Solving Nonlinear Polynomial Systems via Symbolic-Numeric Elimination Method

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Abstract

Consider a general polynomial system S in $x_1, \ldots, x_n$ of degree q and its corresponding vector of monomials of degree less than or equal to q. The system can be written as

$$M_0 \cdot [x_1^q, x_1^{q-1}x_2, \ldots, x_n^2, x_1, \ldots, x_n, 1]^T = [0, 0, \ldots, 0, 0, \ldots, 0, 0]^T \tag{1}$$

in terms of its coefficient matrix M_0. Here and hereafter, $[...]^T$ means the transposition. Further, $[\xi_1, \xi_2, \ldots, \xi_n]$ is one of the solutions of the polynomial system, if and only if

$$[\xi_1^q, \xi_1^{q-1}\xi_2, \ldots, \xi_n^2, \xi_1, \ldots, \xi_n, 1]^T \tag{2}$$

is a null vector of the coefficient matrix M_0.

Since the number of monomials is usually bigger than the number of polynomials, the dimension of the null space can be big. The aim of completion methods, such as ours and those based on Gröbner bases and others [4, 5, 6, 7, 8, 10, 16, 18, 17, 12, 9, 20], is to include additional polynomials belonging to the ideal generated by S, to reduce the dimension to its minima.

The bijection

$$\phi : x_i \leftrightarrow \frac{\partial}{\partial x_i}, \quad 1 \le i \le n, \tag{3}$$

maps the system S to an equivalent system of linear homogeneous PDEs denoted by R. Jet space approaches are concerned with the study of the jet variety

$$V(R) = \left\{ \left(\underset{q}{u}, \underset{q-1}{u}, \ldots, \underset{1}{u}, u \right) \in J^q : R\left(\underset{q}{u}, \underset{q-1}{u}, \ldots, \underset{1}{u}, u \right) = 0 \right\}, \tag{4}$$

where $\underset{j}{u}$ denotes the formal jet coordinates corresponding to derivatives of order exactly j.

A single prolongation of a system R of order q consists of augmenting the system with all possible derivatives of its equations, so that the resulting augmented systems, denoted by DR, has order $q + 1$. Under the bijection ϕ, the equivalent operation for polynomial systems is to multiply by monomials, so that the resulting augmented system has degree $q + 1$.

A single geometric projection is defined as

$$E(R) := \left\{ \left(\underset{q-1}{u}, \ldots, \underset{1}{u}, u \right) \in J^{q-1} : \exists\, \underset{q}{u}, R\left(\underset{q}{u}, \underset{q-1}{u}, \ldots, \underset{1}{u}, u \right) = 0 \right\}. \tag{5}$$

The projection operator E maps a point in J^q to one in J^{q-1} by simply removing the jet variables of order q (i.e. eliminating $\underset{q}{u}$). For polynomial systems of degree q, by the bijection ϕ, the projection is equivalent to eliminating the monomials of the highest degree q. To numerically implement an approximate involutive form method, we proposed in [19, 13, 14, 1] a numeric projection operator $\hat{E}$ based on singular value decomposition.

The system $R = 0$ is said to be (exactly or symbolically) involutive [11] at order k and projected order l, if $E^l(D^kR))$ satisfies the projected elimination test

$$\dim E^l\left(D^kR\right) = \dim E^{l+1}\left(D^{k+1}R\right) \tag{6}$$

and the *symbol* of $E^l(D^kR)$ is involutive.

The symbol space of a system is the Jacobian matrix of the system with respect to its highest order jet coordinates. The definition of the symbol space implies that

$$\dim\left(\text{Symbol}\, E^l\left(D^kR\right)\right) = \dim E^l\left(D^kR\right) - \dim E^{l+1}\left(D^kR\right). \tag{7}$$

By the famous Cartan-Kuranishi Theorem [3, 11], after application of a finite number of prolongations and projections, the algorithm above terminates with an involutive or an inconsistent system.

Suppose that R is involutive at prolonged order k and projected order l, and by the bijection ϕ has corresponding system of polynomials S. Then the dimension of $\hat{E}^l(D^kR)$ allows us to determine the number of approximate solutions of S up to multiplicity. In particular these solutions approximately generate the null space of $\hat{E}^l(D^kR)$. It should be noticed that for polynomial system of finite number of solutions, if $E^l(D^kR)$ is involutive, then $\dim\left(\text{Symbol}\, E^l\left(D^kR\right)\right) = 0$ [15]. Hence, the projected involutive symbol test amounts to verifying whether:

$$\dim E^l\left(D^kR\right) = \dim E^{l+1}\left(D^kR\right) \tag{8}$$

in this case. Moreover, we can form the multiplication matrices from the null space of $E^l\left(D^kR\right)$ and $E^{l+1}\left(D^kR\right)$. The solutions can be obtained by computing eigenvalues and eigenvectors. The details are discussed in the following example given by Stetter in [18].

$$p_1 := -3.8889 + 0.078524\, x + 0.66203\, y + 2.9722\, x^2 - 0.46786\, xy + 1.0277\, y^2,$$
$$p_2 := -3.8889 + 0.66203\, x - 0.078524\, y + 1.0416\, x^2 + 0.70179\, xy + 3.9584\, y^2.$$

Using the methods of [18], this is a difficult problem which required about 30 Digits of precision to obtain 10 correct digits for the y-component if we are using a generic normal set $\{1, x, x^2, x^3\}$.

The method we now describe does not use a normal set, and only needs Digits = 10 for success in Maple 9. Under the bijection ϕ, the system is equivalent to the PDE system R. Applying the symbolic-numeric completion method to R with tolerance 10^{-9}, we obtain the table of dimensions of $\hat{E}^l(D^kR)$ below:

	$k=0$	$k=1$	$k=2$	$k=3$
$l=0$	4	4	4	4
$l=1$	3	4	4	4
$l=2$	1	3	4	4
$l=3$		1	3	4
$l=4$			1	3
$l=5$				1

Applying the approximate version of the involutive test to the example shows that the system involutive after one prolongation and no projection, i.e. $k = 1$, $l = 0$, yielding DR as the sought approximately involutive system.

The involutive system has $\dim(DR) = 4$ and so by the bijection the polynomial system has 4 solutions up to multiplicity, and the monomial bases for these spaces should include the second degree monomials in order to recover all solutions. In the following, we show how to find the solutions without computing normal set w.r.t. a specified order of variables. It is a key improvement on [14] since there a type of normal set was used.

1. Compute an approximate basis of the null space of DR, denoted by a 4×10 matrix B. The 4×6 submatrix B_1 of B by deleting entries corresponding to the third degree monomials is a basis of null space of $\hat{E}(DR)$ since $\dim(\hat{E}(DR)) = \dim(DR) = 4$.
2. Let $\mathcal{N} = \left[x^2, xy, y^2, x, y, 1\right]$ be the set of all monomials of degree less than or equal to 2. For numerical stability, instead of selecting four monomials as the normal set from $\mathcal{N}$, we compute the SVD of $B_1 = U \cdot S \cdot V$. The first four columns of U form the 6×4 submatrix U_s, and guarantee a stable polynomial set $\mathcal{N}_p = U_s^T \cdot \mathcal{N}^T$ (including four quadratic polynomials) for computing multiplication matrices.
3. The multiplication matrices of x, y with respect to $\mathcal{N}_p$ can be formed as $M_x = U_s^T \cdot B_x \cdot V^T \cdot S_i$ and $M_y = U_s^T \cdot B_y \cdot V^T \cdot S_i$, where B_x, B_y are the $1, 2, 3, 5, 6, 8$ and $2, 3, 4, 6, 7, 9$ rows of B corresponding to monomials $x^3, x^2y, xy^2, x^2, xy, x$ and $x^2y, xy^2, y^3, xy, y^2, y$ respectively, and S_i is a well-conditioned diagonal matrix with elements which are inversions of the first four elements of S: 0.99972, 0.95761, 0.64539, 0.58916.
4. Compute the eigenvectors v_p of $M_x - M_y$(or any random linear combination of M_x, M_y), and recover the eigenvector corresponding to the monomial set $\mathcal{N}$ by $v = U_s \cdot v_p$. Since $x, y, 1$ appear as the last three components in $\mathcal{N}$, the solutions of p_1, p_2 can be obtained as $x = v[4, i]/v[6, i], y = v[5, i]/v[6, i]$:

$$\{x = 1.04972, y = -0.80689\}\,; \quad \{x = 1.04972, y = 0.64062\}\,;$$
$$\{x = -1.20441, y = -0.78652\}\,; \quad \{x = -0.76039, y = 1.05888\}\,.$$

Substituting these solutions back to p_1, p_2, we found that the errors are smaller than 10^{-8}. It should be noticed that, for this example, although the first two solutions have the same x values, there is no sincere multiple root. So the step 4 is successful. Otherwise, we could apply a reordered Schur factorization method in [2] to the multiplication matrices $M_x, M_y, ...$ to recover all roots including the multiplicities.

The method has been applied successfully to solve some over-determined problems such as camera pose determination in singular positions [14]. At present, the method can only be used to solve zero-dimensional polynomial systems. Our test suit and Maple implementation are available by request.

References

[1] J. Bonasia, F. Lemaire, G.J. Reid, R. Scott, and L. Zhi. *Determination of approximate symmetries of differential equations*, in Gomez-Ullate, Winternitz, editor, CRM Proceedings and Lecture Notes, **39**, 233-249, Amer, Math, Soc, 2004.

[2] R.M. Corless, P.M. Gianni and B.M. Trager. *A Reordered Schur Factorization Method for Zero-Dimensional Polynomial Systems with Multiple Roots*, Proc. ISSAC, W.W. Küchlin, ed., 133-140, 1997.

[3] M. Kuranishi. *On E. Cartan's Prolongation Theorem of Exterior Differential Systems*, Amer. J. Math, **79**, 1-47, 1957.

[4] B. Mourrain and Ph. Trébuchet. *Solving Projective Complete Intersection Faster*, Proc. ISSAC, C. Traverso, ed., New York, ACM Press, 430-443, 2000.

[5] D. Lazard. *Gaussian Elimination and Resolution of Systems of Algebraic Equations*, Proc. EUROCAL **83**, 146-157, 1993.

[6] J.C. Faugére. *A New Efficient Algorithm for Computing Gröbner Bases without Reduction to Zero(F5).* Proc. ISSAC, T. Mora, ed., New York, ACM Press, 75-83, 2002.

[7] V.P. Gerdt, Y.A. Blinkov. *Involutive Bases of Polynomial Ideals*, Mathematics and Computers in Simulation, **45**, 519-541, 1998.

[8] B. Mourrain. *A New Criterion for Normal Form Algorithms*, Proc. AAECC, Fossorier, M.Imai, H. Shu Lin and Poli, A., eds., LNCS, **1719**, Springer, Berlin, 430-443.

[9] B. Mourrain and Ph. Trébuchet, *Algebraic methods for numerical solving,* In Proceedings of the 3rdth International Workshop on Symbolic and Numeric Algorithms for Scientific Computing, pp. 42–47.

[10] H.M. Möller, T. Sauer. *H-bases for polynomial interpolation and system solving.* Advances Comput. Math., to appear.

[11] J.F. Pommaret. *Systems of Partial Differential Equations and Lie Pseudogroups*, Gordon and Breach Science Publishers, 1978.

[12] Ph. Trébuchet. *Vers une Résolution Stable et Rapide des Équations Algébriques.* Ph.D. Thesis, Université Pierre et Marie Curie, 2002.

[13] G.J. Reid, P. Lin, and A.D. Wittkopf. *Differential Elimination-completion Algorithms for DAE and PDAE*, Studies in Applied Mathematics, **106**(1), 1-45, 2001.

[14] G.J. Reid, J.Tang and L. Zhi. *A Complete Symbolic-Numeric Linear Method for Camera Pose Determination*, Proc. ISSAC, J.R. Sendra, ed., New York, ACM Press, 215-223, 2003.

[15] W.M. Seiler. *Analysis and Application of the Formal Theory of Partial Differential Equations*, Ph.D. Thesis, Lancaster University, 1994.

[16] W. Auzinger, H. Stetter. *An Elimination Algorithm for the Computation of All Zeros of a System of Multivariate Polynomial Equations*, Numerical Mathematics, Proceedings of the International Conference, Singapore, 1988, Vol. 86 of Int. Ser. Numer. Math., 11-30.

[17] H.M. Möller and H. Stetter. *Multivariate Polynomial Equations with Multiple Zeros Solved by Matrix Eigenproblems*, Numer. Math., **70**, 311-329, 1995.

[18] H.J. Stetter. *Numerical Polynomial Algebra*, SIAM, 2004.

[19] A.D. Wittkopf and G.J. Reid. *Fast Differential Elimination in C: The CDiffElim Environment,* Comp. Phys. Comm., **139**(2), 192-217, 2001.

[20] W.T. Wu. *On the Construction of Gröebner Basis of a Polynomial Ideal Based on Riquier-Janet Theory,* MM-Res. Preprints, MMRC, **5**, 5-22,1990; Also in Sys. Sci.& Math. Sci., **4**, 193-207,1991.