Hybrid Method for Solving New Pose Estimation Equation System ⋆

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Abstract. Camera pose estimation is the problem of determining the position and orientation of an internally calibrated camera from known 3D reference points and their images. We introduce a new polynomial equation system for 4-point pose estimation and apply our symbolic-numeric method to solve it stably and efficiently. In particular, our algorithm can also recognize the points near critical configurations and deal with these near critical cases carefully. Numerical experiments are given to show the performance of the hybrid algorithm.

1 Introduction

Given a set of correspondences between 3D reference points and their images, *4-point pose estimation* consists of determining the position and orientation of the camera with respect to four known reference points. It is a classical and common problem in computer vision and photogrammetry and has been studied in the past [1, 2, 6, 8, 11, 20, 23].

The well-known polynomial system (1) corresponding to the *4-point pose estimation* generically has a unique positive solution. It can be found successfully by linear algorithms proposed in [20, 2, 23]. But there are certain degenerate cases for which no unique solution is possible. These *critical configurations* are known precisely and include the following notable degenerate case: a 3D line and a circle in an orthogonal plane touching the line. In [2] an algorithm is presented that solves the problem including the critical configurations, but the relative error and failure rate (backward error) are significantly higher than one would like. In [23], the authors present a new linear algorithm which works well even in the degenerate cases. However, the matrices are much larger 70×90 compared with 24×24 matrices used in [2].

In this paper, we introduce a new variable and transform the polynomial system for *4-point pose estimation* to a new system with only five equations and

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three variables. Our symbolic-numeric method can also be applied to the new system and find solutions in general or critical cases. The matrices we used in the general or critical cases are of size 20×20; while in near critical cases, we are using a matrix of size 35×50 in order to recover the sensitive unique root.

The rest of the paper is organized as follows. In Section 2, we introduce the basic geometry of *the 4-point pose estimation problem*. A new system of equations is introduced. In Section 3, we briefly review the symbolic-numeric method for polynomial system solving. Then, we illustrate how to apply this method to solve the polynomial system corresponding to the critical or near critical cases. In Section 4, the simulated experimental results are given. Some conclusions are given in Section 5.

2 Geometry of camera pose from four points

In the following, we briefly introduce the geometry of camera pose from four points. Let C be the calibrated camera center, and P_1, P_2, P_3, P_4 be the reference points (see Fig. 1). Let $c_{12} = 2\cos\angle(P_1CP_2)$, $c_{13} = 2\cos\angle(P_1CP_3)$, $c_{14} = 2\cos\angle(P_1CP_4)$, $c_{23} = 2\cos\angle(P_2CP_3)$, $c_{24} = 2\cos\angle(P_2CP_4)$, $c_{34} = 2\cos\angle(P_3CP_4)$.

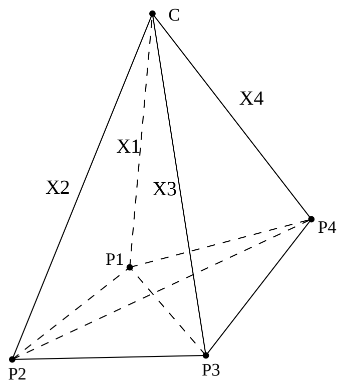

Fig. 1. The 4-point pose estimation problem

From triangles CP_1P_2, CP_1P_3, CP_2P_3, CP_1P_4, CP_2P_4 and CP_3P_4, we obtain the 4$-$*point pose estimation equation system*:

$$\begin{cases} X_1^2 + X_2^2 - c_{12}X_1X_2 - |P_1P_2|^2 = 0, \\ X_1^2 + X_3^2 - c_{13}X_1X_3 - |P_1P_3|^2 = 0, \\ X_2^2 + X_3^2 - c_{23}X_2X_3 - |P_2P_3|^2 = 0, \\ X_2^2 + X_4^2 - c_{24}X_2X_4 - |P_2P_4|^2 = 0, \\ X_3^2 + X_4^2 - c_{34}X_3X_4 - |P_3P_4|^2 = 0, \\ X_1^2 + X_4^2 - c_{14}X_1X_4 - |P_1P_4|^2 = 0. \end{cases} \quad (1)$$

We are only interested in finding the positive solutions for X_1, X_2, X_3, X_4. Since $X_4 = |P_4C|$ is positive, we may make the following variable changes. Let

$$X_1 = x_1X_4,\ X_2 = x_2X_4,\ X_3 = x_3X_4,$$
$$|P_1P_4| = \sqrt{w}X_4,\ |P_1P_2| = \sqrt{aw}X_4,\ |P_1P_3| = \sqrt{bw}X_4,$$
$$|P_2P_3| = \sqrt{cw}X_4,\ |P_2P_4| = \sqrt{dw}X_4,\ |P_3P_4| = \sqrt{ew}X_4.$$

Equation system (1) become the following equivalent equation system:

$$\begin{cases} x_1^2 + x_2^2 - c_{12}x_1x_2 - aw = 0, \\ x_1^2 + x_3^2 - c_{13}x_1x_3 - bw = 0, \\ x_2^2 + x_3^2 - c_{23}x_2x_3 - cw = 0, \\ x_2^2 + 1 - c_{24}x_2 - dw = 0, \\ x_3^2 + 1 - c_{34}x_3 - ew = 0, \\ x_1^2 + 1 - c_{14}x_1 - w = 0. \end{cases} \tag{2}$$

From $x_1^2 + 1 - c_{14}x_1 - w = 0$ and $|c_{14}| < 2$ because $c_{14} = 2\cos\angle(P_1CP_4)$, we have

$$w = (|P_1P_4|/X_4)^2 = x_1^2 + 1 - c_{14}x_1 = (x_1 - c_{14}/2)^2 + 1 - c_{14}^2/4 > 0.$$

X_4 can be uniquely determined by $X_4 = |P_1P_4|/\sqrt{w}$ and the equivalent correspondence is:

$$\begin{array}{l} (x_1, x_2, x_3, \sqrt{w}) \overset{|P_1P_4|=\sqrt{w}X_4}{\longleftarrow - - - - - - \longrightarrow} (x_1, x_2, x_3, X_4) \\ \overset{X_1=x_1X_4, X_2=x_2X_4, X_3=x_3X_4}{\longleftarrow - - - - - - - - - - - \longrightarrow} (X_1, X_2, X_3, X_4) \end{array} \tag{3}$$

Substituting w into above equation system, we have the following equivalent equation system:

$$\begin{cases} (1-a)x_1^2 + x_2^2 - c_{12}x_1x_2 - a(1 - c_{14}x_1) = 0, \\ (1-b)x_1^2 + x_3^2 - c_{13}x_1x_3 - b(1 - c_{14}x_1) = 0, \\ (1-c)x_2^2 + x_3^2 - c_{23}x_2x_3 - c(1 - c_{14}x_1) = 0, \\ (1-d)x_2^2 + 1 - c_{24}x_2 - d(1 - c_{14}x_1) = 0, \\ (1-e)x_3^2 + 1 - c_{34}x_3 - e(1 - c_{14}x_1) = 0. \end{cases} \tag{4}$$

The equation system (4) is simpler than the original system (1), and from the positive solution x_i we can get the coordinates X_i according to the equivalent correspondence. The recovered camera-point distances X_i are used to estimate the coordinates of the 3D reference points in a camera-centered 3D frame: $\bar{P}_i = X_iK^{-1}U_i$ (see [20]). The final step is the absolute orientation determination [21]. The determination of the translation and the scale follow immediately from the estimation of the rotation.

The system (4) is still an overdetermined polynomial system of five equations in 3 variables. The parameters $c_{ij}(1 \le i, j \le 4)$ and a, b, c, d, e are data of limited accuracy. It is still very difficult to use Gröbner basis algorithms [4] or Ritt-Wu's characteristic algorithms [29, 31] to solve such approximate overdetermined polynomial systems. In the following, we briefly introduce our new developed complete linear method [23] for solving such system stably.

3 Linear methods for pose determination from 4 points

Consider a general polynomial system S in $x_1, \ldots, x_n$ of degree q and its corresponding vector of monomials of degree less than or equal to q. The system can be written as

$$M_0 \cdot [x_1^q, x_1^{q-1}x_2, \ldots, x_n^2, x_1, \ldots, x_n, 1]^T = [0, 0, \ldots, 0, 0, \ldots, 0, 0]^T \tag{5}$$

in terms of its coefficient matrix M_0. Here and hereafter, $[...]^T$ means the transposition. Further, $[\xi_1, \xi_2, \ldots, \xi_n]$ is one of the solutions of the polynomial system, if and only if

$$[\xi_1^q, \xi_1^{q-1}\xi_2, \ldots, \xi_n^2, \xi_1, \ldots, \xi_n, 1]^T \tag{6}$$

is a null vector of the coefficient matrix M_0.

Since the number of monomials is usually bigger than the number of polynomials, the dimension of the null space can be big. The aim of completion methods, such as ours and those based on Gröbner bases and others [15, 12, 5, 14, 17, 18, 16, 25, 28], is to include additional polynomials belonging to the ideal generated by S, to reduce the dimension to its minima.

The bijection

$$\phi : x_i \leftrightarrow \frac{\partial}{\partial x_i}, \quad 1 \leq i \leq n, \tag{7}$$

maps the system S to an equivalent system of linear homogeneous PDEs denoted by R. Jet space approaches are concerned with the study of the jet variety

$$V(R) = \left\{ \left(\underset{q}{u}, \underset{q-1}{u}, \ldots, \underset{1}{u}, u \right) \in J^q : R\left(\underset{q}{u}, \underset{q-1}{u}, \ldots, \underset{1}{u}, u \right) = 0 \right\}, \tag{8}$$

where $\underset{j}{u}$ denotes the formal jet coordinates corresponding to derivatives of order exactly j.

A single prolongation of a system R of order q consists of augmenting the system with all possible derivatives of its equations, so that the resulting augmented systems, denoted by DR, has order $q+1$. Under the bijection ϕ, the equivalent operation for polynomial systems is to multiply by monomials, so that the resulting augmented system has degree $q+1$.

A single geometric projection is defined as

$$E(R) := \left\{ \left(\underset{q-1}{u}, \ldots, \underset{1}{u}, u \right) \in J^{q-1} : \exists\, \underset{q}{u}, R\left(\underset{q}{u}, \underset{q-1}{u}, \ldots, \underset{1}{u}, u \right) = 0 \right\}. \tag{9}$$

The projection operator E maps a point in J^q to one in J^{q-1} by simply removing the jet variables of order q (i.e. eliminating $\underset{q}{u}$). For polynomial systems of degree q, by the bijection ϕ, the projection is equivalent to eliminating the monomials of the highest degree q. To numerically implement an approximate involutive form method, we proposed in [30, 23] a numeric projection operator $\hat{E}$ based on singular value decomposition.

By the famous Cartan-Kuranishi Theorem [10, 19, 27], after application of a finite number of prolongations and projections, the algorithm above terminates with an involutive or an inconsistent system.

Suppose that R is involutive at prolonged order k and projected order l, and by the bijection ϕ has corresponding system of polynomials S. Then the dimension of $\hat{E}^l(D^k R)$ allows us to determine the number of approximate solutions of S up to multiplicity. In particular these solutions approximately generate the null space of $\hat{E}^l(D^k R)$. We can compute eigenvalues and eigenvectors to find these solutions. It should be noticed that the above symbolic prolongation and numeric projection method works only for solving the polynomial systems with finite number of solutions.

The following example corresponds to the third singular case as pointed in [2, 23]. In the example the coordinate of the camera point is $(1, 1, 1)$, and the coordinates of the four control points are $(-1, 1, 0)$, $(-1, -1, 0)$, $(1, -1, 0)$ and $(1, 1, 0)$ respectively. The corresponding *4-point pose estimation equation system* is:

$$\begin{cases} p_1 := {x_2}^2 - 2.0\, {x_1}^2 - 0.666667\, x_2 + 1.78885\, x_1 - 1.0, \\ p_2 := {x_3}^2 - {x_1}^2 - 0.894427\, x_3 + 0.894427\, x_1, \\ p_3 := {x_2}^2 - 1.49071\, x_1\, x_2 + 0.894427\, x_1 - 1.0, \\ p_4 := -{x_1}^2 + {x_3}^2 - 0.4\, x_1\, x_3 + 1.78885\, x_1 - 2.0, \\ p_5 := {x_2}^2 + {x_3}^2 - 1.49071\, x_2\, x_3 - {x_1}^2 + 0.894427\, x_1 - 1.0. \end{cases} \tag{10}$$

We show how our symbolic-numeric method can be used to solve (10). Under the bijection $\phi : x_i \leftrightarrow \frac{\partial}{\partial x_i}$ where $i = 1, 2, 3$, the system is equivalent to the PDE system R:

$$\begin{cases} \phi(p_1)u := \frac{\partial^2 u}{\partial {x_2}^2} - 2.0\, \frac{\partial^2 u}{\partial {x_1}^2} - 0.666667\, \frac{\partial u}{\partial x_2} + 1.78885\, \frac{\partial u}{\partial x_1} - 1.0u, \\ \phi(p_2)u := \frac{\partial^2 u}{\partial {x_3}^2} - \frac{\partial^2 u}{\partial {x_1}^2} - 0.894427\, \frac{\partial u}{\partial x_3} + 0.894427\, \frac{\partial u}{\partial x_1}, \\ \phi(p_3)u := \frac{\partial^2 u}{\partial {x_2}^2} - 1.49071\, \frac{\partial^2 u}{\partial x_1\, \partial x_2} + 0.894427\, \frac{\partial u}{\partial x_1} - 1.0u, \\ \phi(p_4)u := -\frac{\partial^2 u}{\partial {x_1}^2} + \frac{\partial^2 u}{\partial {x_3}^2} - 0.4\, \frac{\partial^2 u}{\partial x_1\, \partial x_3} + 1.78885\, \frac{\partial u}{\partial x_1} - 2.0u, \\ \phi(p_5)u := \frac{\partial^2 u}{\partial {x_2}^2} + \frac{\partial^2 u}{\partial {x_3}^2} - 1.49071\, \frac{\partial^2 u}{\partial x_2\, \partial x_3} - \frac{\partial^2 u}{\partial {x_1}^2} + 0.894427\, \frac{\partial u}{\partial x_1} - 1.0u. \end{cases} \tag{11}$$

Applying the symbolic-numeric completion method to R with tolerance 10^{-9}, we obtain the table of dimensions below:

Table 1: $\mathbf{dim(\hat{E}^l D^k R)}$ for (11)

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$l=0$	5	2	2	2	2
$l=1$	4	2	2	2	2
$l=2$	1	2	2	2	2
$l=3$		1	2	2	2
$l=4$			1	2	2
$l=5$				1	2
$l=6$					1

We seek the smallest k such that there exists an $l = 0, ..., k$ with $\hat{E}^l D^k R$ approximately involutive. Passing the approximate projected elimination test amounts to test looking in the table for the first column with an equal entry in the next column on the downwards sloping diagonal (with both entries being on or above the main diagonal $k = l$). This first occurs for $k = 1$ and $l = 0, 1, 2$.

Applying the approximate version of the projected involutive symbol test to the example, shows that it is passed for $k = 1$, $l = 0$, and $l = 1$, so we choose the largest l ($l = 1$), yielding $\hat{E}DR$ as the sought after approximately involutive system.

The involutive system has $\dim(\hat{E}DR) = 2$ and so by the bijection the polynomial system (10) has 2 solutions up to multiplicity. In the following, we apply an eigenvalue method to solve (10).

1. Compute an approximate basis of the null space of DR, denoted by a 20×2 matrix B. Since $\dim(DR) = \dim(\hat{E}DR) = \dim(\hat{E}^2 DR) = 2$, the 4×2 submatrix B_1 and 10×2 submatrix B_2 of B by deleting entries corresponding to the second and third degree monomials are bases of null spaces of $\hat{E}^2 DR$ and $\hat{E}DR$ respectively.
2. Consider the set of all monomials of degree less than or equal to 1:
$$\mathcal{N} = [x_1, x_2, x_3, 1].$$
For numerical stability, we compute the singular value decomposition of B_1
$$U, S, V := \text{SingularValues}(B_1).$$
The first two columns of U form the 2×4 matrix U_s, and guarantee a stable linear polynomial set $\mathcal{N}_p = U_s^T \cdot \mathcal{N}^T$ for computing multiplication matrices.
3. The multiplication matrix of x_i with respect to $\mathcal{N}_p$ can be formed as
$$M_{x_i} = U_s^T \cdot B_{x_i} \cdot V^T \cdot S_i$$
where $B_{x_1}, B_{x_2}, B_{x_3}$ are the $[1, 2, 3, 7], [2, 4, 5, 8]$ and $[3, 5, 6, 9]$ rows of B_2 respectively, and S_i is a diagonal matrix with elements which are inversions of the first two elements of S: $1.95588, 111.524$.
4. The coordinates x_i of the double root can be found as the average of the eigenvalues of M_{x_i} for $i = 1, 2, 3$:
$$x_1 = 2.23607, x_2 = 3.0, x_3 = 2.23607. \tag{12}$$

Substituting the solution (12) into (10), we find $|p_i(\xi_1, \xi_2, \xi_3)| < 0.42 \cdot 10^{-7}$ for $i = 1, 2, \ldots, 5$. If one substitutes the positive solution (12) to the Jacobian matrix

$$\begin{bmatrix} \frac{\partial p_1}{\partial x_1} & \frac{\partial p_1}{\partial x_2} & \frac{\partial p_1}{\partial x_3} \\ \frac{\partial p_2}{\partial x_1} & \frac{\partial p_2}{\partial x_2} & \frac{\partial p_2}{\partial x_3} \\ \vdots & \vdots & \vdots \\ \frac{\partial p_5}{\partial x_1} & \frac{\partial p_5}{\partial x_2} & \frac{\partial p_5}{\partial x_3} \end{bmatrix} \tag{13}$$

then the singular values of the transpose of the Jacobian matrix are

$$11.8865, 5.42001, 0.109804 \cdot 10^{-8}.$$

The Jacobian matrix is near singular. This tells us that the solution is quite unstable for any small perturbations. Suppose we perturb (10) by errors of order 10^{-6}, the number of solutions read from the dimension table will generally become 1.

In general, we obtain the following table:

Table 2: $\dim(\hat{\mathbf{E}}^{\mathbf{l}}\mathbf{D}^{\mathbf{k}}\mathbf{R})$ for near critical case

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$l=0$	5	2	1	1	1
$l=1$	4	2	1	1	1
$l=2$	1	2	1	1	1
$l=3$		1	1	1	1
$l=4$			1	1	1
$l=5$				1	1
$l=6$					1

Applying the projected elimination and involutive symbol tests shows that $\hat{E}^2D^2R$ is approximately involutive. The computed positive root has backward error of order $10^{-6} \sim 10^{-9}$ in general.

In order to compare the difference between general cases, critical cases and near critical cases, in the below, we also show the dimension table corresponding to the general cases.

Table 3: $\dim(\hat{\mathbf{E}}^{\mathbf{l}}\mathbf{D}^{\mathbf{k}}\mathbf{R})$ for general case

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$l=0$	5	1	1	1	1
$l=1$	4	1	1	1	1
$l=2$	1	1	1	1	1
$l=3$		1	1	1	1
$l=4$			1	1	1
$l=5$				1	1
$l=6$					1

From the three different dimension tables, it is easy to deduce the following conclusions. Firstly, in the general case, the unique solution can be recovered from the null vector of the 20×20 matrix generated by $p_i, x_i\, p_j$ for $i, j = 1, 2, 3$. Secondly, if the four points are on the critical configuration, we have to deal it with eigenvalue method after forming the multiplication matrix with respect to x_1, x_2, x_3 separately. Finally, if the points are near the critical configuration, then the solution should be found stably from the null vector of the 35×50 matrix generated by $p_i, x_ip_j, x_ix_jp_k$ for $i, j, k = 1, 2, 3$. The main reason is due to that the dimension of the null space of the 20×20 matrix is two from table 2 in near degenerate cases.

4 Experimental Results

Based on the linear symbolic-numeric method, we may have the following algorithm for the *4-point pose estimation problem*:

- Compute the c_{ij} from the image points and the camera calibration matrix K.
- Compute the inter-point distances $|P_iP_j|$ from the reference points.
- Compute the solution x_1, x_2, x_3 of the polynomial system (4) using the symbolic-numeric method [23].
- Recover the camera-point distances X_1, X_2, X_3, X_4 from the equivalence correspondence (3).
- Estimate the coordinates of the 3D reference points in a camera-centered 3D frame: $\bar{P}_i = X_i K^{-1} U_i$.
- Compute the camera rotation and translation using the absolute orientation $[9, 20, 21]$.

The following experiments are done with Maple 8 in the default setting of digits (Digits=10).

The first experiment is to show the accuracy and stability of the algorithm for the general *4-point pose estimation*. The optical center is located at the origin and the matrix of camera's intrinsic parameters is assumed to be the identity matrix. At each trial, four noncoplanar control points are generated at random within a cube centered at $(0, 0, 50)$ and of dimension $60 \times 60 \times 60$. The orientation Euler angles of the camera are positioned randomly. The control points are projected onto an image plane using the camera pose and internal parameters. We carry out one hundred trials and generate 100 sets of control points randomly for each trial. For a set of solutions, we substitute them into (1) and check the backward error. The backward error of the experimental results is generally less than 10^{-8}.

We check the stability of the algorithm. The relative error of the estimated translation t_i w.r.t. the true t is measured by $2|t_i - t|/(|t_i| + |t|)$. The relative error of the estimated rotation R_i w.r.t. the true R is measured by the sum of the absolute values of the three Euler angles of the relative rotation $R_i R^T$(Fig. 2). We also check the failure rate defined as the percentage of total trials where either the rotation error or the translation error is over 0.5 (Fig. 3).

The second experiment is to show the accuracy and the stability of the algorithm in determining the solutions for the critical configurations. As mentioned in the introduction, the pose problem has some computationally troublesome singular cases. Fig. 4 and Fig. 5 show the relative error and the failure rate for one such critical configuration using our symbolic-numeric linear method.

The data is 4 coplanar points in a square $[-1, 1] \times [-1, 1]$ and the camera starts at position=0, at a singular point directly above their center $(0.5 < h < 1.5)$, where h is the height of the camera. The camera then moves sideways parallel to one edge of the square. At position=$\sqrt{2}$ units it crosses the side of the vertical circular cylinder through the 4 data points, where another singularity occurs. From Fig. 4 and Fig. 5, the relative error and especially the failure rate of the

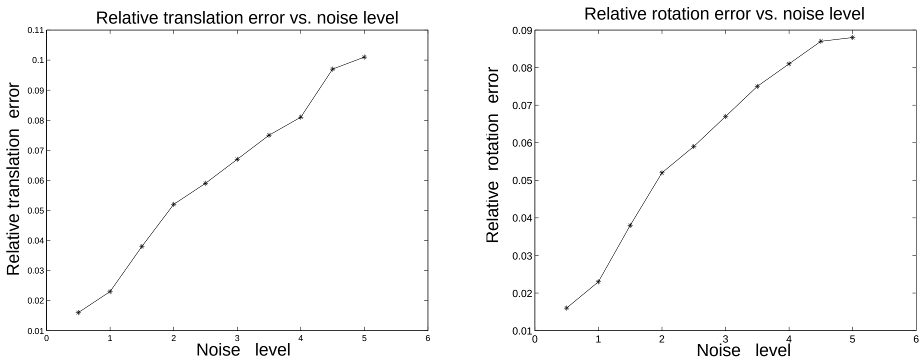

Fig. 2. Relative errors vs. noise level

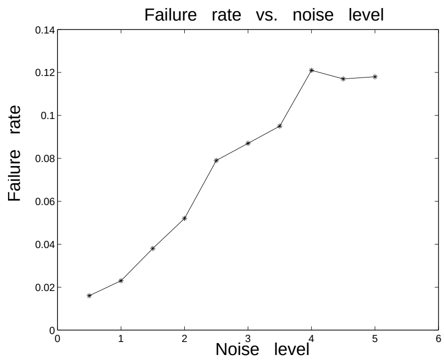

Fig. 3. Failure rate vs. noise level

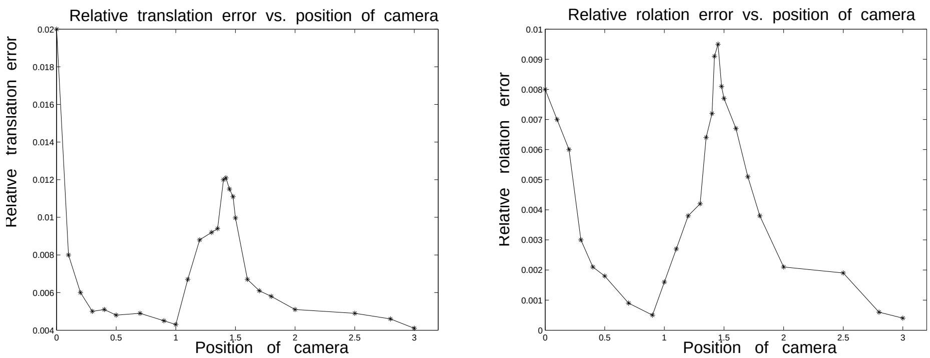

Fig. 4. Relative translation errors vs. noise level for the critical configurations

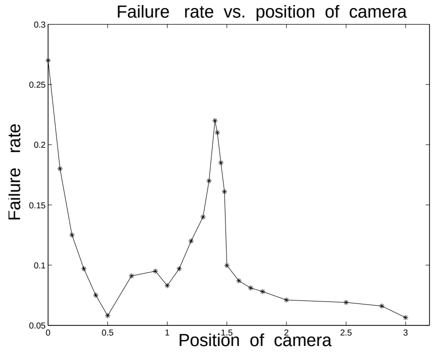

Fig. 5. Failure rate for the critical configurations

algorithm are significantly lower compared with the algorithm in [2]. The relative error and the failure rate of our algorithm are also acceptable. It is natural that the error and failure rate near the position 0 and $\sqrt{2}$ are a little higher than at other positions.

It is clear that the experimental results are very similar to those we have presented in [23]. However, the computation is simpler due to smaller size of the polynomial system.

5 Conclusion

In this paper, we present a stable algorithm to find the numeric solution for *4-point pose estimation*. The algorithm gives a unique solution whenever the control points are not sitting on one of the known critical configurations. When the control points are sitting on or near some known critical configurations, the algorithm also obtains reliable solutions. Compared with other algorithms, the main advantage of our linear algorithm is that it can recognize the critical and near critical cases and deal with different cases in different ways. The matrices in our approach are only bigger than those used in other approaches when the points are near critical configurations. The experiments show that the new simple polynomial system for *4-point pose estimation* is well solvable by our symbolic-numeric method.

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