2.

回顾：

	运算	性质	例	同态映射
集合 S				
群 G	$G\times G\to G$ $(x,y)\to xy$	$(xy)z=x(yz)$ $x\cdot 1=1\cdot x=x$ $x\cdot x^{-1}=x^{-1}x=1$	S_n $GL_n(\mathbb{R})$	$f:G\to G'$ $f(xy)=f(x)f(y)$
环 R	$R\times R\to R$ $(x,y)\to x+y$ $(x,y)\to xy$	$(R,+)$ 交换群 $(R,\cdot)$ 半群 $x(y+z)=xy+xz$	$\mathbb{Z}$ $\mathbb{Q}[X]$	$f:R\to R'$ $f(x+y)=f(x)+f(y)$ $f(xy)=f(x)f(y)$ $(f(1)=1')$
域 F	同上	$(R,+)$ 交换群 $(R\backslash\{0\},\cdot)$ 交换群 $1\neq 0$	$\mathbb{Q}$ $\mathbb{Z}/p\mathbb{Z}$	

模 M：交换群 + 环的作用　　向量空间

交换群（阿贝尔群）：$xy=xy\ \forall\, x,y\in G$.

同构：同态 + 双射　记作：$\cong$

例子（群）：

$$GL_n(F)=\{A\in M_n(F)\mid \det(A)\neq 0\}$$
$$SL_n(F)=\{A\in GL_n(F)\mid \det(A)=1\}.$$

$$O(n) = \left\{ A : \mathbb{R}^n \longrightarrow \mathbb{R}^n \text{ 且 } (A\vec{x} | A\vec{x}) = (\vec{x} | \vec{x}) = \sum_{i=1}^{n} x_i^2 \quad \forall \vec{x} \in \mathbb{R}^n \right\}$$

$$= \{ A \in GL_n(\mathbb{R}) \mid {}^tA \cdot A = E \}.$$ 保范数

$$A \in O(n) \Rightarrow \det(A) = \pm 1.$$

$$SO(n) = \{ A \in O(n) \mid \det(A) = 1 \}$$ 保范数 + 保方向.

$$U(n) = \left\{ A : \mathbb{C}^n \longrightarrow \mathbb{C}^n \text{ 且 } (A\vec{x} | A\vec{x}) = (\vec{x} | \vec{x}) = \sum_{i=1}^{n} x_i \overline{x_i} \quad \forall \vec{x} \in \mathbb{C}^n \right\}.$$

$$= \{ A \in GL_n(\mathbb{C}) \mid {}^*A \cdot A = E \}.$$

$${}^*A = {}^t\overline{A}$$

$$A \in U(n) \Rightarrow |\det(A)| = 1.$$

$$SU(n) = \{ A \in U(n) \mid \det(A) = 1 \}.$$

$n = 1$

$$O(1) = \{\pm 1\}. \quad SO(1) = \{1\}.$$

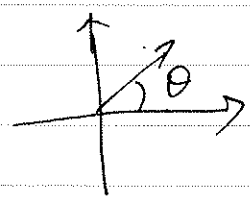
$$U(1) = \{ z \in \mathbb{C} \mid \overline{z} z = 1 \} = \{ e^{i\varphi} \mid 0 \le \varphi < 2\pi \}.$$

$$SU(1) = \{1\}.$$

4.

$n=2$.

$SO(2) = \{ \mathbb{R}^2 上的旋转 \} = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| \, 0 \le \theta < 2\pi \right\}$



$\downarrow$

$U(1)$

$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \longrightarrow \cos\theta + i\sin\theta = e^{i\theta}$

群同态且为双射

$SO(2) \cong U(1)$

几何上: $SO(2) \cong$ ○ $\mathbb{R}^2$ 中的 S^1

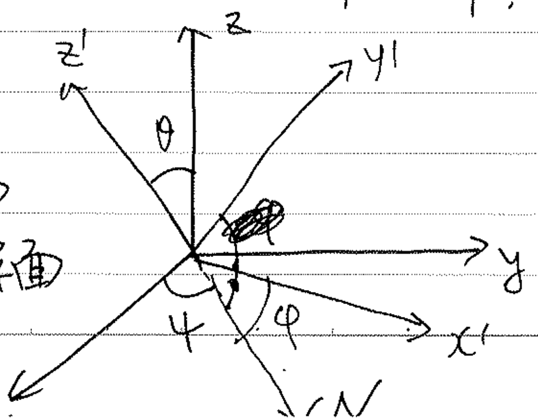
拓扑等价

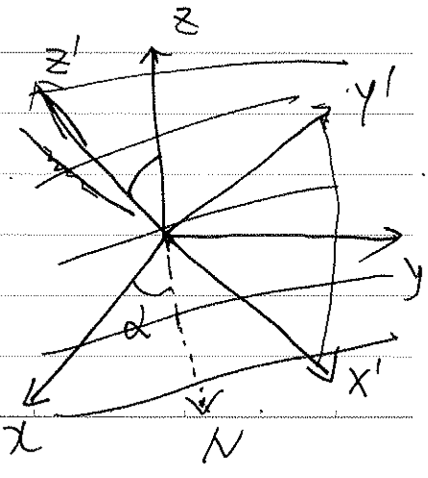
2: $SO(3)$ $SU(2)$ 的参数表示.

$SO(3) = \{ \mathbb{R}^3 上的旋转 \}$. $A \in SO(3)$.

利用Euler角 对A作分解.

$N = xy$-平面 $\cap$ $x'y'$-平面





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(ψ, θ, φ) 称为 Euler 角

$A: Oxyz \longrightarrow Ox'y'z'$

1) z 轴不动, $x \xrightarrow{\psi} N$

2) N 轴不动, $z \xrightarrow{\theta} z'$

3) z' 轴不动 $N \xrightarrow{\varphi} x'$

$$x \xrightarrow{\psi} N \xrightarrow{\varphi} x'$$

$$z \xrightarrow{\theta} z' \qquad y \longrightarrow y'$$

$$A = \underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi & \\ \sin\varphi & \cos\varphi & \\ & & 1 \end{pmatrix}}_{= B_\varphi} \underbrace{\begin{pmatrix} 1 & & \\ & \cos\theta & -\sin\theta \\ & \sin\theta & \cos\theta \end{pmatrix}}_{C_\theta} \begin{pmatrix} \cos\psi & -\sin\psi & \\ \sin\psi & \cos\psi & \\ & & 1 \end{pmatrix}$$

$g \in SU(2)$

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad {}^*g = {}^t\bar{g} = \begin{pmatrix} \bar\alpha & \bar\gamma \\ \bar\beta & \bar\delta \end{pmatrix}$$

$$\alpha\delta - \beta\gamma = 1 \qquad = g^{-1} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$$

$$\Longrightarrow \delta = \bar\alpha \quad \gamma = -\bar\beta$$

$$g \in SU(2) \Longleftrightarrow g = \begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1.$$

6.

$\alpha = \alpha_1 + i\alpha_2 \quad \beta = \beta_1 + i\beta_2$

$|\alpha|^2 + |\beta|^2 = 1 \Rightarrow \alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2 = 1$

$SU(2) \underset{\text{拓扑等价}}{\simeq} S^3$

$$b_\varphi = \begin{pmatrix} e^{-\frac{i\varphi}{2}} & 0 \\ 0 & e^{-\frac{i}{2}\varphi} \end{pmatrix} \in SU(2) \qquad c_\theta = \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \in SU(2).$$

$\forall\ g \in SU(2).\ \exists\ u \in SU(2)$ s.t.

$$g = u\, b_\varphi\, u^{-1}$$

5: $SU(2) \longrightarrow SO(3)$ 存在满同态.

$$M_2^+ = \{ H \in M_2(\mathbb{C}) \mid {}^t\bar{H} = H.\ \mathrm{tr} H = 0 \}.$$

$$= \left\{ \begin{pmatrix} x_3 & x_1 + ix_2 \\ x_1 - ix_2 & -x_3 \end{pmatrix} \middle| x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

$$H_{\vec{x}} = x_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$\Downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\vec{x} = x_1 e_1 + x_2 e_2 + x_3 e_3$

Date

$\Psi: M_2^+ \longrightarrow \mathbb{R}^3$

$H_{\vec{x}} \longrightarrow \vec{x}$

$\Psi(\lambda_1 H_{\vec{x}_1} + \lambda_2 H_{\vec{x}_2}) = \lambda_1 \vec{x}_1 + \lambda_2 \vec{x}_2$

$\quad \overset{\parallel}{\Psi}(H_{\lambda_1\vec{x}_1+\lambda_2\vec{x}_2})$

Ψ是同构.

$(\vec{x}|\vec{x}) = \sum_{i=1}^{3} x_i^2$

$= -\det(H_{\vec{x}})$

令 $g \in SU(2)$.

$\Phi_g^+: M_2^+ \longrightarrow M_2^+$

$H_{\vec{x}} \longrightarrow g H_{\vec{x}} g^{-1}$

· 良定义

${}^*(g H_{\vec{x}} g^{-1}) = {}^*(g^{-1})^* H_{\vec{x}}{}^* g = g H_{\vec{x}} g^{-1}$

$\mathrm{tr}(g H_{\vec{x}} g^{-1}) = \mathrm{tr}\, H_{\vec{x}} = 0.$

· $\Rightarrow g H_{\vec{x}} g^{-1} \in M_2^+$

· 线性.

$g(\lambda_1 H_{\vec{x}_1} + \lambda_2 H_{\vec{x}_2}) g^{-1} = \lambda_1 g H_{\vec{x}_1} g^{-1} + \lambda_2 g H_{\vec{x}_2} g^{-1}.$

Φ_g^+ 为 M_2^+ 上的自同构.

$\Phi_g: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$\vec{x} \longrightarrow \Psi \cdot \Phi_g^+ \Psi^{-1}(\vec{x})$

8. 令 Φ 记 $\Phi_g^+(H_{\vec{x}}) = H_{\vec{y}}$

$$(\Phi_g(\vec{x}) \mid \Phi_g(\vec{x})) = \left(\psi(\Phi_g^+(H_{\vec{x}})) \mid \psi(\Phi_g^+(H_{\vec{x}}))\right)$$

$$= (\psi(H_{\vec{y}}) \mid \psi(H_{\vec{y}}))$$

$$= (\vec{y} \mid \vec{y}) = -\det(H_{\vec{y}})$$

$$= -\det(\Phi_g^+(H_{\vec{x}}))$$

$$= -\det(g H_{\vec{x}} g^{-1}) = -\det(H_{\vec{x}})$$

$$= (\vec{x} \mid \vec{x}).$$

$$\Rightarrow \Phi_g \in O(3)$$

$$\Phi: SU(2) \longrightarrow O(3)$$

$$g \longrightarrow \Phi_g$$

$$\Phi(g_1 g_2)(\vec{x}) = \Phi_{g_1 g_2}(\vec{x}) = \psi \cdot \Phi^+_{g_1 g_2} \psi^{-1}(\vec{x})$$

$$= \psi(\Phi^+_{g_1 g_2}(H_{\vec{x}}))$$

$$= \psi(g_1 g_2 H_{\vec{x}} g_2^{-1} g_1^{-1})$$

Φ是群同态.

$$= \psi(g_1 \Phi^+_{g_2}(H_{\vec{x}}) g_1^{-1})$$

$$= \psi(\Phi^+_{g_1} \cdot \Phi^+_{g_2}(H_{\vec{x}}))$$

$$= \psi \Phi^+_{g_1} \Phi^+_{g_2} \cdot \psi^{-1}(\vec{x})$$

$$= \Phi(g_1) \Phi(g_2)(\vec{x}).$$

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$\mathrm{Ker}(\Phi) = \{ g \in SU(2) \mid \Phi_g = id \}$ 恒同映射

$= \{ g \in SU(2) \mid gH_{\vec{x}} = H_{\vec{x}}g \ \forall H_{\vec{x}} \in M_2^+ \}$

直接验证. $= \{ g \in SU(2) \mid gH_{e_i} = H_{e_i}g,\ i=1,2,3 \}$.

$\Rightarrow \ker(\Phi) = \{ \pm E \}$.

$\mathrm{Img}(\Phi)$: 利用参数表示

$$b_\varphi = \begin{pmatrix} e^{\frac{i}{2}\varphi} & 0 \\ 0 & e^{-\frac{i\varphi}{2}} \end{pmatrix} \in SU(2) \qquad c_\theta = \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \in SU(2)$$

$b_\varphi H_{e_1} b_\varphi^{-1} = \cos\varphi H_{e_1} + \sin\varphi H_{e_2}$

$b_\varphi H_{e_2} b_\varphi^{-1} = -\sin\varphi H_{e_1} + \cos\varphi H_{e_2}$

$b_\varphi H_{e_3} b_\varphi^{-1} = H_{e_3}$.

$$\Phi_{b_\varphi} = B_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi & \\ +\sin\varphi & \cos\varphi & \\ & & 1 \end{pmatrix}$$

同理: $\Phi_{c_\theta} = C_\theta$.

$\forall g \in SU(2) \quad \exists u \in SU(2)$ s.t $g = u b_\varphi u^{-1}$

$\det(\Phi_g) = \det(\Phi_{u b_\varphi u^{-1}}) = \det(\Phi_u) \cdot 1 \cdot \det(\Phi_u)^{-1}$
$= 1$.

$\Phi: SU(2) \longrightarrow SO(3)$.
$\Rightarrow \mathrm{Img}(\Phi) \subset SO(3)$

10.

$\forall\ A \in SO(3) \quad A = B_\varphi C_\theta B_\psi$.

令 $g = b_\varphi c_\theta b_\psi$

$$\Phi(g) = \Phi_g = \Phi_{b_\varphi c_\theta b_\psi} = B_\varphi C_\theta B_\psi = A$$

$\Rightarrow \mathrm{Img}(\Phi) = SO(3)$.

定理1：$\Phi: SU(2) \longrightarrow SO(3)$ 是满同态且

$\qquad g \longmapsto \Phi_g \qquad \mathrm{Ker}(\Phi) = \{\pm E\}$.

推论：$SO(3) \simeq \mathbb{P}^3(\mathbb{R})\ \ \mathbb{RP}^3$.

证明：$SU(2) \simeq \mathbb{S}^3$

$$\mathbb{RP}^3 = \mathbb{S}^3 / \sim \ \simeq SU(2)/\{\pm E\} \simeq SO(3).$$

~~对径点~~ ~~径对映点~~

$\vec{x} \sim \vec{y} \iff \vec{x} + \vec{y} = 0$

注：1) 利用参数表示可直接证 $\mathrm{Img}(SU(2)) = SO(3)$

$\Phi(b_\varphi) = B_\varphi \quad \Phi(c_\theta) = C_\theta$

$\Phi(b_\varphi c_\theta b_\psi) = B_\varphi C_\theta B_\psi$.

2) 定义：$M_2^- = \{H \in M_2(\mathbb{C}) \mid {}^*H = -H,\ \mathrm{tr}\,H = 0\}$.

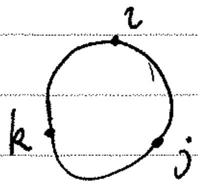
M_2^+ 可用 M_2^- 代替.

4. 四元数

1. i j k 符号.

定义: ~~$1\cdot 1 = 1$~~

×	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1



$(\{\pm 1, \pm i, \pm j, \pm k\}\ \cdot)$ 构成一个群.

$$\mathbb{H} = \{ a\cdot 1 + b\cdot i + c\cdot j + d\cdot k \mid a, b, c, d \in \mathbb{R} \}.$$

• 加法: $a_1 1 + b_1 i + c_1 j + d_1 k + a_2 1 + b_2 i + c_2 j + d_2 k$

$$= (a_1 + a_2) 1 + (b_1 + b_2) i + (c_1 + c_2) j + (d_1 + d_2) k.$$

• 数乘: $\lambda(a1 + bi + cj + dk) = \lambda a 1 + \lambda b i + \lambda c j$
$+ \lambda d k.$

$\mathbb{H}$ 构成 $\mathbb{R}$-向量空间.

• 乘法: 利用群的乘法及

$$(\lambda p) q = p(\lambda q) = \lambda(pq)$$

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$\mathbb{H}$ 构成一个环：$0 = 0\cdot1 + 0i + 0j + 0k$

$1 = 1\cdot1 + 0i + 0j + 0k$.

性质：$\mathbb{R} \hookrightarrow \mathbb{C} \hookrightarrow \mathbb{H}$.

$a \longmapsto a + 0i$

$a + bi \longmapsto a\cdot1 + b\cdot i$

$\mathbb{H}$ 是2维 $\mathbb{C}$-向量空间：

$$q = a1 + bi + cj + dk = (a\cdot1 + bi) + (c + di)j$$

共轭：$\bar{q} = a\cdot1 - bi - cj - dk$

$$N(q) := q\bar{q} = \bar{q}q = a^2 + b^2 + c^2 + d^2$$

$$N(q) = 0 \iff q = 0$$

若 $q \neq 0 \Rightarrow q \cdot \bar{q}/N(q) = \bar{q} \cdot q/N(q) = 1$

$\Rightarrow q$ 可逆

$\mathbb{H}$ 是可除代数

性质：$\underline{q_1, q_2 \in \mathbb{H},\ \lambda_1, \lambda_2 \in \mathbb{R}}$

$$\overline{(\lambda_1 q_1 + \lambda_2 q_2)} = \lambda_1 \bar{q_1} + \lambda_2 \bar{q_2}$$

$$\overline{q_1 q_2} = \bar{q_2} \cdot \bar{q_1}$$

$$N(q_1 q_2) = N(q_1) N(q_2).$$

$q \to \bar{q}$ 是反自同构

$N: q \to N(q)$ 是 $\mathbb{H} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ 的同态.

$$\mathrm{Ker}\, N = \{q \in \mathbb{H} \mid N(q) = 1\}.$$

$\Gamma: \mathbb{H} \longrightarrow M_2(\mathbb{C})$. $\quad c \quad c'$

$q = \alpha + \beta i + \gamma j + \delta k = \alpha + \beta i + (\gamma + \delta i) j$

$$\Gamma(q) = \begin{pmatrix} \alpha + \beta i & \gamma + \delta i \\ -\gamma + \delta i & \alpha - \beta i \end{pmatrix} = \begin{pmatrix} c & c' \\ -\bar{c}' & \bar{c} \end{pmatrix}$$

$* \cancel{\Gamma(q) =}\ \det(\Gamma(q)) = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = N(q)$.

验证:

$$\Gamma(\lambda_1 q_1 + \lambda_2 q_2) = \lambda_1 \Gamma(q_1) + \lambda_2 \Gamma(q_2)$$
$$\Gamma(q_1 q_2) = \Gamma(q_1)\Gamma(q_2)$$
$$\Gamma(1) = E$$

$\Gamma: \mathbb{H}^* \longrightarrow M_2(\mathbb{C})$ 一个同态.

$\Gamma(\mathrm{Ker} N) = SU(2)$.

$$= \left\{ \begin{pmatrix} c & c' \\ -\bar{c}' & \bar{c} \end{pmatrix} \middle| \, |c|^2 + |c'|^2 = 1 \right\}.$$

~~$\Gamma|_{\mathrm{Ker}N}$ 是单射~~ $\Gamma: \mathrm{Ker} N \longrightarrow SU(2)$ 是同构

$\forall\ q \in \mathrm{Ker} N$. 定义. $\psi_q: \mathbb{H} \longrightarrow \mathbb{H}$.

$\psi_q(p) = q p q^{-1}$

令 $\mathbb{H}^- = \{ p \in \mathbb{H} \mid \bar{p} = -p \}$.

$\dim_{\mathbb{R}} \mathbb{H}^- = 3$.

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$\forall\ p \in \mathbb{H}^-$ $\quad \overline{\varphi_q(p)} = \overline{qpq^{-1}} = \overline{q^{-1}}\,\overline{p}\,\overline{q} = q(-p)q^{-1}$

$$= -qpq^{-1} = -\varphi_q(p)$$

$\Rightarrow \varphi_q(p) \in \mathbb{H}^- \Rightarrow \varphi_q : \mathbb{H}^- \to \mathbb{H}^-$ 线性算子

在 $\mathbb{H}^-$ 上定义度量：$|p|^2 := N(q)$

$$|\varphi_q(p)|^2 = N(\varphi_q(p)) = N(qpq^{-1})$$
$$= N(p) = |p|^2$$

$\therefore \varphi_q$ 为保长度的线性算子. i.e $\varphi_q \in O(3)$.

设 $q = \alpha + \beta i + \gamma j + \delta k$

令 $r(t) = \cos\theta_1 t\cos\theta_2 t + \cos\theta_1 t\sin\theta_2 t\, i$

$$+ \sin\theta_1 t\cos\theta_2 t\, j + \sin\theta_1 t\sin\theta_2 t\, k$$

取合适的 θ_1, θ_2 s.t. $r(0) = 1$, $r(1) = q$.

$\because \varphi_{r(t)}$ 关于 t 是连续函数构成的矩阵.

$\therefore \det(\varphi_{r(t)})$ 在 $[0,1]$ 上是连续函数.

而 $\det(\varphi_1) = 1$ 且 $\det(\varphi_{r(t)}) = \pm 1 \Rightarrow \det(\varphi_{r(t)}) = 1$.

$\therefore \varphi_q \in SO(3)$

此外：$\varphi_{q_1 q_2} = \varphi_{q_1}\varphi_{q_2}$

$\therefore \varphi : \ker N \to SO(3)$ 为~~同构~~ 群同态

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$\mathrm{Ker}\psi = \{ q \in \mathrm{Ker} N \mid q p q^{-1} = p \ \forall p \in \mathbb{H}^- \}$

$= \left\{ q \in \mathrm{Ker} N \mid \begin{matrix} qi = iq \\ qj = jq \quad qk = kq \end{matrix} \right\}$

$= \{ \pm 1 \}$.

$\mathrm{Img}\psi = SO(3)$

取 $q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2} i$

$\psi_q(i) = (\cos\frac{\theta}{2} + \sin\frac{\theta}{2} i) i (\cos\frac{\theta}{2} - \sin\frac{\theta}{2} i)$

$= i$

$\psi_q(j) = \cos\theta j + \sin\theta k$

$\psi_q(k) = -\sin\theta j + \cos\theta k$.

$\Rightarrow \psi_q = C_\theta$.

取 $q = \cos\frac{\varphi}{2} + \sin\frac{\varphi}{2} k$

$\psi_q = B_\varphi$.

$\therefore \mathrm{Im}\psi = SO(3)$.

定理1': $\psi : \mathrm{Ker} N \longrightarrow SO(3)$ 是满同态且

$q \longrightarrow \psi_q \quad \mathrm{Ker}\psi = \{\pm 1\}$.

注: $\Gamma(\mathbb{H}^-) = M_2^- = \left\{ H \in M_2(\mathbb{C}) \mid \begin{matrix} {}^t\bar{H} = -H \\ \mathrm{tr} H = 0 \end{matrix} \right\}$.