A Complete Symbolic-Numeric Linear Method for Camera Pose Determination*

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ABSTRACT
Camera pose estimation is the problem of determining the position and orientation of an internally calibrated camera from known 3D reference points and their images. We briefly survey several existing methods for pose estimation, then introduce our new complete linear method, which is based on a symbolic-numeric method from the geometric (Jet) theory of partial differential equations. The method is stable and robust. In particular, it can deal with the points near critical configurations. Numerical experiments are given to show the performance of the new method.

Categories and Subject Descriptors
H.4 [**Information Systems Applications**]

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Calibration, Camera Pose Estimation, Polynomial Solving, Numerical Linear Algebra, Numeric Elimination, Partial Differential Equations, Jet Spaces, Involutive Bases, Numeric Jet Geometry.

1. INTRODUCTION
Given a set of correspondences between 3D reference points and their images, *pose estimation* consists of determining the position and orientation of the camera with respect to the known reference points. It is also called *space resection* in the photogrammetry community. It is a classical and common problem in computer vision and photogrammetry and has been studied in the past. Camera pose estimation impacts many important fields, such as computer vision [15], automation, image analysis and automated cartography [8], robotics [1].

With three points, the problem generically has four possible solutions. Fischer and Boles characterize the problem using a biquadratic polynomial in one unknown. Haralick et al. [13] review many old and new variants of the basic 3-point methods and carefully examine their numerical stability under different order of substitution and elimination. Gao et al. [9] use Wu-Ritt's zero decomposition method to obtain a complete solution classification. Three-point methods intrinsically give multiple solutions. If a unique solution is required, additional information must be given, and a fourth point generally suffices. But there are certain degenerate cases for which no unique solution is possible. These *critical configurations* are known precisely and include the following notable degenerate case: a 3D line and a circle in an orthogonal plane touching the line. Many linear methods have been presented for finding the solution in the unique solution case [29]. Horaud et al. [15] obtain a fourth degree polynomial equation and prove that the problem has at most four possible solutions. In [29], Quan and Lan give a special linear method which finds the unique solution in the generic case. In [2] a method is presented that solves the problem including the critical configurations, but the relative error and failure rate (backward error) are significantly higher than one would like. In Gao & Tang [10], the authors give the triangular decomposition and closed form solution for this problem.

We present a new linear method which works well even in the degenerate cases and obtain the following results:

1. In general, our method gives a linear and unique solution. The experiments show that the unique solution is stable and the new method is robust.
2. For the well-known critical configurations, we first estimate the number of solutions, and then find the linear solutions according to that requirement. It is the first method that can be used to solve the pose estimation problem near critical configurations.
3. The method applies to the case $n > 4$, where n is the number of reference points.

We briefly mention the origin and motivation for our meth-

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ods. Exact elimination methods for exactly given polynomial systems (e.g. Gröbner Bases), usually employ Gaussian Elimination (e.g. linear elimination of monomials). Such exact methods usually depend on the ordering of input (e.g. term ordering in the case of Gröbner Bases), and so are coordinate dependent. Since the order of elimination can force division by small leading entries, such methods are generally unstable, when used on approximate systems. In contrast, exact elimination methods from the geometric theory of PDE (Jet Space methods) are coordinate independent, and this motivated our study of numerical versions of such methods [4, 31, 38] which is continued in this paper (see Tuomela and Arponen [36] for a jet space approach to numerical DAE solving).

We exploit the well-known correspondence between polynomial systems and systems of constant coefficient linear homogeneous PDE. This equivalence has been extensively studied and exploited in the exact case by Gerdt [11] and his co-workers in their development of *involutive bases*. An early example of its use is in [20] and it is heavily used in [7, 22, 33]. This correspondence is used in our paper to write the polynomial systems in the form of PDE systems, to which our new numeric methods are applied. The method depends on viewing the polynomial systems as matrix functions of their monomials, and applying Numerical Linear Algebra (in particular the Singular Value Decomposition), to the null spaces of these maps. Ultimately the solutions are determined by applying eigenvalue-eigenvector techniques to a related eigen-problem. We direct the reader to the work of Stetter [3, 23], who in contrast to our approach, uses the method of border bases, and works directly with the coefficient matrix, in a method which is closer to a numerical translation of the idea of Gröbner Bases. Ultimately Stetter also obtains solutions of zero dimensional problems from a related eigen-problem (also see the work of Mourrain [24, 22, 25, 26] and others [6, 7, 20, 21, 27]).

The rest of the paper is organized as follows. In Section 2, we introduce the basic geometry of the camera pose and *the 4-point pose estimation problem*. In Section 3, we present the method and provide an illustrative example. In Section 4, the simulated experimental results are given. Our conclusions are discussed in Section 5.

2. GEOMETRY OF CAMERA POSE FROM FOUR POINTS

Given a calibrated camera centered at $\mathbf{c}$ and correspondences between four 3D reference points $\mathbf{P_i}$ and their images $\mathbf{u_i}$, each pair of correspondences i and j gives a constraint on the unknown camera-point distances $x_i = |\mathbf{p_i} - \mathbf{c}|$(cf. Figure 1):

$$P_{ij}(x_i, x_j) \equiv x_i^2 + x_j^2 + c_{ij} x_i x_j - d_{ij}^2 = 0$$

$$c_{ij} \equiv -2\cos\theta_{ij}$$

where $d_{ij} = |\mathbf{p_i} - \mathbf{p_j}|$ is the known inter-point distance between the $i-$th and $j-$th reference points and θ_{ij} is the 3D viewing angle subtended at the camera center by the $i-$th

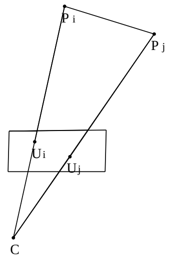

Figure 1: The basic geometry of camera pose determination for each pair of correspondences

and $j-$th points. The cosine of this viewing angle is directly computed from the image points and the calibration matrix $\mathbf{K}$ of the internal parameters of the camera.

In the following, we will mainly discuss the case: $n = 4$, i.e., the *4-point pose estimation problem*. Let P be the calibrated camera center, and A, B, C, D be the reference points (see Figure 2). Let $p = 2\cos\angle(BPC)$, $q = 2\cos\angle(APC)$, $r = 2\cos\angle(APB)$, $s = 2\cos\angle(CPE)$, $t = 2\angle(APE)$, $u = 2\cos\angle(BPE)$.

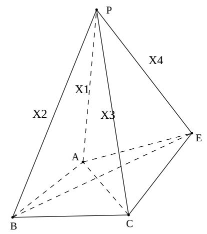

Figure 2: The 4-point pose estimation problem

From triangles PAB, PAC, PBC, PAE, PBE and PCE we obtain the 4$-$*point pose estimation equation system*:

$$\left\{\begin{array}{l} X_1^2 + X_2^2 - rX_1X_2 - |AB|^2 = 0 \\ X_1^2 + X_3^2 - qX_1X_3 - |AC|^2 = 0 \\ X_2^2 + X_3^2 - pX_2X_3 - |BC|^2 = 0 \\ X_1^2 + X_4^2 - sX_1X_4 - |AE|^2 = 0 \\ X_4^2 + X_3^2 - tX_3X_4 - |CE|^2 = 0 \\ X_2^2 + X_4^2 - uX_2X_4 - |BE|^2 = 0 \end{array}\right. \quad (1)$$

The recovered camera-point distances $\mathbf{X_i}$, are used to estimate the coordinates of the 3D reference points in a camera-centered 3D frame: $\bar{\mathbf{P}}_\mathbf{i} = \mathbf{X_i}\mathbf{K}^{-1}\mathbf{u_i}$ (see [29]). The final step is the absolute orientation determination [32]. The determination of the translation and the scale follow immediately from the estimation of the rotation.

The system (1) is an overdetermined polynomial system of six equations in 4 variables. The parameters p, q, r, s, t, u and $|AB|$, $|AC|$, $|BC|$, $|AE|$, $|BE|$, $|CE|$ are data of limited accuracy. It is still very difficult to use Gröbner basis algorithms [5] or Ritt-Wu's characteristic algorithms [37, 39] to solve such approximate polynomial systems. Linear methods in [2, 29] can solve problems in generic cases, but have trouble near degenerate cases. In the following, we propose a complete linear method which solves the above polynomial system including cases where A, B, C, D are near critical configurations.

3. LINEAR METHODS FOR POSE DETERMINATION FROM 4 POINTS

3.1 Symbolic-Numeric completion of polynomial systems

Consider a general polynomial system $S = \{p_1(x_1, \ldots, x_n) = 0, \ldots, p_m(x_1, \ldots, x_n) = 0\}$ of degree q and its corresponding vector of monomials of degree less than or equal to q. For example, if $q = 2$, $n = 2$, $m = 2$ then the monomial vector is the transpose of $[x_1^2, x_1x_2, x_2^2, x_1, x_2, 1]$. In the general case the system can be written as

$$M_0 \cdot \begin{pmatrix} x_1^q \\ x_1^{q-1}x_2 \\ \vdots \\ x_n^2 \\ x_1 \\ \vdots \\ x_n \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \qquad (2)$$

in terms of its coefficient matrix M_0. Further, $[\xi_1, \xi_2, \ldots, \xi_n]$ is one of the solutions of the polynomial system, if and only if the transpose of

$$[\xi_1^q, \xi_1^{q-1}\xi_2, \ldots, \xi_n^2, \xi_1, \ldots, \xi_n, 1] \qquad (3)$$

is a null vector of the coefficient matrix M_0.

Since the number of monomials is usually bigger than the number of polynomials, the dimension of the null space can be much bigger than zero. The aim of completion methods, such as our method and Gröbner Bases, is to include additional polynomials belonging to the ideal generated by S, to reduce these dimensions to their minima.

There are many ways to include new polynomials. For example the authors of [2] suggest computing the null space of the coefficient matrix corresponding to $\{x_ip_j,\ 1 \le i \le n,\ 1 \le j \le m\}$. If the null space is of dimension 1, then the solution of S can be easily recovered from the null vector. The authors of [2] point out that the enlarged matrix of the *4-point pose estimation problem* should generically have one dimensional null space. However they also show that the failure rate of their method is undesirably high near singular cases. The main reason as shown by the example in next subsection is that the dimension of the null space in the singular case is bigger than 1.

The method we now describe is based on the geometric theory of PDE [19, 28] which in the exact case has algorithms described in [30, 34]. Symbolic-numeric completion methods for PDE based on this approach are described in [38]. This method uses a bijection to convert the given polynomial system (1) into an equivalent PDE system, and then solves the system stably (including the singular cases).

In particular the bijection

$$\phi : x_i \leftrightarrow \frac{\partial}{\partial x_i}, \quad 1 \le i \le n, \qquad (4)$$

maps the system S to an equivalent system of linear homogeneous PDE denoted by R. These PDE can also be written in matrix form:

$$M_0 \cdot \begin{pmatrix} \frac{\partial^q u}{\partial x_1^q} \\ \frac{\partial^q u}{\partial x_1^{q-1}\partial x_2} \\ \vdots \\ \frac{\partial^2 u}{\partial x_n^2} \\ \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_n} \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \qquad (5)$$

The approach of Jet Geometry, is to study the above system, by studying the properties of the linear mapping: $R : v \mapsto R(v) = M_0 v$. Then v is regarded as a vector of formal coordinates (no longer derivatives of solutions), in a complex Euclidean space, whose dimension is the same as the total number of derivatives (or equivalently monomials), less than or equal to q. That space is called the jet space of order q, is denoted by $J^q \approx C^{n_q}$ and is easily seen to have dimension $n_q = \begin{pmatrix} n+q \\ q \end{pmatrix}$. In addition, we partition v into groups, $v = [\underset{q}{u}, \underset{q-1}{u}, \ldots, \underset{1}{u}, u]^T$, where $\underset{j}{u}$ denotes the formal jet coordinates corresponding to derivatives of order exactly j. Jet space approaches are concerned with the study of the (jet) variety

$$V(R) = \{(\underset{q}{u}, \underset{q-1}{u}, \ldots, \underset{1}{u}, u) \in J^q : R(\underset{q}{u}, \underset{q-1}{u}, \ldots, \underset{1}{u}, u) = 0\}, \qquad (6)$$

a much easier task than directly studying the solutions of the PDE system.

There are two major coordinate-independent operations in the geometric theory of PDE: prolongation and projection.

A single prolongation (differentiation) of a system R of order q consists of augmenting the system with all possible derivatives of its equations, so that the resulting augmented systems, denoted by DR, has order $q+1$ [28, 34]. Under the bijection ϕ, the equivalent operation for polynomial systems, to multiply by monomials, so that the resulting augmented system has degree $q+1$. For the polynomial system S, in the case where each equation has the same degree q, this yields the equivalent system $\{p_1, \ldots, p_m, x_1p_1, x_1p_2, \ldots, x_np_m\}$. The monomial vectors are updated to

$$[x_1^{q+1}, x_1^q x_2, \ldots, x_n^3, x_1^2, \ldots, x_n, 1]^T.$$

The prolonged system DR can also be written in matrix form

$$M_1 \cdot \begin{pmatrix} \underset{q+1}{u} \\ \underset{q}{u} \\ \vdots \\ \underset{1}{u} \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \qquad (7)$$

using a constant matrix of form $M_1 = \begin{pmatrix} * & * \\ \mathbf{0} & M_0 \end{pmatrix}$. Thus it is a mapping on J^{q+1}. Successive prolongations $D^2R, \ldots$, etc. can similarly be written using larger constant matrices $M_2, \ldots$, etc.

A single geometric projection is defined as

$$E(R) := \{(\underset{q-1}{u}, \ldots, \underset{1}{u}, u) \in J^{q-1} : \exists\, \underset{q}{u}, R(\underset{q}{u}, \underset{q-1}{u}, \ldots, \underset{1}{u}, u) = 0\} \quad (8)$$

The projection operator E maps a point in J^q to one in J^{q-1} by simply removing the jet variables of order q (i.e. eliminating $\underset{q}{u}$). For polynomial systems of degree q, by the bijection ϕ, the projection is equivalent to eliminating the monomials of the highest degree q.

The system $R = 0$ is said to be (exactly or symbolically) involutive [28] at order k and projected order l, if $E^l(D^k(R))$ satisfies the projected elimination test

$$\dim E^l(D^k R) = \dim E^{l+1}(D^{k+1} R) \quad (9)$$

and the *symbol* of $E^l(D^k R)$ is involutive(see [28], [30] for details). An algorithm to obtain such a form with exact input and exact intermediate computations is to compute $D^k R$, $k = 0, 1, \ldots$ until some projection $E^l D^k R$, $l = 0, 1, \ldots, k$ is exactly involutive. This is the exact algorithm underlying our numeric method.

The symbol space of a system is the jacobian matrix of the system with respect to its highest order jet coordinates. The definition of the symbol space implies that

$$\dim(\text{Symbol } E^l(D^k R)) = \dim E^l(D^k R) - \dim E^{l+1}(D^k R)$$
$$\text{Rank}(\text{Symbol } E^l(D^k R)) = n_q - \dim \text{Symbol } E^l(D^k R) \quad (10)$$

When $\dim(\text{Symbol } E^l(D^k R)) = 0$, or equivalently when that the symbol has full rank, it is easily shown that the symbol of $E^l(D^k R)$ is involutive [28], so that the projected involutive symbol test amounts to test:

$$\dim E^l(D^k R) = \dim E^{l+1}(D^k R) \quad (11)$$

in this case (which occurs in this paper). The cases in which the dimension of the symbol is not zero, need a finer analysis of the structure of the symbol space and involve the determination of the dimensions of certain subspaces (the Cartan Characters).

By the famous Cartan-Kuranishi Theorem [19], after application of a finite number of prolongations and projections, the algorithm above terminates with an involutive or an inconsistent system. Involutive systems are locally solvable and contain all their integrability conditions. They allow an existence and uniqueness theorem for local analytic solutions of the original system R by Cartan-Kähler Theorem [18]; which via the bijection ϕ gives us information on the number and existence of solutions of the corresponding polynomial system.

Exact involutive form algorithms, use symbolic differentiation to compute the prolongations of a system of PDE, and exact polynomial elimination methods to compute the eliminations (e.g. Gröbner basis algorithms [5] or Ritt-Wu's characteristic algorithms [37]). However, they are usually unstable when applied with approximate data, such as the application considered in this paper.

To numerically implement an approximate involutive form method, in [4, 38], we proposed a numeric projection operator $\hat{E}$. First, a singular value decomposition [12] of M_k is computed to find the numeric rank of M_k and a basis for its null space. Then, the components corresponding to the highest order jet coordinates are deleted, which yields an approximate projected basis. This generates an approximate spanning set for $\hat{E}(D^k R)$ (and similarly for $\hat{E}^l(D^k R)$). Application of the singular value decomposition to the spanning yields the approximate null space of $\hat{E}(D^k R)$. In this way we obtain the dimensions of which allowing us to determine the dimensions needed in the elimination test (9). Discussion of the involutive symbol test is given in [4, 38].

Suppose R is involutive at prolonged order k and projected order l, and by the bijection ϕ has corresponding system of polynomials S. Then the dimension of $\hat{E}^l(D^k R)$ allows us to determine the number of approximate solutions of S up to multiplicity. In particular these solutions approximately generate the null space of $\hat{E}^l(D^k R)$. We can compute eigenvalues and eigenvectors to find these solutions. The details are discussed in the following example.

3.2 An example for pose estimation from 4 points

The following example corresponds to the third singular case as pointed in [2]. In the example the coordinate of the camera point is $(1, 1, 1)$, and the coordinates of the four control points are $(-1, 1, 0)$, $(-1, -1, 0)$, $(1, -1, 0)$ and $(1, 1, 0)$ respectively. The corresponding *4-point pose estimation equation system* is:

$$\left\{\begin{array}{lcl} p_1 & = & x_1^2 + x_2^2 - 1.49071 x_1 x_2 - 4 \\ p_2 & = & x_1^2 + x_3^2 - .400000 x_1 x_3 - 8 \\ p_3 & = & x_1^2 + x_4^2 - .894427 x_1 x_4 - 4 \\ p_4 & = & x_2^2 + x_3^2 - 1.49071 x_2 x_3 - 4 \\ p_5 & = & x_2^2 + x_4^2 - .666667 x_2 x_4 - 8 \\ p_6 & = & x_3^2 + x_4^2 - .894427 x_3 x_4 - 4 \end{array}\right. \quad (12)$$

We show how our symbolic-numeric method can be used to solve (12). Under the bijection $\phi : x_i \leftrightarrow \frac{\partial}{\partial x_i}$ where $i = 1, 2, 3, 4$, the system is equivalent to the PDE system R:

$$\left\{\begin{array}{l} \phi(p_1)u = \frac{\partial^2 u}{\partial {x_1}^2} + \frac{\partial^2 u}{\partial {x_2}^2} - 1.49071 \frac{\partial^2 u}{\partial x_1 \partial x_2} - 4u \\ \phi(p_2)u = \frac{\partial^2 u}{\partial {x_1}^2} + \frac{\partial^2 u}{\partial {x_3}^2} - .400000 \frac{\partial^2 u}{\partial x_1 \partial x_3} - 8u \\ \phi(p_3)u = \frac{\partial^2 u}{\partial {x_1}^2} + \frac{\partial^2 u}{\partial {x_4}^2} - .894427 \frac{\partial^2 u}{\partial x_1 \partial x_4} - 4u \\ \phi(p_4)u = \frac{\partial^2 u}{\partial {x_2}^2} + \frac{\partial^2 u}{\partial {x_3}^2} - 1.49071 \frac{\partial^2 u}{\partial x_2 \partial x_3} - 4u \\ \phi(p_5)u = \frac{\partial^2 u}{\partial {x_2}^2} + \frac{\partial^2 u}{\partial {x_4}^2} - .666667 \frac{\partial^2 u}{\partial x_2 \partial x_4} - 8u \\ \phi(p_6)u = \frac{\partial^2 u}{\partial {x_3}^2} + \frac{\partial^2 u}{\partial {x_4}^2} - .894427 \frac{\partial^2 u}{\partial x_3 \partial x_4} - 4u \end{array}\right. \quad (13)$$

Applying the symbolic-numeric completion method to R with tolerance 10^{-9}, we obtain the table of dimensions below:

Table 1: dim $\hat{\mathbf{E}}^l \mathbf{D}^k \mathbf{R}$ for (13)

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$l=0$	9	7	4	4	4
$l=1$	5	7	4	4	4
$l=2$	1	3	4	4	4
$l=3$		1	3	4	4
$l=4$			1	3	4
$l=5$				1	3
$l=6$					1

We seek the smallest k such that there exists an $l = 0, ..., k$

with $\hat{E}^l D^k R$ approximately involutive (choosing the largest such l if there are several such values for the given k). Passing the approximate projected elimination test (9), amounts to test looking in the table for the first column with an equal entry in the next column on the downwards sloping diagonal (with both entries being on or above the main diagonal $k = l$). This first occurs for $k = 2$ and $l = 0, 1, 2$.

Applying the approximate version of the projected involutive symbol test to the example, shows that it is passed for $k = 2$, $l = 0$, and $l = 1$, so we choose the largest l ($l = 1$), yielding $\hat{E}D^2R$ as the sought after approximately involutive system.

The involutive system has dim $\hat{E}(D^2R) = 4$ and so by the bijection the polynomial system (12) has 4 solutions up to multiplicity, and the monomial bases for these spaces should include the second degree monomials in order to recover all solutions. In the following, we apply an eigenvalue method to solve (12).

1. Compute an approximate basis of the null space of D^2R, denoted by a 4×70 matrix B. Since $\dim(D^2R) = \dim(ED^2R) = \dim(E^2D^2R) = 4$, the 4×15 submatrix B_1 and 4×35 submatrix B_2 of B by deleting entries corresponding to the third and fourth degree monomials are bases of null spaces of E^2D^2R and ED^2R respectively.

2. Consider the set of all monomials of degree less than or equal to 2:

   $$Mons = [x_1^2, x_1x_2, x_1x_3, x_1x_4, \cdots, x_4^2, x_1, x_2, x_3, x_4, 1]$$

   For numerical stability, we choose 4 random linear combinations of $Mons$ obtained by multiplying $Mons$ by a 15×4 random matrix A_{rand} and obtain the polynomial basis $\mathcal{B} = \{q_1, q_2, q_3, q_4\}$:

   $$\begin{aligned} q_1 &= -0.5\,{x_1}^2 + \cdots - 0.9\,x_1\,x_2 - 0.4\,x_1\,x_3 \\ &\quad -0.9\,{x_2}^2 + \cdots - 0.9\,{x_4}^2 - 0.5 \\ q_2 &= 0.7\,{x_1}^2 + \cdots - 0.9\,x_3\,x_2 - 0.2\,x_3\,x_4 \\ &\quad +0.3\,{x_2}^2 + \cdots + 0.8\,x_3 - 0.4\,x_4, \\ q_3 &= 0.7\,{x_1}^2 + \cdots + 0.2\,x_3\,x_2 - 0.1\,x_3\,x_4 \\ &\quad 0.5\,{x_2}^2 + \cdots + 0.1\,x_4 + 0.7, \\ q_4 &= -0.2\,{x_1}^2 + \cdots - 0.4\,x_3\,x_2 + 0.6\,x_3\,x_4 \\ &\quad -0.5\,{x_2}^2 - 0.7\,x_3 + \cdots + 0.7. \end{aligned}$$

   We note that it is impossible to randomly choose any four distinct monomials in $Mons$ to form a stable monomial basis. The random matrix A_{rand} is necessary to guarantee a stable polynomial basis for the multiplication matrix.

3. The multiplication matrix of x_1 with respect to $\mathcal{B}$ can be formed as $M_{x_1} = (B_1 \cdot A_{\text{rand}})^{-1} B_{21} \cdot A_{\text{rand}}$:

   $$\begin{bmatrix} -9.60694 & -4.22074 & 5.69427 & -4.71943 \\ 4.22540 & 3.30275 & -2.42708 & 1.78735 \\ -1.22749 & -6.83466 & -2.93355 & 1.01883 \\ 13.2366 & -2.60838 & -12.9602 & 9.23773 \end{bmatrix} \quad (14)$$

   where B_{21} is the submatrix of B_2 with columns corresponding to monomials $x_1 \cdot Mons$ (and in detail consists of the columns numbered $1, \cdots, 10, 21, 22, 23, 24, 31$ in B_{21}).

   Since B_1 is of rank four, we always can choose the random matrix A_{rand} ensure that $B_1 \cdot A_{\text{rand}}$ is well-conditioned.

The eigenvalues of M_{x_1} are:

$$2.236067977,\ 2.236067993,\ -2.236067977,\ -2.236067955. \quad (15)$$

Thus there is a pair of positive and a pair of negative eigenvalues coincident up to 8 digits. Form the multiplication matrices w.r.t. x_2, x_3, x_4 independently, and compute the eigenvalues. The similar phenomena as that in (15) can be observed. This tells us that (12) has one positive and one negative double root. We choose one set of positive real eigenvalues which correspond to the distances $|PA|$, $|PB|$, $|PC|$, $|PE|$:

$$\begin{cases} \xi_1 &= 2.236067977 \\ \xi_2 &= 3.000000000 \\ \xi_3 &= 2.236067961 \\ \xi_4 &= 1.000003584 \end{cases} \quad (16)$$

Substituting the solution into (12), we find $|p_i(\xi_1, \xi_2, \xi_3, \xi_4)| < 10^{-7}$ for $i = 1, 2, \ldots, 6$.

If one substitutes the positive solution (16) to the Jacobian matrix

$$\begin{bmatrix} \frac{\partial p_1}{\partial x_1} & \frac{\partial p_1}{\partial x_2} & \frac{\partial p_1}{\partial x_3} & \frac{\partial p_1}{\partial x_4} \\ \frac{\partial p_2}{\partial x_1} & \frac{\partial p_2}{\partial x_2} & \frac{\partial p_2}{\partial x_3} & \frac{\partial p_2}{\partial x_4} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial p_6}{\partial x_1} & \frac{\partial p_6}{\partial x_2} & \frac{\partial p_6}{\partial x_3} & \frac{\partial p_6}{\partial x_4} \end{bmatrix} \quad (17)$$

then the singular values of the Jacobian matrix are

$$6.5319707, 6.1967714, 3.5777055, 0.92618581 \cdot 10^{-5}, 0, 0$$

The Jacobian matrix is near singular. This tells us that the solution is quite unstable for any small perturbations. But our method can deal with this singular case well. Suppose we perturb (12) by errors of order 10^{-6}, the number of solutions read from the dimension table will generally become 2.

In detail, we obtain the following table:

Table 2: dim $\hat{\mathbf{E}}^{\mathbf{l}}\mathbf{D}^{\mathbf{k}}\mathbf{R}$ for perturbation of (13)

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$l=0$	9	7	3	2	2
$l=1$	5	7	3	2	2
$l=2$	1	3	3	2	2
$l=3$		1	3	2	2
$l=4$			1	2	2
$l=5$				1	2
$l=6$					1

Applying the projected elimination and involutive symbol tests shows that $\hat{E}^3D^3R$ is approximately involutive. The computed positive root has backward error of order $10^{-6} \sim 10^{-9}$ in general.

Since the null space of the 24×24 matrix in [2] has dimension 2, their linear method has difficulty dealing with this degenerate case. From the dimension table, using the approximate involution test, the system becomes approximately involutive after 3 prolongations and 3 projections. The matrix we used to solve for the example is of order 70×90 which is quite large compared with the matrices in [2]. Since the polynomial system (12) is of symmetric structure, both x_i and $-x_i$ are solutions. But only positive roots are meaningful as they corresponding to the distances. We will try to reduce the size of our matrices in future using such symmetries.

4. EXPERIMENTAL RESULTS

We first demonstrate the accuracy and stability of the new linear method for the generic cases. Then, we also check our method for points near the *critical configurations.* The following experiments are done with Maple 8 in the default setting of digits (Digits=10).

The first experiment is to show the stability of the method in predicting the number of solutions. The optical center is located at the origin and the matrix of camera's intrinsic parameters is assumed to be the identity matrix. At each trial, four noncoplanar control points are generated at random within a cube centered at $(0, 0, 50)$ and of dimension $60 \times 60 \times 60$. The orientation Euler angles of the camera are positioned randomly. The control points are projected onto an image plane using the camera pose and internal parameters. One hundred trials are carried out and 100 sets of control points are generated for each trial. For each set of control points, two results are computed: one with the original control points; the other with the control points perturbed by random noise at a certain level. In trial i, let n_i be the number of the control points such that the two results are the same and let $\frac{\|n_i - n\|}{n}$ (here $n = 100$) be the relative error. The following Figure 3 gives the relative errors with respect to varying noise. We observe that the computation is robust to variation in noise.

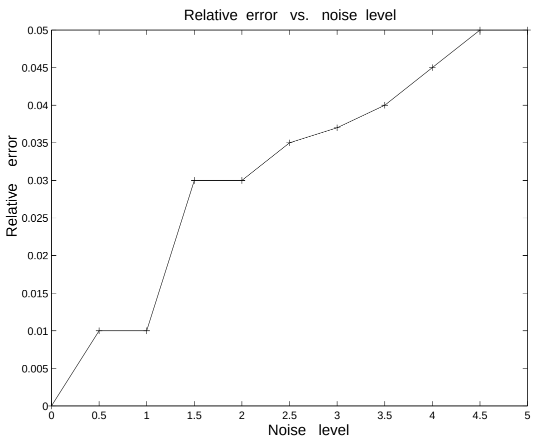

Figure 3. The relative error w.r.t. varying noise

The second experiment is to show the accuracy and stability of the method for the general *4-point pose estimation equation system.* We carry out one hundred trials and generate 100 sets of control points randomly for each trial. For a set of solutions, we substitute them into (1) and check the backward error. The backward error of the experimental results is generally less than 10^{-8}.

In order to check the stability of the method, for each set of control points, two results are computed: one with the original control points X; the other with the control points perturbed by random noise at a certain level $\tilde{X}$. Figure. 4 shows the relative solution errors $\frac{\|\tilde{X} - X\|}{\|X\|}$ w.r.t. noise level. We also check the failure rate defined as the percentage of total trials where the absolute solution error is over 0.5 (Figure 5).

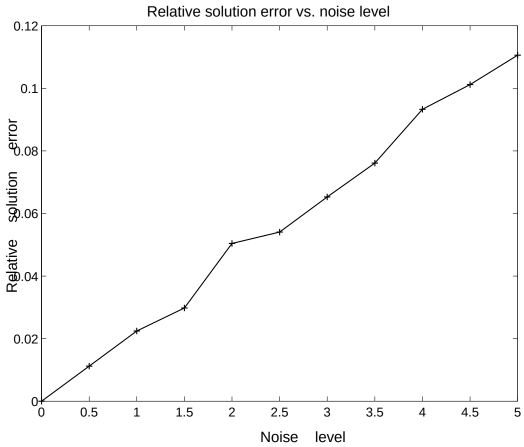

Figure 4. Relative solution errors vs. noise level

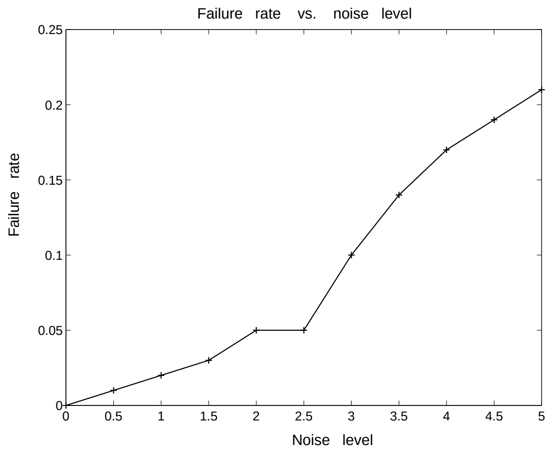

Figure 5. Failure rate vs. noise level

The third experiment is to show the accuracy and the stability of the method in determining the solutions for the critical configurations. As mentioned in the introduction, the pose problem has some computationally troublesome singular cases. Most methods [10, 29] do not consider these singular cases. The only method we found dealing with the singular cases is [2], but the relative error and the failure rate (backward error) reported in that paper are higher than one would like. Figure 6 shows the relative error and the failure rate for one such critical configuration using our symbolic-numeric linear method. The data is 4 coplanar points in a square $[-1, 1] \times [-1, 1]$ and the camera starts at position=0, at a singular point directly above their center ($0.5 < h < 1.5$), where h is the height of the camera. The camera then moves sideways parallel to one edge of the square. At position=$\sqrt{2}$ units it crosses the side of the vertical circular cylinder through the 4 data points, where another singularity occurs. From Figure 6, the relative error and especially the failure rate of the method are significantly lower compared with the method in [2]. The failure rate here is defined as the percentage of total trials where the solution errors are over 10^{-8}. The relative error and the failure rate of our method are also acceptable. It is natural that the error and failure rate near the position 0 and $\sqrt{2}$ are a little higher than at other positions.

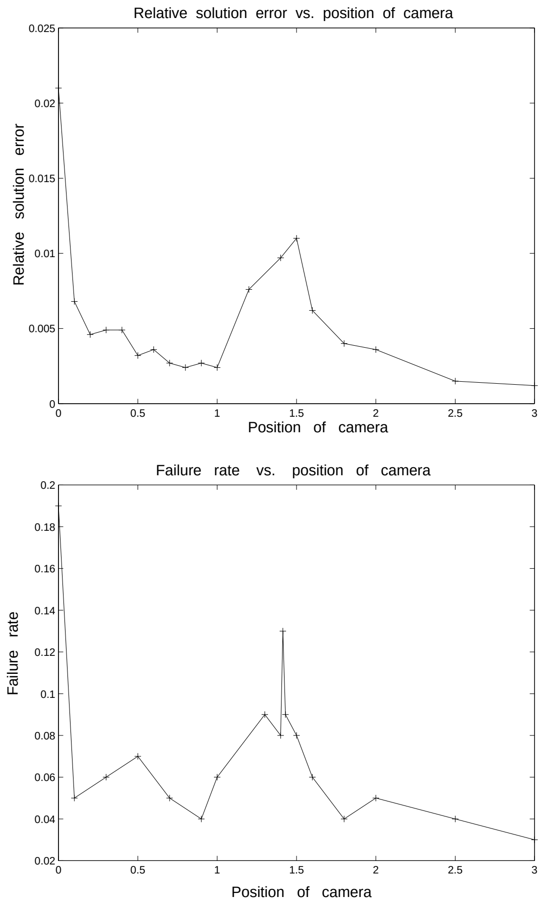

Figure 6. Experimental results for the critical configurations

5. CONCLUSION

In this paper, we present a stable method to find the numeric solutions for pose estimation determination. The linear method gives a linear and unique solution whenever the control points are not sitting on one of the known critical configurations. When the control points are sitting on or near some known critical configurations, the method also obtains reliable solutions. Compared with other methods, the main advantages of our linear method are: it is more stable and for critical configurations, the method still can find the solutions up to certain accuracy. The method developed in this paper can be easily applied to other problem in computer vision with overconstrained polynomial systems. The matrices in our approach are larger than those used in other approaches. In the future we shall try to reduce the size of the matrices by exploiting the structure of the problems and making use of structured singular value decomposition.

This paper represents a beginning step in the applications of the new area of Numeric Jet Geometry. It illustrates the potential usefulness of such methods by giving a complete problem to solution treatment of a problem of practical interest. An experimental analysis (including backward error analysis) of the method is given. Much remains to be done, for example an analytical backward error analysis in terms of an appropriate error metric is an important future task.

Our differential-algebraic method is easily reformulated and implemented by the bijection ϕ in terms of pure linear algebra on monomials (this correspondence is well-known going back at least to [20] and also see [7, 22, 33]). Such an implementation would be more efficient than our current differential method in Maple since it does not have the additional overhead for differentiation.

The algebraic method which has a close relation to our method is the method of H-bases [21], which also focuses on the dimensions of the vector spaces of generated by monomials. Example 2.4 of [21] is an H-basis of degree 4, but can be shown to become involutive only after prolongation to degree 7 (when it also becomes a Gröbner Basis). However this H-basis is *minimally formally integrable*, in the sense defined in [30, Appendix A]. In future work we will investigate the relation between H-bases and minimal formal integrability which unlike H-bases applies to the more general case of differential system.

Under-determined systems (i.e. positive dimension systems) can be treated by extracting their Hilbert polynomial from their Cartan Characters (e.g. see [34]). This allows the determination of the top dimensional positive dimensional components, and the construction of an appropriate random linear subspace, which when intersected with these components, cuts out generic points on those components (i.e. by using a variation of the methods of Sommese, Verschelde and Wampler [35]). Such generic points can be calculated with the eigen-method of this paper.

In addition to the application of numerical linear algebra, the other major direction we are taking in our exploration of Numeric Jet Geometry, is in the extension of the Homotopy Continuation Methods of Sommese, Verschelde and Wampler [35] to polynomially nonlinear systems of PDE (see the works [31, 14]). Even in that nonlinear case, analysis of linear homogenous PDE, such as by the methods of this paper, are needed for determination of the property of involutivity of the symbol of the nonlinear systems.

Further, the innocuous fact that the derivative of a nonlinear PDE becomes linear in its highest derivative, lies at the heart of theory and completion methods for systems of PDE and sets them apart from purely algebraic equations. Exploiting this structure with efficient linear methods in the leading derivatives, and consequently shrinking the size of the nonlinear systems that must be reduced, has been at the centre of the efforts of a number of authors in the developement of efficient symbolic algorithms the exact case. We expect this dichotomy to continue in the numerical case, with numerical linear algebra being exploited for leading derivatives, and smaller nonlinear systems being treated with nonlinear methods such as homotopy methods.

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