V(f-g) = [\psi_1 \cdots \psi_k]^T := \psi, \qquad \psi_i = \frac{1}{(k-1)!}(c^{i-1}f(c))^{(k-1)}.$$

ψ can also be expressed as the following:

$$\psi = \Omega J \eta,$$

where

$$\Omega = (\Omega_{ij}), \qquad \Omega_{ij} = \frac{1}{(k-j)!}\binom{i-1}{j-1}c^{i-j}, \qquad J = (\delta_{i,k+1-j})$$

$$\eta = [f(c)f'(c)\cdots f^{(k-1)}(c)]^T. \tag{2.7}$$

Therefore,

$$\mathcal{N}_m = \eta^* J\Omega^*(VV^*)^{-1}\Omega J\eta = \eta^*(J\Omega^{-1}VV^*(\Omega^{-1})^* J)^{-1}\eta. \tag{2.8}$$

Let

$$\Lambda := J\Omega^{-1}VV^*(\Omega^{-1})^* J = \Big(\lambda_{ij}\Big), \tag{2.9}$$

then

$$\lambda_{ij} = \frac{\partial^{i+j-2}}{\partial c^{i-1}\partial\overline{c}^{j-1}}q, \qquad q = \sum_{j=0}^{m-1}(c\overline{c})^j. \tag{2.10}$$

Hence

$$\mathcal{N}_m = \eta^*\Lambda^{-1}\eta. \tag{2.11}$$

which is true for any k. We will write it with index k as

$$\mathcal{N}_m^{(k)} = \eta_k^*\Lambda_k^{-1}\eta_k. \tag{2.12}$$

Equation (2.12) is the expression for the quadratic form used in the following discussion.

3. First Derivative

For any matrix or vector $M = (m_{ij})$, we will use the notation $\frac{\partial M}{\partial c} = \left(\frac{\partial m_{ij}}{\partial c}\right)$.

Theorem 1.

$$\frac{\partial\mathcal{N}_m^{(k)}}{\partial c} = \frac{1}{(\det\Lambda_k)^2}P_{k+1}\overline{P_k}, \tag{3.1}$$

where

$$P_i := \det\begin{pmatrix}\Lambda_{i-1} & \eta_{i-1}\\ \omega_{i-1}^* & f^{(i-1)}\end{pmatrix}, \qquad i = k, k+1, \tag{3.2}$$

and ω_{i-1}^* *is the* i*th row of* Λ_i *upon deletion of its last element.*

Proof. Let $\xi_k := \Lambda_k^{-1}\eta_k$, i.e. $\eta_k = \Lambda_k\xi_k$. Now

$$\frac{\partial\Lambda_k}{\partial c}\xi_k + \Lambda_k\frac{\partial\xi_k}{\partial c} - \frac{\partial\eta_k}{\partial c} = 0,$$

$$\Lambda_k\frac{\partial\xi_k}{\partial c} = -\frac{\partial\Lambda_k}{\partial c}\Lambda_k^{-1}\eta_k + \frac{\partial\eta_k}{\partial c}.$$

For $i < k$, the ith row of $\frac{\partial\Lambda_k}{\partial c}$ is the $(i+1)$th row of Λ_k; the kth row of $\frac{\partial\Lambda_k}{\partial c}$ is the $(k+1)$th row of Λ_{k+1} (upon deletion of its last element) which we denote by ω_k^*. Therefore,

$$\begin{aligned}\Lambda_k\frac{\partial\xi_k}{\partial c} &= -\begin{bmatrix}0 & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1\\ & & \omega_k^*\Lambda_k^{-1} & & \end{bmatrix}\eta_k + \frac{\partial\eta_k}{\partial c}\\ &= -[f'(c)\ \cdots\ f^{(k-1)}(c)\ \omega_k^*\Lambda_k^{-1}]^T + [f'(c)\ \cdots\ f^{(k-1)}(c)\ f^{(k)}(c)]^T\\ &= [0\ \cdots\ 0\ f^{(k)}(c) - \omega_k^*\Lambda_k^{-1}\eta_k],\end{aligned}$$

$$
\begin{aligned}
\frac{\partial \mathcal{N}_m^{(k)}}{\partial c} &= \frac{\partial}{\partial c}(\eta_k^* \xi_k) = \eta_k^* \Lambda_k^{-1} [0 \ \cdots \ 0 \ \ f^{(k)}(c) - \omega_k^* \Lambda_k^{-1} \eta_k]^T \\
&= (f^{(k)}(c) - \omega_k^* \Lambda_k^{-1} \eta_k) \frac{\det \Lambda_{k-1}}{\det(\Lambda_k)} (f^{(k-1)}(c) - \omega_{k-1}^* \Lambda_{k-1}^{-1} \eta_{k-1})^* \\
&= \frac{1}{(\det \Lambda_k)^2} \det \begin{pmatrix} \Lambda_k & \eta_k \\ \omega_k^* & f^{(k)}(c) \end{pmatrix} \left(\det \begin{pmatrix} \Lambda_{k-1} & \eta_{k-1} \\ \omega_{k-1}^* & f^{(k-1)(c)} \end{pmatrix} \right)^* \\
&= \frac{1}{(\det \Lambda_k)^2} P_{k+1} \overline{P_k}.
\end{aligned}
$$

So we have equation (3.1). □

4. Recursive Relations

In the following, we will prove some recursive relations between Λ_k and P_k.

Theorem 2.

$$
\det \Lambda_k \frac{\partial^2}{\partial c \partial \overline{c}} \det \Lambda_k - \frac{\partial}{\partial c} \det \Lambda_k \frac{\partial}{\partial \overline{c}} \det \Lambda_k = \det \Lambda_{k-1} \det \Lambda_{k+1}. \tag{4.1}
$$

Proof. We have

$$
\begin{aligned}
\det \Lambda_{k+1} &= \det \begin{pmatrix} \Lambda_{k-1} & u_1 & u_2 \\ u_1^* & \alpha & \xi \\ u_2^* & \beta & \eta \end{pmatrix} = \det \Lambda_{k-1} \left(\begin{pmatrix} \alpha & \xi \\ \beta & \eta \end{pmatrix} - \begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix} \Lambda_{k-1}^{-1} (u_1 \ u_2) \right) \\
&= \det \Lambda_{k-1} (\alpha - u_1^* \Lambda_{k-1}^{-1} u_1)(\eta - u_2^* \Lambda_{k-1}^{-1} u_2) \\
&\quad - \det \Lambda_{k-1} (\beta - u_2^* \Lambda_{k-1}^{-1} u_1)(\xi - u_1^* \Lambda_{k-1}^{-1} u_2) \\
\det \Lambda_k &= \det \begin{pmatrix} \Lambda_{k-1} & u_1 \\ u_1^* & \alpha \end{pmatrix} = \det \Lambda_{k-1} (\alpha - u_1^* \Lambda_{k-1}^{-1} u_1).
\end{aligned}
$$

In $\det \Lambda_k$, the partial derivative of the ith row with respect to c is the $(i+1)$th row for $i < k$, and the kth row of $\frac{\partial \det \Lambda_k}{\partial c}$ is the $(k+1)$th row of $\det \Lambda_{k+1}$ upon deletion of its last element; similarly for the derivatives of the columns with respect to $\overline{c}$. Hence,

$$
\begin{aligned}
\frac{\partial}{\partial c} \det \Lambda_k &= \det \begin{pmatrix} \Lambda_{k-1} & u_1 \\ u_2^* & \beta \end{pmatrix} = \det \Lambda_{k-1} (\beta - u_2^* \Lambda_{k-1}^{-1} u_1), \\
\frac{\partial}{\partial \overline{c}} \det \Lambda_k &= \det \begin{pmatrix} \Lambda_{k-1} & u_2 \\ u_1^* & \xi \end{pmatrix} = \det \Lambda_{k-1} (\xi - u_1^* \Lambda_{k-1}^{-1} u_2), \\
\frac{\partial^2}{\partial c \partial \overline{c}} \det \Lambda_k &= \det \begin{pmatrix} \Lambda_{k-1} & u_2 \\ u_2^* & \eta \end{pmatrix} = \det \Lambda_{k-1} (\eta - u_2^* \Lambda_{k-1}^{-1} u_2).
\end{aligned}
$$

Thus, we get equation (4.1). □

In an analogous fashion, we derive Theorems 3 and 4.

Theorem 3.

$$
P_k \frac{\partial^2}{\partial c \partial \overline{c}} P_k - \frac{\partial}{\partial c} P_k \frac{\partial}{\partial \overline{c}} P_k = P_{k-1} P_{k+1}. \tag{4.2}
$$

THEOREM 4.

$$\det\Lambda_k \frac{\partial}{\partial c}P_k - P_k\frac{\partial}{\partial c}\det\Lambda_k = \det\Lambda_{k-1}P_{k+1}. \tag{4.3}$$

5. The Useful Factor

Let $c = a + ib$, where a and b are real, and consider $\mathcal{N}_m^{(k)}$ as a real rational function of the real variables a and b. The problem is to find the real solutions of the system

$$\frac{\partial \mathcal{N}_m^{(k)}}{\partial a} = 0, \qquad \frac{\partial \mathcal{N}_m^{(k)}}{\partial b} = 0. \tag{5.1}$$

Because

$$\frac{\partial \mathcal{N}_m^{(k)}}{\partial c} = \frac{1}{2}\left(\frac{\partial \mathcal{N}_m^{(k)}}{\partial a} - i\frac{\partial \mathcal{N}_m^{(k)}}{\partial b}\right), \qquad \frac{\partial \mathcal{N}_m^{(k)}}{\partial \overline{c}} = \frac{1}{2}\left(\frac{\partial \mathcal{N}_m^{(k)}}{\partial a} + i\frac{\partial \mathcal{N}_m^{(k)}}{\partial b}\right), \tag{5.2}$$

it is sufficient to consider

$$\frac{\partial \mathcal{N}_m^{(k)}}{\partial c} = 0 \tag{5.3}$$

for determining c. The trace T and determinant D of the Hessian matrix can be expressed as

$$T = 4\frac{\partial^2 \mathcal{N}_m^{(k)}}{\partial c\partial\overline{c}}, \qquad D = 4\left(\left(\frac{\partial^2 \mathcal{N}_m^{(k)}}{\partial c\partial\overline{c}}\right)^2 - \frac{\partial^2 \mathcal{N}_m^{(k)}}{\partial c^2}\frac{\partial^2 \mathcal{N}_m^{(k)}}{\partial \overline{c}^2}\right). \tag{5.4}$$

By Theorem 1, we consider $P_k = 0$ and $P_{k+1} = 0$ separately.

THEOREM 5. *$\mathcal{N}_m^{(k)}$ attains its local minimum at c satisfying $P_k = 0$ if*

$$(\det\Lambda_{k-1})^4 P_{k+1}\overline{P_{k+1}} - (\det\Lambda_k)^4 P_{k-1}\overline{P_{k-1}} > 0. \tag{5.5}$$

PROOF. If $P_k = 0$, we have:

$$\frac{\partial P_k}{\partial c} = \frac{\det\Lambda_{k-1}P_{k+1}}{\det\Lambda_k}, \frac{\partial\overline{P_k}}{\partial c} = -P_{k-1}P_{k+1}/\frac{\partial P_k}{\partial c} = -\frac{\det\Lambda_k}{\det\Lambda_{k-1}}P_{k-1},$$

$$T = 4\frac{P_{k+1}}{(\det\Lambda_k)^2}\frac{\partial\overline{P_k}}{\partial\overline{c}} = 4\frac{P_{k+1}}{(\det\Lambda_k)^2}\frac{\det\Lambda_{k-1}\overline{P_{k+1}}}{\det\Lambda_k} = 4\frac{\det\Lambda_{k-1}}{(\det\Lambda_k)^3}P_{k+1}\overline{P_{k+1}} \geq 0,$$

$$D = 4\left(\left(\frac{\det\Lambda_{k-1}}{(\det(\Lambda_k))^3}P_{k+1}\overline{P_{k+1}}\right)^2 - \frac{P_{k-1}\overline{P_{k-1}}P_{k+1}\overline{P_{k+1}}}{(\det\Lambda_k)^2(\det\Lambda_{k-1})^2}\right)$$

$$= 4\frac{P_{k+1}\overline{P_{k+1}}}{(\det\Lambda_{k-1})^2(\det\Lambda_k)^6}((\det\Lambda_{k-1})^4P_{k+1}\overline{P_{k+1}} - (\det\Lambda_k)^4P_{k-1}\overline{P_{k-1}}).$$

So $\mathcal{N}_m^{(k)}$ attains its local minimum at c satisfying $P_k = 0$ if $D > 0$, i.e.,

$$(\det\Lambda_{k-1})^4 P_{k+1}\overline{P_{k+1}} - (\det\Lambda_k)^4 P_{k-1}\overline{P_{k-1}} > 0. \quad \square$$

For $P_{k+1} = 0$ we have the following conclusion.

THEOREM 6. *For c satisfying $P_{k+1} = 0$, the trace is non-positive.*

Table 1. $\mathcal{N}_m = 0.1763296120$.

Zeros of h	Zeros of f
$0.580\,685\,7529$ (double)	0, 1
$-1.050\,883\,646$	-1
$-0.0747\,216\,6958 + 0.9804\,509\,313i$	i
$-0.0747\,216\,6958 - 0.980\,450\,9313i$	$-i$

Table 2. $\mathcal{N}_m = 0.5075634529$.

Zeros of h	Zeros of f
$0.3642294253 + 0.3642294253i$ (triple)	$0, 1, i$
$-0.9744637807 + 0.03279153516i$	-1
$0.03279153516 - 0.9744637807i$	$-i$

Proof. If $P_{k+1} = 0$ then

$$\frac{\partial^2 \mathcal{N}_m^{(k)}}{\partial c \partial \overline{c}} = \frac{\overline{P_k}}{(\det \Lambda_k)^2} \frac{\partial P_{k+1}}{\partial \overline{c}},$$

$$\frac{\partial P_{k+1}}{\partial \overline{c}} \det \Lambda_{k-1} = \frac{\partial \det \Lambda_k}{\partial \overline{c}} \frac{\partial P_k}{\partial c} + \det \Lambda_k \frac{\partial^2 P_k}{\partial c \partial \overline{c}} - \frac{\partial P_k}{\partial \overline{c}} \frac{\partial \det \Lambda_k}{\partial c} - P_k \frac{\partial^2 \det \Lambda_k}{\partial c \partial \overline{c}},$$

$$\frac{\partial \det \Lambda_k}{\partial \overline{c}} \frac{\partial P_k}{\partial c} - P_k \frac{\partial^2 \det \Lambda_k}{\partial c \partial \overline{c}} = \frac{P_k}{\det \Lambda_k} \frac{\partial \det \Lambda_k}{\partial c} \frac{\partial \det \Lambda_k}{\partial \overline{c}} - P_k \frac{\partial^2 \det \Lambda_k}{\partial c \partial \overline{c}} = -P_k \frac{\det \Lambda_{k+1} \det \Lambda_{k-1}}{\det \Lambda_k},$$

$$\det \Lambda_k \frac{\partial^2 P_k}{\partial c \partial \overline{c}} - \frac{\partial P_k}{\partial \overline{c}} \frac{\partial \det \Lambda_k}{\partial c} = \det \Lambda_k \frac{\partial^2 P_k}{\partial c \partial \overline{c}} - \frac{\det \Lambda_k}{P_k} \frac{\partial P_k}{\partial c} \frac{\partial P_k}{\partial \overline{c}} = \frac{\det \Lambda_k}{P_k} P_{k-1} P_{k+1} = 0.$$

So we have

$$T = 4 \frac{\partial^2 \mathcal{N}_m^{(k)}}{\partial c \partial \overline{c}} = -4 \frac{\det \Lambda_{k+1}}{(\det \Lambda_k)^3} P_k \overline{P_k} \leq 0. \quad \square$$

We conclude that it is sufficient to consider those zeros of $P_k = 0$ which are not zeros of $P_{k+1} = 0$. As for zeros of $P_{k+1} = 0$, they are candidates for c with a multiplicity higher than k. By a repeated use of Theorems 3 and 4, the common zeros of $P_k = 0$ and $P_{k+1} = 0$ are zeros of $P_j = 0$ for all $m \geq j > k$.

6. Numerical Examples

Example 1. $f = x^5 - x$.

For $k = 2$ (cf. Zhi Lihong and Wu Wenda (1997)), there are four nearest singular polynomials due to the geometry of the zeros of f; one of them is

$$h \approx x^5 + 0.03895547966x^4 + 0.06708530296x^3 + 0.1155277233x^2 -0.8010494959x + 0.3426130279.$$

The zeros of h are shown in Table 1. The other three cases can be obtained by rotation with an angle $\pi/2$, π and $3\pi/2$ respectively.

For $k = 3$, there are four nearest singular polynomials; one of them is

$$h \approx x^5 + (-0.1510160305 - 0.1510160305i)x^4 - 0.311275054ix^3 + (0.3858582928 \\ -0.3858582928i)x^2 - 0.5746952045x + 0.09187090467 + 0.09187090467i$$

with the roots shown in Table 2. $c = 0$ is the common zero of P_4 and P_5, for $k = 4, 5$; the nearest singular polynomial is

$$h = x^5, \qquad \text{and} \qquad \mathcal{N}_m = 1.$$

Example 2.

$$\begin{aligned} f &= (x - 0.89 - 0.03i)(x - 0.88 + 0.02i)(x - 0.87)(x - 1) \\ &= x^4 + (-3.64 - 0.01i)x^3 + (4.9637 + 0.0273i)x^2 + (-3.005606 - 0.024782i)x \\ &\quad +0.681906 + 0.007482i. \end{aligned}$$

For $k = 2$ (cf. Zhi Lihong and Wu Wenda (1997)), the nearest singular polynomial of f is unique:

$$h \approx x^4 + (-3.639999897 - 0.01000012076i)x^3 + (4.963700115 + 0.02729986094i)x^2 \\ +(-3.005605870 - 0.02478216008i)x + 0.6819061456 + 0.007481815756i,$$

with these roots shown in Table 3.

For $k = 3$, the nearest singular polynomial with a zero of multiplicity 3 is also unique:

$$h \approx x^4 + (-3.639968566 - 0.01002119406i)x^3 + (4.963698969 + 0.02730077333i)x^2 \\ +(-3.005622319 - 0.02477096306i)x + 0.6819004541 + 0.007485892787i.$$

The zeros of h are shown in Table 4. For $k = 4$, the nearest singular polynomial with a zero of multiplicity 4 is:

$$h \approx x^4 + (-3.637528548 - 0.009999075252i)x^3 + (4.961817732 + 0.02727894127i)x^2 \\ +(-3.008080147 - 0.02480691942i))x + 0.6838580852 + 0.007519618582i.$$

The zeros of h are shown in Table 5.

Table 3. $\mathcal{N}_m = 0.1552760144 \times 10^{-12}$.

Zeros of h	Zeros of f
$0.8768135619 - 0.01006779565i$ (double)	$0.88 - 0.02i$, 0.87
$0.8866786823 + 0.02982563187i$	$0.89 + 0.03i$
$0.9996940915 + 0.0003100788900i$	1

Table 4. $\mathcal{N}_m = 0.3311925673 \times 10^{-8}$.

Zeros of h	Zeros of f
$0.8817735725 + 0.002337412033i$ (triple)	$0.89 + 0.03i$, $0.88 - 0.02i$, 0.87
$0.9946478484 + 0.003008957959i$,	1

Table 5. $\mathcal{N}_m = 0.004425554008$.

Zeros of h	Zeros of f
$0.9093821369 + 0.002499768813i$	$0.89 + 0.03i$, $0.88 - 0.02i$, 0.87, 1

Acknowledgement

The authors thank Professor H. J. Stetter for valuable suggestions.

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Originally Received 23 July 1997
Accepted 12 December 1997