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Quasi-equivalence of heights in algebraic function fields of one variable ☆

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ABSTRACT

For points (a, b) on an algebraic curve over a field K with height $\mathfrak{h}$, the asymptotic relation between $\mathfrak{h}(a)$ and $\mathfrak{h}(b)$ has been extensively studied in diophantine geometry. When $K = \overline{k(t)}$ is the field of algebraic functions in t over a field k of characteristic zero, Eremenko in 1998 proved the following quasi-equivalence for an absolute logarithmic height $\mathfrak{h}$ in K: Given $P \in K[X, Y]$ irreducible over K and $\epsilon > 0$, there is a constant C only depending on P and ϵ such that for each $(a, b) \in K^2$ with $P(a, b) = 0$,

$$(1 - \epsilon)\deg(P, Y)\mathfrak{h}(b) - C \leq \deg(P, X)\mathfrak{h}(a) \leq (1 + \epsilon)\deg(P, Y)\mathfrak{h}(b) + C.$$

In this article, we shall give an explicit bound for the constant C in terms of the total degree of P, the height of P and ϵ. This result is expected to have applications in some other areas

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such as symbolic computation of differential and difference equations.

1. Introduction

In diophantine geometry, heights are often used to express the discreteness of algebraic points on an algebraic variety. They play an important role in diophantine geometry as well as other areas such as the theory of transcendental numbers. The study of the functorial property of heights can be tracked back to the date of Siegel, who gave the first asymptotic estimate of $\mathfrak{h}(f(\mathbf{c}))$ in terms of $\deg(f)$ when $\mathfrak{h}(\mathbf{c})$ is large enough, where $\mathbf{c}$ is a point on a projective algebraic curve and f is a nonconstant rational function on this curve. Later, Siegel's result was improved by many authors (see for example Néron [16], Bombieri [4], Habegger [10], Abouzaid [2] and Bartolome [3]) who gave error terms to the asymptotic estimates. For instance, in [16], Néron proved the following quasi-equivalence of heights: Let $P \in \overline{\mathbb{Q}}[X,Y]$ be irreducible with $m = \deg(P,X) \geq 1$ and $n = \deg(P,Y) \geq 1$, then there is a constant $c(P)$ such that if $(a,b) \in \overline{\mathbb{Q}}^2$ with $P(a,b) = 0$, the bound

$$\left|\frac{\mathfrak{h}(a)}{n} - \frac{\mathfrak{h}(b)}{m}\right| \leq c(P)\sqrt{\max\left\{\frac{\mathfrak{h}(a)}{n}, \frac{\mathfrak{h}(b)}{m}\right\}}.$$

An explicit estimate of the constant $c(P)$ is of particular interest in an effective version of Runge's theorem on the integer solutions of certain diophantine equations. In [11], Habegger gave an explicit bound for the constant $c(P)$ and applied this bound to Runge's theorem. Other related height estimates may also be found in [2,3].

The heights appearing in the above results are all defined in algebraic number fields. As to an absolute logarithmic height defined in function fields (see Section 3 of Chapter 3 in [14] for definition), Eremenko in 1998 proved quasi-equivalence of the following type, where k is an algebraically closed field of characteristic zero.

Proposition 1.1 *(Lemma 2 of [9]). Let $P \in \overline{k(t)}[X,Y]$ be an irreducible polynomial of degree m with respect to X and of degree n with respect to Y. Given $\epsilon > 0$ there exists a constant C depending on P and ϵ such that for every $a, b \in \overline{k(t)}$ satisfying $P(a,b) = 0$ we have*

$$(1-\epsilon)n\mathfrak{h}(b) - C \leq m\mathfrak{h}(a) \leq (1+\epsilon)n\mathfrak{h}(b) + C.$$

One can see from Remark 2.6 that if $a \in k(t)$ then $\mathfrak{h}(a)$ defined in the above proposition is exactly the degree of a, i.e. the maximum of the degrees of the numerator and denominator of a. Eremenko applied the above result to show that rational solutions of a first order algebraic ordinary differential equation (AODE) $F = 0$ are of degree not greater than a constant only depending on F. From the viewpoint of algorithms, an explicit estimate of the constant C is usually necessary to guarantee the termination of algorithms for computing rational solutions of AODEs. Meanwhile, such explicit estimate has potential applications in the algorithmic aspect of computing rational points on an algebraic variety over $k(t)$. In this article, we shall give an explicit bound for the constant C in terms of the total degree of P, the height of P and ϵ. We obtain this explicit bound by computing the explicit expressions for constants appearing at each step of the proof of the above proposition given by Eremenko. In particular, we give bounds for the heights of the coefficients of a certain nonzero element in the Riemann-Roch space of a divisor. Precisely, suppose that $L = K(x, y)$ is an algebraic function field of one variable over a field K, where x is transcendental over K and y is algebraic over $K(x)$. Then each element $A \in L$ can be presented as a polynomial in y with coefficients in $K(x)$, i.e. $A = \frac{1}{q(x)} \sum_{i=0}^{n-1} \sum_{j=0}^{m} a_{i,j} x^j y^i$ with $a_{i,j} \in K, q \in K[X]$ and $n = [L : K(x)]$. For a certain nonzero element A in the Riemann-Roch space of a divisor, we give a bound for the height of the projective point $\mathbf{a} = (\cdots : a_{i,j} : \cdots)$ (see Proposition 3.11) as well as a bound for the height of $q(X)$. Note that Schmidt in [17] presented a bound for m, a degree bound for q and a bound for the absolute values of the coefficients of the Puiseux series expansion of A when K is the field of algebraic numbers. Although it is possible to obtain a bound for the height of $\mathbf{a}$ by the results (mainly Theorem C2) presented in [17], we do not take this approach because the absolute logarithmic height under consideration in this paper satisfies the triangle inequality, i.e. $\mathfrak{h}(a+b) \le \mathfrak{h}(a) + \mathfrak{h}(b)$, which is not usually satisfied for absolute logarithmic heights defined in algebraic number fields. The triangle inequality enables us to obtain a simpler expression for the constant C. Finally, let us remark that the construction of the Riemann-Roch space of a divisor is one of the fundamental problems in the theory of algebraic function fields. Many algorithms have already been developed for this problem, see for example [1,6,8,12,13,15,19].

The article is organized as follows. In Section 2, we introduce some basic concepts and notations about algebraic function fields of one variable and heights used in the later sections. In Section 3, we estimate the heights of the coefficients for a certain nonzero element in the Riemann-Roch space of a given divisor. Finally, in Section 4, we present an explicit bound for the constant C.

As usual, for a polynomial $P(X_1, \ldots, X_m)$, we use $\mathrm{tdeg}(P)$ and $\deg(P, X_i)$ to denote the total degree of P and the degree of P with respect to X_i respectively. $\mathbb{P}^m(\cdot)$ denotes the projective space of dimension m over a field and $(a_0 : \cdots : a_m)$ denotes a point in $\mathbb{P}^m(\cdot)$ with coordinates a_i.

2. Algebraic function fields of one variable and heights

In this section, we will introduce some basic concepts and notations of algebraic function fields of one variable and heights. Readers are referred to [5,14,18,20] for details.

2.1. Algebraic function fields of one variable

Throughout this subsection, K always stands for an algebraically closed field of characteristic zero. Let L be an algebraic function field of one variable over K. Assume that $L = K(x, y)$ where x is transcendental over K and y satisfies

$$P(x, y) = A_0(x)y^n + A_1(x)y^{n-1} + \cdots + A_n(x) = 0, \ A_i \in K[X],$$

where $P \in K[X, Y]$ is irreducible. Denote by $K((z))$ the quotient field of the ring of formal power series in z. Let $\mathbf{x}(z) = \sum_{i=r}^{\infty} c_i z^i \in K((z))$ with $r \in \mathbb{Z}, c_i \in K, c_r \neq 0$, then $\text{ord}_z(\mathbf{x}(z))$ is defined to be r. We call $(\mathbf{x}(z), \mathbf{y}(z)) \in K((z))^2$ a parametrization of $P(X, Y) = 0$ provided $P(\mathbf{x}(z), \mathbf{y}(z)) = 0$ and $\mathbf{x}(z)$ or $\mathbf{y}(z)$ does not belong to K. If there is an integer $s \geq 2$ such that $\mathbf{x}(z), \mathbf{y}(z) \in K((z^s))$ then the parametrization $(\mathbf{x}(z), \mathbf{y}(z))$ is said to be reducible, otherwise irreducible. Two parametrizations $(\mathbf{x}(z), \mathbf{y}(z))$ and $(\tilde{\mathbf{x}}(z), \tilde{\mathbf{y}}(z))$ are said to be equivalent if there is $\mathbf{w}(z) \in K((z))$ with $\text{ord}_z(\mathbf{w}(z)) = 1$ such that

$$\mathbf{x}(z) = \tilde{\mathbf{x}}(\mathbf{w}(z)) \text{ and } \mathbf{y}(z) = \tilde{\mathbf{y}}(\mathbf{w}(z)).$$

Definition 2.1. An equivalent class of irreducible parametrizations is called a place of $P(X, Y) = 0$.

It was shown on page 95 of [20] that an irreducible parametrization of $P(X, Y) = 0$ is equivalent to the one of the type

$$(a + z^{\mu}, z^{\nu}(b_0 + b_1 z^{\ell_1} + \cdots)) \tag{1}$$

where $a \in K, b_i \in K \setminus \{0\}, \mu \in \mathbb{Z} \setminus \{0\}, \nu, \ell_i \in \mathbb{Z}$, $0 < \ell_1 < \ell_2 < \cdots$ and $\mu, \nu, \nu + \ell_1, \nu + \ell_2, \cdots$ have no common factor greater than 1, moreover if $\mu < 0$ then $a = 0$. In the rest of this article, all irreducible parametrizations of $P(X, Y) = 0$ will be of the type (1). Let $\mathfrak{p}$ be a place of the form (1). We say that $\mathfrak{p}$ lies above $x - a$ if $\mu > 0$, and lies above $1/x$ if $\mu < 0$. The integer $|\mu|$ is called the ramification index of $\mathfrak{p}$ with respect to $K(x)$, denoted by $e_{\mathfrak{p}, K(x)}$. Suppose that $f \in L \setminus \{0\}$. The order of f at $\mathfrak{p}$, denoted by $\text{ord}_{\mathfrak{p}}(f)$, is defined to be $\text{ord}_z(f(z^{\mu} + a, z^{\nu}(b_0 + \cdots)))$. If $\text{ord}_{\mathfrak{p}}(f) > 0$, $\mathfrak{p}$ is called a zero of f and if $\text{ord}_{\mathfrak{p}}(f) < 0$, $\mathfrak{p}$ is called a pole of f. It is well-known that a nonzero f admits only finitely many zeros and poles. We set $\text{ord}_{\mathfrak{p}}(0) = \infty$. For $f, g \in L$, one can verify that

$$\text{ord}_{\mathfrak{p}}(fg) = \text{ord}_{\mathfrak{p}}(f) + \text{ord}_{\mathfrak{p}}(g), \ \text{ord}_{\mathfrak{p}}(f + g) \geq \min\{\text{ord}_{\mathfrak{p}}(f), \text{ord}_{\mathfrak{p}}(g)\}$$

where the equality in the last formula holds if $\text{ord}_{\mathfrak{p}}(f) \neq \text{ord}_{\mathfrak{p}}(g)$.

Denote $V(\mathfrak{p}) = \{f \in L \mid \text{ord}_{\mathfrak{p}}(f) \geq 0\}$. One can check that $V(\mathfrak{p})$ is a discrete valuation ring of L. One can also check that for $f \in V(\mathfrak{p})$ there is a unique $c_f \in K$ such that $\text{ord}_{\mathfrak{p}}(f - c_f) > 0$. We define a map $\pi_{\mathfrak{p}} : V(\mathfrak{p}) \to K$ given by $f \mapsto c_f$. Then $\pi_{\mathfrak{p}}$ is a K-homomorphism. For convenience, we set $\pi_{\mathfrak{p}}(f) = \infty$ if $\text{ord}_{\mathfrak{p}}(f) < 0$. The point $(\pi_{\mathfrak{p}}(x), \pi_{\mathfrak{p}}(y))$ is called the center of $\mathfrak{p}$.

Remark 2.2. In [5], a place is presented by the unique maximal ideal of a discrete valuation ring of L over K. Precisely, let $\mathfrak{p}$ be a place of $P(X,Y) = 0$ and let $V(\mathfrak{p})$ be as above. Set $\mathfrak{m}_{\mathfrak{p}} = \{f \in L \mid \text{ord}_{\mathfrak{p}}(f) > 0\}$. One sees that $\mathfrak{m}_{\mathfrak{p}}$ is the unique maximal ideal of $V(\mathfrak{p})$, which is the "place" defined in [5] corresponding to $\mathfrak{p}$. Conversely, given a discrete valuation ring V of L over K with $\mathfrak{m}$ as its unique maximal ideal, we can construct a unique place of $P(X,Y) = 0$ corresponding to V. Let z be a uniformizing variable at V. Expanding x, y as Puiseux series in z yields an irreducible parametrization and thus a place of $P(X,Y) = 0$. Furthermore, different choices of uniformizing variables at V induce equivalent irreducible parametrizations and so the same place. Additionally, the order of f at $\mathfrak{m}_{\mathfrak{p}}$ defined in [5] is nothing else but $\text{ord}_{\mathfrak{p}}(f)$.

Definition 2.3. A divisor D of L is a finite formal sum of places of $P(X,Y) = 0$ with integer coefficients, i.e. $D = \sum_{\mathfrak{p}} d_{\mathfrak{p}}\mathfrak{p}$ where $d_{\mathfrak{p}} \in \mathbb{Z}$ and $d_{\mathfrak{p}} = 0$ for all but finitely many places $\mathfrak{p}$. If all $d_{\mathfrak{p}} = 0$, we call D the divisor zero and write $D = 0$.

Suppose that $D = \sum_{\mathfrak{p}} d_{\mathfrak{p}}\mathfrak{p}$ is a divisor of L. We call $\sum_{\mathfrak{p}} d_{\mathfrak{p}}$, denoted by $\deg(D)$, the degree of D. The set of places $\mathfrak{p}$ with $d_{\mathfrak{p}} \neq 0$ is called the support of D, denoted by $\text{supp}(D)$. We call D an integral divisor if $d_{\mathfrak{p}} \geq 0$ for all $\mathfrak{p}$, denoted by $D \geq 0$. For a nonzero $f \in L$, denote

$$\text{div}(f) = \sum_{\mathfrak{p}} \text{ord}_{\mathfrak{p}}(f)\mathfrak{p},$$

which is called the divisor of f. Each divisor D can be uniquely written as $D^+ - D^-$ where D^+, D^- are integral divisors and $\text{supp}(D^+) \cap \text{supp}(D^-) = \emptyset$. Given a divisor D, denote

$$\mathcal{L}_K(D) = \{f \in L \mid \text{div}(f) + D \geq 0\} \cup \{0\}.$$

We call $\mathcal{L}_K(D)$ the Riemann-Roch space of D which is a K-vector space of finite dimension, and we denote its dimension by $\ell(D)$. By the Riemann-Roch theorem, $\ell(D) > 0$ if $\deg(D)$ is not less than the genus of L over K.

2.2. Heights in an algebraic function field of one variable

Throughout this subsection, $k(t)$ stands for the field of rational functions in t with coefficients in an algebraically closed field k of characteristic zero, and $\overline{k(t)}$ for the algebraic closure of $k(t)$. Let $L \subset \overline{k(t)}$ be a finite extension of $k(t)$. Then L is an algebraic function field of one variable over k. Places in this subsection will be presented by maximal ideals of discrete valuation rings of L over k or equivalent classes of irreducible parameterizations of $P(X,Y)=0$, where $P(X,Y)=0$ is an algebraic curve whose function field coincides with L. Let us first define an absolute logarithmic height of a point in $\mathbb{P}^m(\overline{k(t)})$. Note that $\mathrm{ord}_{\mathfrak{p}}(0)=\infty$.

Definition 2.4. Given $\mathbf{a}=(a_0:\cdots:a_m)\in\mathbb{P}^m(\overline{k(t)})$, let L be a finite extension of $k(t)$ containing all a_i. The *absolute logarithmic height* (or simply *height*) of $\mathbf{a}$, denoted by $\mathfrak{h}(\mathbf{a})$, is defined to be

$$\frac{\sum_{\mathfrak{p}}\max_{i=0}^m\{-\mathrm{ord}_{\mathfrak{p}}(a_i)\}}{[L:k(t)]}$$

where $\mathfrak{p}$ ranges over all places of L over k.

From Section 3 of Chapter 3 in [14], $\mathfrak{h}(\cdot)$ is a logarithmic height function. Actually it is an absolute logarithmic height function i.e. a logarithmic height function independent of the choices of the field L. To see this, let $\tilde{L}$ be a finite extension of L and suppose that $\mathfrak{p}$ is a place of L over k. Then there are finitely many places $\mathfrak{P}$ of $\tilde{L}$ over k lying above $\mathfrak{p}$, i.e. $\mathfrak{P}\cap L=\mathfrak{p}$. For brevity, denote by $\mathfrak{P}|\mathfrak{p}$ a place $\mathfrak{P}$ lying above $\mathfrak{p}$. Note that the relative degree of $\mathfrak{P}$ is 1 because k is algebraically closed, and due to Theorem 1 on page 52 of [5], for a fixed $\mathfrak{p}$, $\sum_{\mathfrak{P}|\mathfrak{p}}e_{\mathfrak{P},L}=[\tilde{L}:L]$. Moreover $\mathrm{ord}_{\mathfrak{P}}(a)=e_{\mathfrak{P},L}\mathrm{ord}_{\mathfrak{p}}(a)$ for any $a\in L$ and any $\mathfrak{P}$ lying above $\mathfrak{p}$. These imply that for a fixed $\mathfrak{p}$,

$$\begin{aligned}\sum_{\mathfrak{P}|\mathfrak{p}}\max_i\{-\mathrm{ord}_{\mathfrak{P}}(a_i)\}&=\sum_{\mathfrak{P}|\mathfrak{p}}\max_i\{-e_{\mathfrak{P},L}\mathrm{ord}_{\mathfrak{p}}(a_i)\}=\sum_{\mathfrak{P}|\mathfrak{p}}e_{\mathfrak{P},L}\max_i\{-\mathrm{ord}_{\mathfrak{p}}(a_i)\}\\&=[\tilde{L}:L]\max_i\{-\mathrm{ord}_{\mathfrak{p}}(a_i)\}.\end{aligned}$$

From this, one easily sees that $\mathfrak{h}(\cdot)$ is independent of the choices of L. Using Definition 2.4, it is natural to define the height of an element in $\overline{k(t)}$ and a polynomial in $\overline{k(t)}[X_1,\dots,X_m]$ as follows.

Definition 2.5.

1. For $a\in\overline{k(t)}$, we define the height of a to be $\mathfrak{h}((1:a))$, denoted by $\mathfrak{h}(a)$.
2. Let Q be a nonzero polynomial in $\overline{k(t)}[X_1,\dots,X_m]$. We define the height of Q to be

$$\mathfrak{h}(Q)=\begin{cases}0 & Q \text{ contains exactly one term}\\ \mathfrak{h}(\mathbf{a}) & \text{otherwise}\end{cases},$$

   where $\mathbf{a}$ is a point in some projective space whose coordinates are the coefficients of Q.

In the following, we set $a/0=\infty$ for any $a\in\overline{k(t)}\setminus\{0\}$ and $\mathfrak{h}(\infty)=0$.

Remark 2.6. Assume that $\mathbf{a} = (a_0 : \cdots : a_m) \in \mathbb{P}^m(\overline{k(t)})$.

1. Suppose that $a_0 = 1$, then

$$\mathfrak{h}(\mathbf{a}) = \frac{\sum_{\mathfrak{p}} \max\{0, -\mathrm{ord}_{\mathfrak{p}}(a_1), \ldots, -\mathrm{ord}_{\mathfrak{p}}(a_m)\}}{[L : k(t)]} \geq 0.$$

2. Let $a \in \overline{k(t)}$ and $L = k(t, a)$. Let $Q(X, Y)$ be a nonzero irreducible polynomial over k such that $Q(t, a) = 0$. It is clear that $\mathfrak{h}(a) = 0$ if $a \in k$. Now assume that $a \notin k$ and $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ are all distinct poles of a in L, then

$$\mathfrak{h}(a) = \frac{-\sum_{i=1}^{s} \mathrm{ord}_{\mathfrak{p}_i}(a)}{[L : k(t)]} = \frac{[L : k(a)]}{[L : k(t)]} = \frac{\deg(Q, X)}{\deg(Q, Y)}.$$

   In particular, if $a \in k(t)$ then $\mathfrak{h}(a) = \deg(a)$ which is defined to be the maximum of the degrees of the denominator and numerator of a.

The height given in Definition 2.4 has the following properties.

Proposition 2.7. $\mathfrak{h}(a^n) = \mathfrak{h}(a^{-n}) = n\mathfrak{h}(a)$, $a \in \overline{k(t)} \setminus \{0\}$, $n \geq 0$.

Proof. Let $L = k(t, a)$. For each place $\mathfrak{p}$ of L over k,

$$\max\{0, -\mathrm{ord}_{\mathfrak{p}}(a^n)\} = \max\{0, -n\mathrm{ord}_{\mathfrak{p}}(a)\} = n \max\{0, -\mathrm{ord}_{\mathfrak{p}}(a)\}.$$

By definition, $\mathfrak{h}(a^n) = n\mathfrak{h}(a)$. For the first equality, it suffices to show that $\mathfrak{h}(a) = \mathfrak{h}(1/a)$. As $(1 : a) = (1/a : 1)$, one sees that

$$\mathfrak{h}(a) = \mathfrak{h}((1 : a)) = \mathfrak{h}((1/a : 1)) = \mathfrak{h}(1/a). \quad \square$$

Suppose that $\Phi = (\phi_0 : \cdots : \phi_m)$ is a morphism, namely

$$\Phi : \mathbb{P}^{s_1}(\overline{k(t)}) \times \cdots \times \mathbb{P}^{s_r}(\overline{k(t)}) \longrightarrow \mathbb{P}^m(\overline{k(t)})$$
$$\mathbf{b} \longmapsto (\phi_0(\mathbf{b}) : \cdots : \phi_m(\mathbf{b})),$$

where $\phi_i \in \overline{k(t)}[X_{1,0}, \ldots, X_{1,s_1}, \ldots, X_{r,0}, \ldots, X_{r,s_r}]$ is a nonzero polynomial homogeneous in $X_{j,0}, \ldots, X_{j,s_j}$ of degree d_j for all $j = 1, \ldots, r$. Write $\phi_i = \sum_{j=1}^{s} c_{i,j}\mathbf{m}_j$, where $c_{i,j} \in \overline{k(t)}$ and $\mathbf{m}_1, \ldots, \mathbf{m}_s$ are all monomials in $X_{1,0}, \ldots,\ X_{1,s_1}, \ldots, X_{r,0}, \ldots, X_{r,s_r}$ of total degree $\sum_{j=1}^{r} d_j$.

Definition 2.8. We define the height of Φ, denoted by $\mathfrak{h}(\Phi)$, to be

$$\mathfrak{h}((c_{0,1} : \cdots : c_{i,j} : \cdots : c_{m,s})).$$

The following proposition will play a key role in the rest of this paper. Although it is a trivial generalization of an existing result (see Proposition on page 15 of [18] or Lemma 1.6 on page 80 of [14] for the case with $r = 1$), we reprove this result for completeness and give an explicit estimate of the error term c in the case of heights in algebraic function fields of one variable.

Proposition 2.9. *Let $\Phi = (\phi_0 : \cdots : \phi_m)$ be as above. Suppose $(\mathbf{a}_1, \ldots, \mathbf{a}_r) \in \mathbb{P}^{s_1}(\overline{k(t)}) \times \cdots \times \mathbb{P}^{s_r}(\overline{k(t)})$ is a point on which Φ is defined. Then*

$$\mathfrak{h}(\Phi(\mathbf{a}_1, \ldots, \mathbf{a}_r)) \le \sum_{i=1}^{r} d_i \mathfrak{h}(\mathbf{a}_i) + \mathfrak{h}(\Phi).$$

Proof. Write $\mathbf{a}_i = (a_{i,0} : \cdots : a_{i,s_i}), \mathbf{j} = (j_{1,0}, \ldots, j_{1,s_1}, \ldots, j_{r,0}, \ldots, j_{r,s_r})$ and

$$\phi_i = \sum_{\mathbf{j}} c_{i,\mathbf{j}} X_{1,0}^{j_{1,0}} \cdots X_{l,l'}^{j_{l,l'}} \cdots X_{r,s_r}^{j_{r,s_r}}$$

with $c_{i,\mathbf{j}} \in \overline{k(t)}$ and $\sum_{l'=0}^{s_l} j_{l,l'} = d_l$. Let L be a finite extension of $k(t)$ containing all $a_{i,j}$ and all $c_{i,\mathbf{j}}$. For each place $\mathfrak{p}$ of L over k, one has that

$$\begin{aligned}
-\mathrm{ord}_{\mathfrak{p}}(c_{i,\mathbf{j}} a_{1,0}^{j_{1,0}} \cdots a_{l,l'}^{j_{l,l'}} \cdots a_{r,s_r}^{j_{r,s_r}}) &= -\mathrm{ord}_{\mathfrak{p}}(c_{i,\mathbf{j}}) - \sum_{l=1}^{r} \sum_{l'=0}^{s_l} j_{l,l'} \mathrm{ord}_{\mathfrak{p}}(a_{l,l'}) \\
&\le -\mathrm{ord}_{\mathfrak{p}}(c_{i,\mathbf{j}}) + \sum_{l=1}^{r} \sum_{l'=0}^{s_l} j_{l,l'} \max\{-\mathrm{ord}_{\mathfrak{p}}(a_{l,0}), \ldots, -\mathrm{ord}_{\mathfrak{p}}(a_{l,s_l})\} \\
&\le \max_{i,\mathbf{j}}\{-\mathrm{ord}_{\mathfrak{p}}(c_{i,\mathbf{j}})\} + \sum_{l=1}^{r} d_l \max\{-\mathrm{ord}_{\mathfrak{p}}(a_{l,0}), \ldots, -\mathrm{ord}_{\mathfrak{p}}(a_{l,s_l})\}.
\end{aligned}$$

This implies that for each i,

$$\begin{aligned}
-\mathrm{ord}_{\mathfrak{p}}(\phi_i(\mathbf{a}_1, \ldots, \mathbf{a}_r)) &\le \max_{\mathbf{j}}\{-\mathrm{ord}_{\mathfrak{p}}(c_{i,\mathbf{j}} a_{1,0}^{j_{1,0}} \ldots a_{l,l'}^{j_{l,l'}} \ldots a_{r,s_r}^{j_{r,s_r}})\} \\
&\le \max_{i,\mathbf{j}}\{-\mathrm{ord}_{\mathfrak{p}}(c_{i,\mathbf{j}})\} + \sum_{l=1}^{r} d_l \max\{-\mathrm{ord}_{\mathfrak{p}}(a_{l,0}), \ldots, -\mathrm{ord}_{\mathfrak{p}}(a_{l,s_l})\}.
\end{aligned}$$

By definition, one sees that $\mathfrak{h}(\Phi(\mathbf{a}_1, \ldots, \mathbf{a}_r)) \le \sum_{l=1}^{r} d_l \mathfrak{h}(\mathbf{a}_l) + \mathfrak{h}(\Phi)$. □

The above proposition has the following corollaries.

Corollary 2.10.

1. *Suppose that Q is a polynomial in $\mathbb{Q}[X_1, \ldots, X_m]$ with degree d_i in X_i for all i. Let $b_1, \ldots, b_m \in \overline{k(t)}$. Then*

$$\mathfrak{h}(Q(b_1,\dots,b_m)) \le \sum_{i=1}^{m} d_i \mathfrak{h}(b_i).$$

In particular, $\mathfrak{h}(a+b), \mathfrak{h}(ab) \le \mathfrak{h}(a)+\mathfrak{h}(b)$ *for any* $a, b \in \overline{k(t)}$.

2. *Suppose that* $c_1, c_2, c_3, c_4 \in \mathbb{Q}$ *satisfy that* $c_1c_4 - c_2c_3 \neq 0$. *Then*

$$\mathfrak{h}\left(\frac{c_1a+c_2}{c_3a+c_4}\right) = \mathfrak{h}(a)$$

for any $a \in \overline{k(t)}$.

Proof. 1. Homogenizing Q, we obtain $\bar{Q} \in \mathbb{Q}[X_{1,0}, X_{1,1}, \dots, X_{m,0}, X_{m,1}] \setminus \{0\}$ homogenous in $X_{i,0}, X_{i,1}$ of degree d_i for all i such that

$$\bar{Q}(\mathbf{b}_1,\dots,\mathbf{b}_m) = Q(b_1,\dots,b_m)$$

where $\mathbf{b}_i = (1 : b_i)$. In Proposition 2.9, if we take $\phi_0 = \prod_{i=1}^m X_{i,0}^{d_i}, \phi_1 = \bar{Q}, \mathbf{a}_i = \mathbf{b}_i$ then we have that

$$\mathfrak{h}(Q(b_1,\dots,b_m)) = \mathfrak{h}(\bar{Q}(\mathbf{b}_1,\dots,\mathbf{b}_m)) \le \sum_{i=1}^{m} d_i \mathfrak{h}(\mathbf{b}_i) = \sum_{i=1}^{m} d_i \mathfrak{h}(b_i).$$

2. If $c_3a + c_4 = 0$, then $c_1a + c_2 \neq 0$ since $c_1c_4 - c_2c_3 \neq 0$. One sees that $\mathfrak{h}((c_1a + c_2)/(c_3a + c_4)) = \mathfrak{h}(\infty) = 0$ and $\mathfrak{h}(a) = \mathfrak{h}(-c_4/c_3) = 0$, then the desired equality holds. Now assume $c_3a + c_4 \neq 0$, we take $r = 1, s_1 = 1$ and

$$\Phi = (c_3X_{1,1} + c_4X_{1,0}, c_1X_{1,1} + c_2X_{1,0}),\ \mathbf{a}_1 = (1 : a)$$

in Proposition 2.9. Then one has that

$$\mathfrak{h}\left(\frac{c_1a+c_2}{c_3a+c_4}\right) = \mathfrak{h}\left(\left(1 : \frac{c_1a+c_2}{c_3a+c_4}\right)\right) = \mathfrak{h}(\Phi(\mathbf{a}_1)) \le \mathfrak{h}(a).$$

Conversely, let $b = (c_1a + c_2)/(c_3a + c_4)$. Then $a = (c_4b - c_2)/(c_1 - c_3b)$. A similar argument implies that $\mathfrak{h}(a) \le \mathfrak{h}(b)$. Thus $\mathfrak{h}(a) = \mathfrak{h}(b)$. □

Corollary 2.11.

1. *Let* $\mathbf{b}_i = (b_{i,0} : \dots : b_{i,n_i}) \in \mathbb{P}^{n_i}(\overline{k(t)})$ *for* $i = 1, 2$. *Suppose that* $b_{1,0} = b_{2,0} = 1$ *and set*

$$\mathbf{c} = (b_{1,0} : \dots : b_{1,n_1} : b_{2,0} : \dots : b_{2,n_2}) \in \mathbb{P}^{n_1+n_2+1}(\overline{k(t)}).$$

Then $\mathfrak{h}(\mathbf{c}) \le \mathfrak{h}(\mathbf{b}_1) + \mathfrak{h}(\mathbf{b}_2)$.

2. *Suppose that* $\mathbf{b} = (b_0 : \cdots : b_n) \in \mathbb{P}^n(\overline{k(t)})$. *Then* $\mathfrak{h}(\mathbf{b}) \leq \sum_{i=0}^{n} \mathfrak{h}(b_i)$.

Proof. 1. We take $r = 2, s_1 = n_1, s_2 = n_2$, $\Phi = (\phi_0 : \cdots : \phi_{n_1+n_2+1})$ with

$$\phi_i = \begin{cases} X_{1,i}X_{2,0} & i = 0, \ldots, n_1 \\ X_{1,0}X_{2,i-n_1-1} & i = n_1+1, \ldots, n_1+n_2+1 \end{cases}$$

and $\mathbf{a}_1 = \mathbf{b}_1, \mathbf{a}_2 = \mathbf{b}_2$. Then $\Phi(\mathbf{a}_1, \mathbf{a}_2) = \mathbf{c}$ and $\mathfrak{h}(\mathbf{c}) \leq \mathfrak{h}(\mathbf{b}_1) + \mathfrak{h}(\mathbf{b}_2)$ because of Proposition 2.9.

2. In Proposition 2.9, take $r = n+1, s_1 = \cdots = s_{n+1} = 1$, $\Phi = (\phi_0, \ldots, \phi_n)$ with $\phi_i = X_{i+1,1} \prod_{j=1, j\neq i+1}^{n+1} X_{j,0}$, $\mathbf{a}_i = (1 : b_{i-1})$ for all $i = 1, \ldots, n+1$. □

Next, we estimate the height of the resultant of two polynomials. One can refer to Section 6 of Chapter 3 in [7] for the definition and properties of resultant.

Corollary 2.12. *Assume that* $P_1, P_2 \in \overline{k(t)}[X_1, \ldots, X_m, Y] \setminus \{0\}$. *Then*

$$\mathfrak{h}(\mathrm{res}_Y(P_1, P_2)) \leq \deg(P_2, Y)\mathfrak{h}(P_1) + \deg(P_1, Y)\mathfrak{h}(P_2)$$

where $\mathrm{res}_Y(P_1, P_2)$ *is the resultant of* P_1 *and* P_2 *with respect to* Y.

Proof. The assertion is clear if $\mathrm{res}_Y(P_1, P_2) = 0$. In the following, suppose that $\mathrm{res}_Y(P_1, P_2) \neq 0$. Assume $\deg(P_i, Y) = n_i$, $i = 1, 2$. Denote $\vec{X} = (X_1, \ldots, X_m)$ and $\vec{X}^{\mathbf{d}} = \prod_{i=1}^{m} X_i^{d_i}$ for $\mathbf{d} = (d_1, \ldots, d_m) \in \mathbb{Z}^m$. Write

$$P_1 = \sum_{i=0}^{n_1} a_i(\vec{X})Y^i, \quad P_2 = \sum_{i=0}^{n_2} b_i(\vec{X})Y^i$$

where $a_i(\vec{X}), b_j(\vec{X}) \in \overline{k(t)}[\vec{X}]$. Then

$$\mathrm{res}_Y(P_1, P_2) = \begin{vmatrix} a_{n_1} & a_{n_1-1} & \cdots & a_0 & & \\ & \ddots & \ddots & & \ddots & \\ & & a_{n_1} & a_{n_1-1} & \cdots & a_0 \\ b_{n_2} & b_{n_2-1} & \cdots & b_0 & & \\ & \ddots & \ddots & & \ddots & \\ & & b_{n_2} & b_{n_2-1} & \cdots & b_0 \end{vmatrix}.$$

Denote by C_1, C_2 the points in $\mathbb{P}^s(\overline{k(t)})$ whose coordinates are the coefficients in $\vec{X}, Y$ of P_1 and P_2 respectively, where s is the maximum of the numbers of terms of P_1 and P_2. By the definition of determinant, we can write

$$\mathrm{res}_Y(P_1, P_2) = \sum_{\mathbf{d}} \left(\sum_{j=1}^{\ell_{\mathbf{d}}} \beta_{\mathbf{d},j} \mathbf{m}_{\mathbf{d},j} \mathbf{n}_{\mathbf{d},j} \right) \vec{X}^{\mathbf{d}}$$

where $\beta_{\mathbf{d},j}, \ell_{\mathbf{d}} \in \mathbb{Z}, \ell_{\mathbf{d}} \geq 0$, $\mathbf{m}_{\mathbf{d},j}$ is a monomial in the coordinates of C_1 of total degree n_2 and $\mathbf{n}_{\mathbf{d},j}$ is a monomial in the coordinates of C_2 of total degree n_1. Viewing $\sum_{j=1}^{\ell_{\mathbf{d}}} \beta_{\mathbf{d},j}\mathbf{m}_{\mathbf{d},j}\mathbf{n}_{\mathbf{d},j}$ as a polynomial homogeneous in the coordinates of C_1 of degree n_2 and homogeneous in the coordinates of C_2 of degree n_1 with coefficients in $\mathbb{Z}$, by Proposition 2.9, one has that

$$\mathfrak{h}(\mathrm{res}_Y(P_1, P_2)) = \mathfrak{h}\left(\left(\cdots : \sum_{j=1}^{\ell_{\mathbf{d}}} \beta_{\mathbf{d},j}\mathbf{m}_{\mathbf{d},j}\mathbf{n}_{\mathbf{d},j} : \cdots\right)\right)$$
$$\leq n_2\mathfrak{h}(C_1) + n_1\mathfrak{h}(C_2) = n_2\mathfrak{h}(P_1) + n_1\mathfrak{h}(P_2). \quad \square$$

Corollary 2.13. *Suppose that* $P \in \overline{k(t)}[X, Y] \setminus \{0\}$ *and* $a, b \in \overline{k(t)}$. *Then*

$$\mathfrak{h}(P(X + a, Y + b)) \leq \mathfrak{h}(P) + \deg(P, X)\mathfrak{h}(a) + \deg(P, Y)\mathfrak{h}(b).$$

Proof. Denote $d_1 = \deg(P, X)$ and $d_2 = \deg(P, Y)$ and write

$$P = \sum_{i=0}^{d_1}\sum_{j=0}^{d_2} c_{i,j}X^iY^j,\ c_{i,j} \in \overline{k(t)}.$$

An easy calculation yields that

$$P(X + a, Y + b) = \sum_{l_1=0}^{d_1}\sum_{l_2=0}^{d_2}\left(\sum_{i=l_1}^{d_1}\sum_{j=l_2}^{d_2}\binom{i}{l_1}\binom{j}{l_2}c_{i,j}a^{i-l_1}b^{j-l_2}\right)X^{l_1}Y^{l_2}.$$

Note that $\sum_{i=l_1}^{d_1}\sum_{j=l_2}^{d_2}\binom{i}{l_1}\binom{j}{l_2}c_{i,j}a^{i-l_1}b^{j-l_2}$ is homogeneous in $c_{i,j}$ of degree 1, homogeneous in $1, a$ of degree d_1 and homogeneous in $1, b$ of degree d_2 with coefficients in $\mathbb{Z}$. By Proposition 2.9,

$$\mathfrak{h}(P(X + a, Y + b)) = \mathfrak{h}\left(\left(\cdots : \sum_{i=l_1}^{d_1}\sum_{j=l_2}^{d_2}\binom{i}{l_1}\binom{j}{l_2}c_{i,j}a^{i-l_1}b^{j-l_2} : \cdots\right)\right)$$
$$\leq \mathfrak{h}(P) + d_1\mathfrak{h}(a) + d_2\mathfrak{h}(b). \quad \square$$

When $\mathfrak{h}$ is an absolute logarithmic height defined in an algebraic number field, the results in Corollaries 2.12 and 2.13 with error terms have already been proved in [2].

Corollary 2.14. *Suppose that* $M = (a_{i,j})$ *is an* $l \times m$ *matrix with* $a_{i,j} \in \overline{k(t)}$ *and* $\mathfrak{h}(a_{i,j}) \leq \kappa$. *Assume that the linear system* $M\mathbf{x} = 0$ *has a nonzero solution. Then* $M\mathbf{x} = 0$ *has a nonzero solution* $\mathbf{a}$ *with* $\mathfrak{h}(\mathbf{a}) \leq r^2(r + 1)\kappa$, *where* $\mathbf{a}$ *is viewed as a point in* $\mathbb{P}^{m-1}(\overline{k(t)})$ *and* $r = \mathrm{rank}(M)$.

Proof. The assertion is clear in the case $M = 0$. Suppose that $M \neq 0$ and $r = \text{rank}(M)$. Without loss of generality, assume that the first r-rows of M are linearly independent over $\overline{k(t)}$ and denote by $\tilde{M}$ the $r \times m$ matrix formed by those rows. Then M and $\tilde{M}$ have the same solution space and thus there is no harm to replace M by $\tilde{M}$. Since $\tilde{M}\mathbf{x} = 0$ has a nonzero solution, $r < m$. Without loss of generality, we further assume that the matrix M_1 formed by the first r-columns of $\tilde{M}$ is invertible. Denote by $\mathbf{b}$ the $(r+1)$-th column of $\tilde{M}$. Using Cramer's rule, $(D_1/\det(M_1), \ldots, D_r/\det(M_1))^t$ is the solution of $M_1\mathbf{x} = -\mathbf{b}$, where D_i is the determinant of the matrix obtained by replacing the i-th column of M_1 by $-\mathbf{b}$, and $(\cdot)^t$ denotes the transpose of a vector. Set

$$\mathbf{a} = (D_1, \ldots, D_r, \det(M_1), \underbrace{0, \ldots, 0}_{m-r-1})^t.$$

Then $\mathbf{a}$ is a solution of $\tilde{M}\mathbf{x} = 0$. Note that D_i and $\det(M_1)$ are homogeneous in $a_{1,1}, \ldots, a_{r,r+1}$ of degree r. By Proposition 2.9, $\mathfrak{h}(\mathbf{a}) \leq r\mathfrak{h}((a_{1,1} : \cdots : a_{r,r+1}))$. By Corollary 2.11,

$$\mathfrak{h}((a_{1,1} : \cdots : a_{r,r+1})) \leq \sum_{i,j} \mathfrak{h}(a_{i,j}) \leq r(r+1)\kappa.$$

So $\mathfrak{h}(\mathbf{a}) \leq r^2(r+1)\kappa$. $\square$

Note that all valuations constructed by places of L over k are non-archimedean (see page 62 of [14] for the construction). By Proposition 2.4 on page 57 in [14] with $s = 0$, one sees that if G and H are polynomials in $\overline{k(t)}[X_1, \ldots, X_m]$, then

$$\mathfrak{h}(GH) = \mathfrak{h}(G) + \mathfrak{h}(H),$$

from which we have the following proposition.

Proposition 2.15.

1. *Suppose that $G, H \in \overline{k(t)}[X_1, \ldots, X_m]$ and G divides H. Then $\mathfrak{h}(G) \leq \mathfrak{h}(H)$.*
2. *Suppose that H is a nonzero polynomial in $\overline{k(t)}[X]$ and a is a zero of H in $\overline{k(t)}$. Then $\mathfrak{h}(a) \leq \mathfrak{h}(H)$.*

Proof. The first assertion is clear. The second one follows from the facts that $X - a$ divides H and $\mathfrak{h}(X - a) = \mathfrak{h}(a)$. $\square$

The following result is claimed on page 13 of [18]. We present a proof here for completeness.

Proposition 2.16. *Suppose that* $\mathbf{a} = (a_0 : \cdots : a_n) \in \mathbb{P}^n(\overline{k(t)})$*. Let* $\mathbf{b}$ *be a point in* $\mathbb{P}^{\binom{d+n}{n}-1}(\overline{k(t)})$ *with all monomials in* $a_0, \ldots, a_n$ *of total degree* d *as coordinates. Then* $\mathfrak{h}(\mathbf{b}) = d\mathfrak{h}(\mathbf{a})$*.*

Proof. Due to Proposition 2.9, one sees that $\mathfrak{h}(\mathbf{b}) \leq d\mathfrak{h}(\mathbf{a})$. It remains to prove the converse. Let $L = k(t, a_0, \ldots, a_n)$. For each place $\mathfrak{p}$ of L over k, one has that

$$\begin{aligned}\max\{-\mathrm{ord}_{\mathfrak{p}}(a_0^{s_0} \cdots a_n^{s_n}) \mid s_i \geq 0, s_0 + \cdots + s_n = d\} &\geq \max_{i=0}^{n}\{-\mathrm{ord}_{\mathfrak{p}}(a_i^d)\} \\ &= d \max_{i=0}^{n}\{-\mathrm{ord}_{\mathfrak{p}}(a_i)\}.\end{aligned}$$

By definition, $\mathfrak{h}(\mathbf{b}) \geq d\mathfrak{h}(\mathbf{a})$. So $\mathfrak{h}(\mathbf{b}) = d\mathfrak{h}(\mathbf{a})$. □

3. The Riemann-Roch spaces

Throughout this section, let K be an algebraically closed field of characteristic zero with an absolute logarithmic height $\mathfrak{h}$. The heights of elements in K and nonzero polynomials with coefficients in K are defined as in Definition 2.5. Furthermore, we assume that

(A1) Propositions 2.7 and 2.9 hold for $\mathfrak{h}$ and K in place of $\overline{k(t)}$. Consequently, Corollaries 2.10, 2.11, 2.12, 2.13 and 2.14 hold for $\mathfrak{h}$ and K in place of $\overline{k(t)}$.

(A2) Propositions 2.15 and 2.16 also hold for $\mathfrak{h}$ and K in place of $\overline{k(t)}$.

Remark 3.1. Under the assumption that Corollary 2.10 holds for $\mathfrak{h}$ and K, one has that $\mathfrak{h}(a) = 0$ for all $a \in \mathbb{Q}$. To see this, we first have that $\mathfrak{h}(m) = 0$ for all $m \in \mathbb{Z}$. Then for $m_1, m_2 \in \mathbb{Z} \setminus \{0\}$,

$$\mathfrak{h}\left(m_1/m_2\right) \leq \mathfrak{h}(m_1) + \mathfrak{h}(1/m_2) = \mathfrak{h}(m_1) + \mathfrak{h}(m_2) = 0.$$

So $\mathfrak{h}(a) = 0$ for all $a \in \mathbb{Q}$.

Let L be an algebraic function field of one variable over K and D a divisor of L. Suppose that $\mathcal{L}_K(D) \neq \{0\}$. In this section, we are going to give bounds for the degrees and height of a certain nonzero element in $\mathcal{L}_K(D)$. Let us start with two lemmas.

Lemma 3.2. *Let* $f_i = \sum_{s\geq 0} a_{i,s} z^s \in K[[z]]$ *for* $i = 1, \ldots, r$*. Write* $\prod_{i=1}^r f_i = \sum_{s\geq 0} c_s z^s$ *with* $c_s \in K$*. Then*

$$\mathfrak{h}(c_s) \leq \sum_{i=1}^{r} (s+1) \max_{j=0}^{s}\{\mathfrak{h}(a_{i,j})\}.$$

Proof. One can easily check that

$$c_s = \sum_{\substack{0 \le l_1, \ldots, l_r \le s, \\ l_1 + \cdots + l_r = s}} a_{1,l_1} \cdots a_{r,l_r}.$$

By Corollary 2.10, one has that

$$\mathfrak{h}(c_s) \le \sum_{i=1}^{r} \sum_{j=0}^{s} \mathfrak{h}(a_{i,j}) \le \sum_{i=1}^{r} (s+1) \max_{j=0}^{s} \{\mathfrak{h}(a_{i,j})\}. \quad \square$$

Lemma 3.3. *Suppose that $Q \in K[z, Y]$ and $f = \sum_{i \ge 0} a_i z^i \in K[[z]]$ with $Q(z, f) = 0$. Then for $i \ge 0$,*

$$\mathfrak{h}(a_i) \le (\deg(Q, Y) + 1)^i \mathfrak{h}(Q).$$

Proof. Denote $n = \deg(Q, Y)$. Dividing Q by some power of z if necessary, we may assume that $z \nmid Q$. Note that this operation does not change the height of Q. It is easy to verify that $\mathfrak{h}(Q(0, Y)) \le \mathfrak{h}(Q)$. Since $Q(0, Y) \ne 0$ and $Q(0, a_0) = 0$, by Proposition 2.15, $\mathfrak{h}(a_0) \le \mathfrak{h}(Q(0, Y)) \le \mathfrak{h}(Q)$. Now set $Q_1 = Q(z, a_0 + zY)$. Then by Corollary 2.13,

$$\begin{aligned} \mathfrak{h}(Q_1) &= \mathfrak{h}(Q(z, a_0 + zY)) = \mathfrak{h}(Q(z, a_0 + Y)) \\ &\le \mathfrak{h}(Q) + n\mathfrak{h}(a_0) \le \mathfrak{h}(Q) + n\mathfrak{h}(Q) = (n+1)\mathfrak{h}(Q). \end{aligned}$$

Again, we may assume that $z \nmid Q_1$. One sees that $a_1 + a_2 z + \cdots$ is a solution of $Q_1(z, Y) = 0$. Using a similar argument, one has that $\mathfrak{h}(a_1) \le \mathfrak{h}(Q_1) \le (n+1)\mathfrak{h}(Q)$. Set $Q_{i+1} = Q_i(z, a_i + zY)$ for $i = 1, 2, \ldots$. Repeating the previous process yields that $\mathfrak{h}(a_i) \le (n+1)^i \mathfrak{h}(Q)$. $\square$

Now suppose that $L = K(x, y)$ where x is transcendental over K and $[L : K(x)] < \infty$. Furthermore, assume that $P \in K[X, Y]$ is a nonzero irreducible polynomial such that $P(x, y) = 0$. Let us first adapt a result given in [17] on the degree bound for a basis of a Riemann-Roch space. For this, we need to recall some notations introduced in [17]. Write

$$P(X, Y) = A_0(X)Y^n + A_1(X)Y^{n-1} + \cdots + A_n(X), \tag{2}$$

where $A_i \in K[X]$, $A_0 \ne 0$ and $\deg(P, Y) = n$. Set

$$y_1 = 1, \; y_2 = A_0(x)y, \; \ldots, \; y_n = A_0(x)y^{n-1} + \cdots + A_{n-2}(x)y.$$

Then the y_i's are integral over $K[x]$. To see this, note that y_i is integral over $K[x]$ if and only if y_i has no pole lying above $x - c$ for any $c \in K$. Suppose that $\mathfrak{p}$ is a pole of some

y_i lying above $x - c$ for some $c \in K$. Then $\mathfrak{p}$ is a pole of y and thus a zero of y^{-1}. On the other hand,

$$y_i = A_0(x)y^{i-1} + \cdots + A_{i-2}(x)y = -A_{i-1}(x) - A_i(x)y^{-1} - \cdots - A_n(x)y^{-n+i-1}$$

which implies that $\mathfrak{p}$ is not a pole of y_i, a contradiction. Let $\mathbf{d}(X)$ be the discriminant of P with respect to Y. Let $D = \sum_{\mathfrak{p}} d_{\mathfrak{p}}\mathfrak{p}$ be a divisor of L. It is clear that $\mathcal{L}_K(0) = K$. Thus we assume $D \neq 0$ in the rest of the article. Additionally, we adopt the following notations, where $\mathrm{tdeg}(\cdot)$ denotes the total degree of a polynomial.

Notation 3.4.

$$\delta_D = \sum_{\mathfrak{p}} |d_{\mathfrak{p}}|,$$

$$\rho = \mathrm{tdeg}(P),$$

$$\mathbf{q}_D(X) = \mathbf{d}(X)^{\rho(\rho+\delta_D)} \prod_{\substack{\mathfrak{p}\in \mathrm{supp}(D),\\ \mathrm{ord}_{\mathfrak{p}}(x)\geq 0}} (X - \pi_{\mathfrak{p}}(x))^{\rho(\rho+\delta_D)},$$

$$U = \mathrm{supp}(\mathrm{div}(x)^-) \cup \mathrm{supp}(\mathrm{div}(y)^-) \cup \mathrm{supp}(\mathrm{div}(\mathbf{q}_D(x))),$$

$$h(D) = \max\{\mathfrak{h}(P), \max\{\mathfrak{h}(\pi_{\mathfrak{p}}(x)) \mid \forall\, \mathfrak{p} \in U\}\}.$$

Remark 3.5. Note that $\mathrm{supp}(D) \subset U$. To see this, for $\mathfrak{p} \in \mathrm{supp}(D)$ with $\mathrm{ord}_{\mathfrak{p}}(x) \geq 0$, $\mathfrak{p}$ is a zero of $x - \pi_{\mathfrak{p}}(x)$ and thus a zero of $\mathbf{q}_D(x)$. So $\mathfrak{p} \in \mathrm{supp}(\mathrm{div}(\mathbf{q}_D(x))) \subset U$. For $\mathfrak{p} \in \mathrm{supp}(D)$ with $\mathrm{ord}_{\mathfrak{p}}(x) < 0$, $\mathfrak{p} \in \mathrm{supp}(\mathrm{div}(x)^-)$ which is obvious in U.

In [17], when $K = \mathbb{C}$, Schmidt gave a degree bound for a basis of the Riemann-Roch space $\mathcal{L}_{\mathbb{C}}(D)$ of a divisor D. Moreover, he proved that if P has coefficients in a subfield k of $\mathbb{C}$ and D is defined over k then there is a basis of $\mathcal{L}_{\mathbb{C}}(D)$ whose elements are in $k(x, y)$. After small modifications of Schmidt's results, we are able to prove that Schmidts's result on degree bound is also valid for the algebraically closed field K. Suppose that $k \subset K$ is an algebraically closed subfield such that $P(X, Y) \in k[X, Y]$. By Definition 2.3, one sees that each divisor of $k(x, y)$ is also a divisor of $K(x, y)$. Suppose that $f \in k(x, y) \setminus \{0\}$. To avoid confusion, we denote by $\mathrm{div}_k(f)$ the divisor of f viewed as an element in $k(x, y)$ and by $\mathrm{div}_K(f)$ the divisor of f viewed as an element in $K(x, y)$.

Lemma 3.6. *Suppose that $k \subset K$ is an algebraically closed subfield such that $P(X, Y) \in k[X, Y]$. Then for every $f \in k(x, y) \setminus \{0\}$, $\mathrm{div}_k(f) = \mathrm{div}_K(f)$.*

Proof. If $f \in k$ then there is nothing to prove. Suppose that $f \notin k$. Since a zero (resp. pole) of f in $k(x, y)$ is also a zero (resp. pole) of f in $K(x, y)$, $f \notin K$. Due to Theorem 4 on page 18 of [5],

$$\deg(\mathrm{div}_k(f)^+) = \deg(\mathrm{div}_k(f)^-) = [k(x,y) : k(f)].$$

Similarly, one has that

$$\deg(\mathrm{div}_K(f)^+) = \deg(\mathrm{div}_K(f)^-) = [K(x,y) : K(f)].$$

As k is algebraically closed, $[k(x,y) : k(f)] = [K(x,y) : K(f)]$. This implies that $\deg(\mathrm{div}_k(f)^+) = \deg(\mathrm{div}_K(f)^+)$ and $\deg(\mathrm{div}_k(f)^-) = \deg(\mathrm{div}_K(f)^-)$. On the other hand, one has that $\mathrm{div}_k(f)^+, \mathrm{div}_k(f)^-$ are divisors of $K(x,y)$ and $\mathrm{div}_K(f)^+ - \mathrm{div}_k(f)^+ \geq 0$, $\mathrm{div}_K(f)^- - \mathrm{div}_k(f)^- \geq 0$. Therefore $\mathrm{div}_k(f)^+ = \mathrm{div}_K(f)^+$ and $\mathrm{div}_k(f)^- = \mathrm{div}_K(f)^-$. Consequently, $\mathrm{div}_k(f) = \mathrm{div}_K(f)$. □

Proposition 3.7. *Let $\rho, \delta_D, \mathbf{q}_D$ be as in Notation 3.4. Then there are integers $\pi_1, \ldots, \pi_n$, and a monic factor q of $\mathbf{q}_D$ with $\deg(q) < \rho(\rho + \delta_D)$, $B_{i,j} \in K[X]$ with $\deg(B_{i,j}) < 2\rho(\rho + 2\delta_D)$ such that $\mathcal{L}_K(D)$ has a basis of the type*

$$x^l \left(\sum_{j=1}^{n} \frac{B_{i,j}(x)}{q(x)} y_j \right) \tag{3}$$

where i runs over all integers $s \in \{1, 2, \ldots, n\}$ satisfying $\pi_s \geq 0$ and l runs over all integers in $\{0, 1, \ldots, \pi_i\}$.

Proof. If $\mathcal{L}_K(D) = \{0\}$, we take all $\pi_i < 0$ for $i = 1, \ldots, n$ and the assertion is obvious. Now assume $\mathcal{L}_K(D) \neq \{0\}$ and $a_1, \ldots, a_m \in L$ is a basis of $\mathcal{L}_K(D)$ over K. Let $k \subset K$ be a field finitely generated over $\mathbb{Q}$ such that $P \in k[X,Y]$, $a_1, \ldots, a_m \in k(x,y)$ and the center of $\mathfrak{p}$ has coordinates in $k \cup \{\infty\}$ for every place $\mathfrak{p}$ in $\mathrm{supp}(D)$. Then the irreducible parametrization corresponding to $\mathfrak{p}$ has coordinates in $\bar{k}((z))$ for any $\mathfrak{p} \in \mathrm{supp}(D)$, where $\bar{k}$ is the algebraic closure of k. We embed $\bar{k}$ into $\mathbb{C}$ and view P as a polynomial in $\mathbb{C}[X,Y]$. Then P is irreducible over $\mathbb{C}$ because P is irreducible over $\bar{k}$. Denote by $\tilde{L}$ the field of fractions of $\mathbb{C}[X,Y]/(P)$, where (P) stands for the ideal in $\mathbb{C}[X,Y]$ generated by P. Then $\bar{k}(x,y)$ can be viewed as a subfield of $\tilde{L}$. Note that D is still a divisor of both $\bar{k}(x,y)$ and $\tilde{L}$. By Lemma 3.6, $\mathcal{L}_{\bar{k}}(D) \subset \mathcal{L}_{\mathbb{C}}(D)$. Since D is defined over $\bar{k}$, by Theorems A2 and B2 of [17], $\mathcal{L}_{\mathbb{C}}(D)$ has a basis of the type (3) with $B_{i,j} \in \bar{k}[X]$, $\deg(q) \leq \deg(P,Y)\delta_D + \deg(\mathbf{d})/2$ and $\deg(B_{i,j}) \leq \deg(P,Y)(\deg(P,X) + 3\delta_D) + \deg(q)$. Note that $\deg(\mathbf{d}) \leq (2\deg(P,Y) - 1)\deg(P,X) < 2\rho^2$. One sees that $\deg(q) < \rho(\rho + \delta_D)$ and $\deg(B_{i,j}) < 2\rho(\rho + 2\delta_D)$. Due to Theorem 1 on page 90 of [5], the vector spaces $\mathcal{L}_{\bar{k}}(D)$ and $\mathcal{L}_{\mathbb{C}}(D)$ have the same dimension and then $\mathcal{L}_{\bar{k}}(D)$ has a basis of the type (3) with $B_{i,j} \in \bar{k}[X]$. Since $\mathcal{L}_{\bar{k}}(D) \subset \mathcal{L}_K(D)$ by Lemma 3.6 and $a_1, \ldots, a_m \in k(x,y)$, $\mathcal{L}_{\bar{k}}(D)$ and $\mathcal{L}_K(D)$ have the same dimension by Theorem 1 on page 90 of [5]. These imply that $\mathcal{L}_K(D)$ has a basis of the type (3) with $B_{i,j} \in \bar{k}[X] \subset K[X]$. □

Corollary 3.8. *Let ρ, δ_D, q be as in Proposition 3.7 and $\tilde{D} = D - \mathrm{div}(q(x))$. Suppose that $\mathcal{L}_K(\tilde{D}) \neq \{0\}$. Then $\mathcal{L}_K(\tilde{D})$ contains a nonzero element of the type*

$$\sum_{j=0}^{n-1} \tilde{B}_j(x)y^j$$

where $\tilde{B}_j \in K[X]$ *with* $\deg(\tilde{B}_j) < 4\rho(\rho + \delta_D)$.

Proof. By Proposition 3.7, there are integers $\pi_1, \dots, \pi_n$, and $B_{i,j} \in K[X]$ with $\deg(B_{i,j}) < 2\rho(\rho + 2\delta_D)$ such that $\mathcal{L}_K(D)$ has a basis of the type

$$x^l \left(\sum_{j=1}^{n} \frac{B_{i,j}(x)}{q(x)} y_j \right) \tag{4}$$

where i runs over all integers $s \in \{1, 2, \dots, n\}$ satisfying $\pi_s \geq 0$ and l runs over all integers in $\{0, 1, \dots, \pi_i\}$. Note that $\mathcal{L}_K(D) = \frac{1}{q(x)}\mathcal{L}_K(\tilde{D})$. Thus $\mathcal{L}_K(D) \neq \{0\}$ which implies that not all π_i are negative. Suppose that $\pi_{i_0} \geq 0$. Setting $l = 0$ in (4) yields that $\sum_{j=1}^{n} \frac{B_{i_0,j}(x)}{q(x)} y_j$ is an element in $\mathcal{L}_K(D)$. Write $\sum_{j=1}^{n} B_{i_0,j}(x)y_j = \sum_{j=0}^{n-1} \tilde{B}_j(x)y^j$ where $\tilde{B}_j \in K[X]$. Note that $y_1 = 1$ and $y_j = \sum_{s=1}^{j-1} A_{j-1-s}(x)y^s$ for $j > 1$. One has that

$$\begin{aligned} \sum_{s=0}^{n-1} \tilde{B}_s(x)y^s &= B_{i_0,1}(x) + \sum_{j=2}^{n} \sum_{s=1}^{j-1} A_{j-1-s}(x)B_{i_0,j}(x)y^s \\ &= B_{i_0,1}(x) + \sum_{s=1}^{n-1} \left(\sum_{j=s+1}^{n} A_{j-1-s}(x)B_{i_0,j}(x) \right) y^s. \end{aligned}$$

Therefore $\tilde{B}_0(x) = B_{i_0,1}(x)$ and $\tilde{B}_s(x) = \sum_{j=s+1}^{n} A_{j-1-s}(x)B_{i_0,j}(x)$ for $s \geq 1$ and so

$$\deg(\tilde{B}_s) \leq \max_j \deg(B_{i_0,j}) + \max_j \deg(A_j) \leq 2\rho(\rho + 2\delta_D) + \rho < 4\rho(\rho + \delta_D).$$

The corollary then follows from the fact that $\mathcal{L}_K(D) = \frac{1}{q(x)}\mathcal{L}_K(\tilde{D})$. □

In the rest of this section, let us estimate the heights of the coefficients of $\tilde{B}_j$. We first estimate the heights of the coefficients of a place represented by an irreducible parametrization of $P(X, Y) = 0$.

Proposition 3.9. *Let* $\rho = \operatorname{tdeg}(P)$. *Suppose that* $(z^\mu + a, z^\nu(c_0 + c_{\ell_1} z^{\ell_1} + \cdots))$ *is a place of* $P(X, Y) = 0$. *Then*

$$\mathfrak{h}(c_i) \leq (\rho + 1)^{i+1} \max\{\mathfrak{h}(P), \mathfrak{h}(a)\}$$

where $c_i = 0$ *if* $i \neq \ell_j$ *for all* $j \geq 1$ *and* $i \neq 0$.

Proof. We first consider the case $\mu > 0$. Set $\bar{P}(z, Y) = z^d P(z^\mu + a, z^\nu Y)$ where d is the integer such that $\bar{P} \in K[z, Y]$ and $z \nmid \bar{P}$. By Corollary 2.13, one can verity that

$$\mathfrak{h}(\bar{P}) \le \mathfrak{h}(P) + \deg(P, X)\mathfrak{h}(a) \le (\rho + 1)\max\{\mathfrak{h}(P), \mathfrak{h}(a)\}.$$

As $c_0+c_{\ell_1}z^{\ell_1}+\cdots$ is a solution of $\bar{P}(z, Y) = 0$ and $\deg(P, Y) = \deg(\bar{P}, Y)$, by Lemma 3.3, one sees that

$$\mathfrak{h}(c_i) \le (\deg(\bar{P}, Y) + 1)^i\mathfrak{h}(\bar{P}) \le (\rho + 1)^{i+1}\max\{\mathfrak{h}(P), \mathfrak{h}(a)\}.$$

Suppose that $\mu < 0$. Similarly, set $\bar{P}(z, Y) = z^d P(z^\mu, z^\nu Y)$ where d is the integer such that $\bar{P} \in K[z, Y]$ and $z \nmid \bar{P}$. Then $\mathfrak{h}(P) = \mathfrak{h}(\bar{P})$ and $\deg(P, Y) = \deg(\bar{P}, Y)$. Since $c_0 + c_{\ell_1}z^{\ell_1} + \cdots$ is a solution of $\bar{P}(z, Y) = 0$, by Lemma 3.3, $\mathfrak{h}(c_i) \le (\rho+1)^{i+1}\mathfrak{h}(P)$. □

For a place $\mathfrak{p} = (z^\mu + a, z^\nu(c_0 + c_1z + \cdots))$ of $P(X, Y) = 0$, the series $(z^\mu + a)^l(z^\nu(c_0 + c_1z + \cdots))^j$ is called the expansion of x^ly^j at $\mathfrak{p}$, denoted by $x^ly^j|_{(x,y)=\mathfrak{p}}$ for brevity.

Lemma 3.10. *Let $\rho = \operatorname{tdeg}(P)$. For $l \ge 0, j \in \{0, \ldots, n-1\}$ and a place $\mathfrak{p}$, x^ly^j has an expansion at $\mathfrak{p}$ of the type*

$$x^ly^j\big|_{(x,y)=\mathfrak{p}} = z^{d_{\mathfrak{p},l,j}}\sum_{s=0}^{\infty}\beta_{\mathfrak{p},l,j,s}z^s$$

where $d_{\mathfrak{p},l,j}$ is an integer greater than $-l\rho - \rho^2$ and $\beta_{\mathfrak{p},l,j,s} \in K$ with

$$\mathfrak{h}(\beta_{\mathfrak{p},l,j,s}) \le ((s + 1)^2(\rho + 1)^{s+2} + l)\max\{\mathfrak{h}(P), \mathfrak{h}(\pi_{\mathfrak{p}}(x))\}.$$

Proof. Suppose that

$$\mathfrak{p} = (\mathbf{x}(z), \mathbf{y}(z)) = (z^\mu + a, z^\nu(c_0 + c_{\ell_1}z^{\ell_1} + \cdots)).$$

Then $\mathfrak{h}(\pi_{\mathfrak{p}}(x)) = \mathfrak{h}(a)$. To see this, if $\mu > 0$ then $\pi_{\mathfrak{p}}(x) = a$ and we are done, if $\mu < 0$ then $\pi_{\mathfrak{p}}(x) = \infty$ and $a = 0$ and thus $\mathfrak{h}(\pi_{\mathfrak{p}}(x)) = 0 = \mathfrak{h}(a)$. By Proposition 3.9, for $i \ge 0$,

$$\mathfrak{h}(c_i) \le (\rho + 1)^{i+1}\max\{\mathfrak{h}(P), \mathfrak{h}(a)\}$$

where $c_i = 0$ if $i \ne l_j$ for all $j \ge 1$ and $i \ne 0$. Write $\mathbf{y}(z)^j = z^{j\nu}\sum_{s\ge0} b_{j,s}z^s$. By Lemma 3.2 with $f_i = \sum_{s\ge0} c_sz^s$, one sees that

$$\begin{aligned}\mathfrak{h}(b_{j,s}) &\le \sum_{i=1}^{j}(s + 1)\max_{\gamma=0}^{s}\{\mathfrak{h}(c_\gamma)\} \le j(s + 1)\max_{\gamma=0}^{s}\{\mathfrak{h}(c_\gamma)\}\\ &\le (s + 1)(\rho + 1)^{s+2}\max\{\mathfrak{h}(P), \mathfrak{h}(a)\}.\end{aligned}$$

The last inequality holds because $j \le n - 1 < \rho + 1$.

We first consider the case $\mu > 0$. Note that $(z^\mu + a)^l = \sum_{s=0}^{l} \binom{l}{s} a^{l-s} z^{s\mu}$. This implies that $(z^\mu + a)^l \mathbf{y}(z)^j$ has an expansion of the type $z^{e_{l,j}} \sum_{s\geq 0} \beta_{\mathfrak{p},l,j,s} z^s$ at $z = 0$, where $e_{l,j} = j\nu$ and

$$\beta_{\mathfrak{p},l,j,s} = \sum_{i=0}^{l} b_{j,s-i\mu} \binom{l}{i} a^{l-i}$$

with $b_{j,i} = 0$ if $i < 0$. Therefore by Corollary 2.10,

$$\mathfrak{h}(\beta_{\mathfrak{p},l,j,s}) \leq \sum_{i=0}^{s} \mathfrak{h}(b_{j,i}) + l\mathfrak{h}(a) \leq ((s+1)^2(\rho+1)^{s+2} + l) \max\{\mathfrak{h}(P), \mathfrak{h}(a)\}.$$

Set $d_{\mathfrak{p},l,j} = j\nu$. Then we have the required expansion for $x^l y^j$ at $\mathfrak{p}$. Finally, as $|\nu| \leq |\mathrm{ord}_{\mathfrak{p}}(y)| \leq \rho$, one has that $d_{\mathfrak{p},l,j} > -\rho^2 \geq -\rho l - \rho^2$.

Now suppose that $\mu < 0$. In this case, one easily sees that $d_{\mathfrak{p},l,j} = j\nu + l\mu$ and $\beta_{\mathfrak{p},l,j,s} = b_{j,s}$. As $|\mu| \leq |\mathrm{ord}_{\mathfrak{p}}(x)| \leq \rho$, one has that $d_{\mathfrak{p},l,j} > -l\rho - \rho^2$. □

Let $\mathbf{c} = (\ldots, c_{l,j}, \ldots)$ be a vector with indeterminate coordinates and set

$$g(\mathbf{c}) = \sum_{j=0}^{n-1} \sum_{l=0}^{4\rho(\rho+\delta_D)-1} c_{l,j} x^l y^j.$$

Proposition 3.11. *Let $\rho, \delta_D, h(D)$ be as in Notation 3.4. Let $\tilde{D}$ be as in Corollary 3.8. Suppose that $\mathcal{L}_K(\tilde{D}) \neq \{0\}$. Then $\mathcal{L}_K(\tilde{D})$ contains a nonzero element of the type*

$$g(\mathbf{a}) = \sum_{j=0}^{n-1} \sum_{l=0}^{4\rho(\rho+\delta_D)-1} a_{l,j} x^l y^j \tag{5}$$

with

$$\mathfrak{h}(\mathbf{a}) \leq 1600(\rho + \delta_D)^6 (\rho+1)^{5(\rho+\delta_D)^3 - 11} h(D),$$

where $a_{l,j} \in K$, at least one of $a_{l,j}$ equals 1 and $\mathbf{a}$ is viewed as a projective point.

Proof. Let U be as in Notation 3.4. By Lemma 3.10, for each place $\mathfrak{p}$, $j = 0, \ldots, n-1$ and $l \geq 0$, $x^l y^j$ has an expansion at $\mathfrak{p}$ of the type

$$x^l y^j \big|_{(x,y)=\mathfrak{p}} = z^{d_{\mathfrak{p},l,j}} \sum_{s=0}^{\infty} \beta_{\mathfrak{p},l,j,s} z^s$$

where $d_{\mathfrak{p},l,j}$ is an integer greater than $-l\rho - \rho^2$ and $\beta_{\mathfrak{p},l,j,s} \in K$ with

$$\mathfrak{h}(\beta_{\mathfrak{p},l,j,s}) \leq ((s+1)^2(\rho+1)^{s+2}+l)\max\{\mathfrak{h}(P),\mathfrak{h}(\pi_{\mathfrak{p}}(x))\}.$$

Set $o=\min_{\mathfrak{p},l,j}\{d_{\mathfrak{p},l,j}\}$ and write

$$x^l y^j|_{(x,y)=\mathfrak{p}} = z^o \sum_{s=0}^{\infty} \alpha_{\mathfrak{p},l,j,s} z^s$$

One can easily see that $\alpha_{\mathfrak{p},l,j,s}=0$ if $s<d_{\mathfrak{p},l,j}-o$ and $\alpha_{\mathfrak{p},l,j,s}=\beta_{\mathfrak{p},l,j,s+o-d_{\mathfrak{p},l,j}}$ if $s\geq d_{\mathfrak{p},l,j}-o$. Therefore for $s\geq d_{\mathfrak{p},l,j}-o$,

$$\mathfrak{h}(\alpha_{\mathfrak{p},l,j,s})=\mathfrak{h}(\beta_{\mathfrak{p},l,j,s+o-d_{\mathfrak{p},l,j}}) \leq ((s+1)^2(\rho+1)^{s+2}+l)\max\{\mathfrak{h}(P),\mathfrak{h}(\pi_{\mathfrak{p}}(x))\},$$

where the last inequality holds since $s+o-d_{\mathfrak{p},l,j}\leq s$. Then for each place $\mathfrak{p}$, $g(\mathbf{c})$ has an expansion at $\mathfrak{p}$ of the type

$$g(\mathbf{c})|_{(x,y)=\mathfrak{p}} = z^o \sum_{s\geq 0}\left(\sum_{l,j} c_{l,j}\alpha_{\mathfrak{p},l,j,s}\right) z^s.$$

Suppose that $\bar{\mathbf{c}}=(\cdots:\bar{c}_{l,j}:\cdots)$ where $\bar{c}_{l,j}\in K$. Note that a pole of $g(\bar{\mathbf{c}})$ is either a pole of x or a pole of y and so all poles of $g(\bar{\mathbf{c}})$ are in U. Write $\tilde{D}=\sum_{\mathfrak{p}} m_{\mathfrak{p}}\mathfrak{p}$. Then $g(\bar{\mathbf{c}})\in\mathcal{L}_K(\tilde{D})$ if and only if $\mathrm{ord}_{\mathfrak{p}}(g(\bar{\mathbf{c}}))\geq -m_{\mathfrak{p}}$ for every $\mathfrak{p}\in\mathrm{supp}(\tilde{D})$ and $\mathrm{ord}_{\mathfrak{p}}(g(\bar{\mathbf{c}}))\geq 0$ for every $\mathfrak{p}\in U\setminus\mathrm{supp}(\tilde{D})$, i.e. $\mathrm{ord}_z(g(\bar{\mathbf{c}})|_{(x,y)=\mathfrak{p}})\geq -m_{\mathfrak{p}}$ for every $\mathfrak{p}\in\mathrm{supp}(\tilde{D})$ and $\mathrm{ord}_z(g(\bar{\mathbf{c}})|_{(x,y)=\mathfrak{p}})\geq 0$ for every $\mathfrak{p}\in U\setminus\mathrm{supp}(\tilde{D})$. Equivalently, $g(\bar{\mathbf{c}})\in\mathcal{L}_K(\tilde{D})$ if and only if $\bar{\mathbf{c}}$ is a solution of the following linear system

$$\begin{aligned}&\left\{\sum_{l,j} c_{l,j}\alpha_{\mathfrak{p},l,j,s}=0 \mid s=0,\ldots,-m_{\mathfrak{p}}-o-1,\,\mathfrak{p}\in\mathrm{supp}(\tilde{D})\right\}\bigwedge\\ &\left\{\sum_{l,j} c_{l,j}\alpha_{\mathfrak{p},l,j,s}=0 \mid s=0,\ldots,-o-1,\,\mathfrak{p}\in U\setminus\mathrm{supp}(\tilde{D})\right\}.\end{aligned} \tag{6}$$

Note that $\tilde{D}=D-\mathrm{div}(q(x))$. By Remark 3.5, $\mathrm{supp}(D)\subset U$ and thus $\mathrm{supp}(\tilde{D})\subset U$. By definition, $\mathfrak{h}(\pi_{\mathfrak{p}}(x))\leq h(D)$ for all $\mathfrak{p}\in U$. So for $l\leq 4\rho(\rho+\delta_D)-1$ and $\mathfrak{p}\in U$,

$$\begin{aligned}\mathfrak{h}(\alpha_{\mathfrak{p},l,j,s}) &\leq ((s+1)^2(\rho+1)^{s+2}+l)h(D)\\ &\leq (\rho+\delta_D)(s+1)^2(\rho+1)^{s+3}h(D).\end{aligned}$$

The second inequality holds because $(s+1)^2(\rho+1)^{s+3}-4\rho\geq(s+1)^2(\rho+1)^{s+2}$. In what follows, we shall estimate $-m_{\mathfrak{p}}$ when $m_{\mathfrak{p}}<0$. Note that

$$\deg(\mathrm{div}(q(x))^+)=\deg(q)\deg(\mathrm{div}(x)^+)=\deg(q)n\leq\rho^2(\rho+\delta_D).$$

Hence $|m_{\mathfrak{p}}| \leq \delta_D + \deg(\mathrm{div}(q(x))^+) < (\rho+1)^2(\rho+\delta_D)$. Since $o > -\rho l - \rho^2$ and $l \leq 4\rho(\rho+\delta_D) - 1$, one has that

$$\begin{aligned} -m_{\mathfrak{p}} - o - 1 &< (\rho+1)^2(\rho+\delta_D) + \rho l + \rho^2 \\ &\leq 5\rho^2(\rho+\delta_D) + 2\rho(\rho+\delta_D) + \rho^2 + \delta_D \\ &= 5(\rho+1)^2(\rho+\delta_D) - 7\rho^2 - 8\rho\delta_D - 5\rho - 4\delta_D \\ &\leq 5(\rho+1)^2(\rho+\delta_D) - 24. \end{aligned}$$

Therefore the heights of the coefficients of the system (6) are not greater than

$$\begin{aligned} T &\triangleq (\rho+\delta_D)(5(\rho+1)^2(\rho+\delta_D) - 23)^2(\rho+1)^{5(\rho+1)^2(\rho+\delta_D)-21}h(D) \\ &\leq 25(\rho+\delta_D)^3(\rho+1)^{5(\rho+\delta_D)^3-17}h(D). \end{aligned}$$

The system (6) contains $4n\rho(\rho+\delta_D) \leq 4\rho^2(\rho+\delta_D)$ variables and thus the rank of the system (6) is not greater than $4\rho^2(\rho+\delta_D)$. Notice that the system (6) has a nonzero solution due to Corollary 3.8. Therefore by Corollary 2.14, the system (6) has a nonzero solution $\bar{\mathbf{c}}$ with

$$\mathfrak{h}(\bar{\mathbf{c}}) \leq (4\rho^2(\rho+\delta_D))^2(4\rho^2(\rho+\delta_D)+1)T \leq 1600(\rho+\delta_D)^6(\rho+1)^{5(\rho+\delta_D)^3-11}h(D).$$

Let λ be a nonzero coordinate of $\bar{\mathbf{c}}$ and set $\mathbf{a} = \bar{\mathbf{c}}/\lambda$. Then $g(\mathbf{a})$ is the desired element. □

Proposition 3.12. *Let* $\mathbf{a}$ *be as in Proposition 3.11 and let* $\rho, \delta_D, h(D)$ *be as in Notation 3.4. Suppose that* $Q_1 \in K[X,Z], Q_2 \in K[Y,Z]$ *are nonzero irreducible polynomials such that* $Q_1(x, g(\mathbf{a})/q(x)) = 0$ *and* $Q_2(y, g(\mathbf{a})/q(x)) = 0$. *Then*

$$\mathfrak{h}(Q_1), \mathfrak{h}(Q_2) \leq 1600(\rho+\delta_D)^6(\rho+1)^{5(\rho+\delta_D)^3-9}h(D).$$

Proof. Suppose that $\mathbf{a} = (\dots, a_{l,j}, \dots)$. Set

$$G(X,Y,Z) = q(X)Z - \sum_{j=0}^{n-1} \sum_{l=0}^{4\rho(\rho+\delta_D)-1} a_{l,j}X^lY^j.$$

Then $\deg(G,X) \leq 4\rho(\rho+\delta_D) - 1$ and $g(\mathbf{a})/q(x)$ is a solution of $G(x,y,Z) = 0$. Denote $R = \mathrm{res}_Y(P,G)$. Since $P(X,Y)$ is irreducible and it does not divide $G(X,Y,Z)$, R is nonzero. Furthermore $R(x, g(\mathbf{a})/q(x)) = 0$ and then Q_1 divides R. Now let us estimate the height of $q(X)$. Suppose that $b_1, \dots, b_d$ are all roots of $q(X) = 0$ where $d = \deg(q)$. Then b_i is either a zero of $\mathbf{d}(X)$ or it is equal to $\pi_{\mathfrak{p}}(x)$ for some $\mathfrak{p} \in \mathrm{supp}(D)$. In the first case, $\mathfrak{h}(b_i) \leq (2\rho-1)\mathfrak{h}(P)$ by Corollary 2.12 and in the second case $\mathfrak{h}(b_i) \leq h(D)$. Therefore $\mathfrak{h}(b_i) \leq (2\rho-1)h(D)$. Each coefficient of $q(X)$ is a homogeneous polynomial in $1, b_1, \ldots, b_d$ of degree d. By Proposition 2.9, $\mathfrak{h}(q(X)) \le d\mathfrak{h}((1 : b_1 : \cdots : b_d))$. Due to Corollary 2.11, $\mathfrak{h}((1 : b_1 : \cdots : b_d)) \le \sum_{i=1}^{d} \mathfrak{h}(b_i)$. Since $d = \deg(q) \le \rho(\rho + \delta_D)$,

$$\mathfrak{h}(q(X)) \le d \sum_{i=1}^{d} \mathfrak{h}(b_i) \le d^2(2\rho - 1)h(D) \le (\rho + \delta_D)^2(2\rho^3 - \rho^2)h(D).$$

Let $\mathbf{c}$ be a point in some projective space with the coefficients of $q(X)$ and all $a_{l,j}$ as coordinates. By Corollary 2.11,

$$\begin{aligned} \mathfrak{h}(\mathbf{c}) &\le \mathfrak{h}(q) + \mathfrak{h}(\mathbf{a}) \\ &\le (\rho + \delta_D)^2(2\rho^3 - \rho^2)h(D) + 1600(\rho + \delta_D)^6(\rho + 1)^{5(\rho+\delta_D)^3-11}h(D) \\ &\le 1600(\rho + \delta_D)^6(\rho + 1)^{5(\rho+\delta_D)^3-10}h(D). \end{aligned}$$

Equivalently, $\mathfrak{h}(G) \le 1600(\rho+\delta_D)^6(\rho+1)^{5(\rho+\delta_D)^3-10}h(D)$. Due to Proposition 2.15 and Corollary 2.12, one has that

$$\begin{aligned} \mathfrak{h}(Q_1) \le \mathfrak{h}(R) &\le \deg(G, Y)\mathfrak{h}(P) + \deg(P, Y)\mathfrak{h}(G) \\ &\le (\rho - 1)h(D) + 1600\rho(\rho + \delta_D)^6(\rho + 1)^{5(\rho+\delta_D)^3-10}h(D) \\ &\le 1600(\rho + \delta_D)^6(\rho + 1)^{5(\rho+\delta_D)^3-9}h(D). \end{aligned}$$

Using a similar argument, one has that

$$\mathfrak{h}(Q_2) \le \deg(G, X)\mathfrak{h}(P) + \deg(P, X)\mathfrak{h}(G)$$

which is also less than $1600(\rho + \delta_D)^6(\rho + 1)^{5(\rho+\delta_D)^3-9}h(D)$. □

4. Main result

Throughout this section, let $K, \mathfrak{h}$ be as in Section 3 and L stands for an algebraic function field of one variable over K, i.e. $L = K(x, y)$ where x is transcendental over K and $[L : K(x)] < \infty$. Suppose that $\mathfrak{p}$ is a place of L over K. Let $\pi_{\mathfrak{p}}$ be defined as in Section 2.1. We start with a height inequality for points on an algebraic curve of special type. This inequality is an easy corollary of Proposition on page 14 of [18]. For completeness, we present a detailed proof and estimate the constant term.

Proposition 4.1. *Suppose that Q is a nonzero polynomial in $K[X, Y]$ satisfying* $\deg(Q, Y) = \mathrm{tdeg}(Q)$. *Then for each $(\alpha, \beta) \in K^2$ with $Q(\alpha, \beta) = 0$,*

$$\mathfrak{h}(\beta) \le \mathfrak{h}(\alpha) + \mathfrak{h}(Q).$$

Proof. Suppose that $n = \mathrm{tdeg}(Q)$. Let $\mathbf{m}_0, \ldots, \mathbf{m}_\ell$ be all monomials in X, Y of total degrees not greater than n. Without loss of generality, we assume that $\mathbf{m}_0 = 1$ and $\mathbf{m}_\ell = Y^n$. Write $Q = cY^n + \sum_{i=0}^{\ell-1} b_i \mathbf{m}_i$ where $c, b_i \in K$ and $c \neq 0$. In Proposition 2.9, we take $r = 1, s_1 = \ell - 1, \Phi = (X_{1,0} : \cdots : X_{1,\ell-1} : -\frac{1}{c}\sum_{i=0}^{\ell-1} b_i X_{1,i})$ and $\mathbf{a} = (1 : \mathbf{m}_1(\alpha, \beta) : \cdots : \mathbf{m}_{\ell-1}(\alpha, \beta))$. Then $\Phi(\mathbf{a}) = (\mathbf{m}_0(\alpha, \beta) : \cdots : \mathbf{m}_\ell(\alpha, \beta))$ and by Propositions 2.16 and 2.9, one has that

$$n\mathfrak{h}((1 : \alpha : \beta)) = \mathfrak{h}(\Phi(\mathbf{a})) \leq \mathfrak{h}(\mathbf{a}) + \mathfrak{h}(\Phi) = \mathfrak{h}(\mathbf{a}) + \mathfrak{h}(Q). \tag{7}$$

Let $\mathbf{n}_0, \ldots, \mathbf{n}_m$ be all monomials in X, Y of total degrees not greater than $n - 1$. One can check that there are $\mathbf{n}_{d_1}, \ldots, \mathbf{n}_{d_n}$ such that $X\mathbf{n}_{d_i} \neq \mathbf{n}_j$ for any i, j and

$$\{\mathbf{m}_i \mid i = 0, \ldots, \ell - 1\} = \{\mathbf{n}_i \mid i = 0, \ldots, m\} \cup \{X\mathbf{n}_{d_i} \mid i = 1, \ldots, n\},$$

where $\ell = m + n + 1$. In Proposition 2.9, we take $r = 2, s_1 = 1, s_2 = m$, $\Phi = (\phi_0 : \cdots : \phi_{\ell-1})$ with $\phi_i = X_{1,0}X_{2,i}$ for $i = 0, \ldots, m$ and $\phi_i = X_{1,1}X_{2,d_{i-m}}$ for $i = m+1, \ldots, \ell-1$, $\mathbf{a}_1 = (1 : \alpha)$ and $\mathbf{a}_2 = (\mathbf{n}_0(\alpha, \beta) : \cdots : \mathbf{n}_m(\alpha, \beta))$. Reordering the subscripts if necessary, we may assume that

$$(\mathbf{m}_0, \ldots, \mathbf{m}_{m+n}) = (\mathbf{n}_0, \ldots, \mathbf{n}_m, X\mathbf{n}_{d_1}, \ldots, X\mathbf{n}_{d_n}).$$

We then have that $\Phi(\mathbf{a}_1, \mathbf{a}_2) = \mathbf{a}$ and

$$\mathfrak{h}(\mathbf{a}) = \mathfrak{h}(\Phi(\mathbf{a}_1, \mathbf{a}_2)) \leq \mathfrak{h}(\mathbf{a}_1) + \mathfrak{h}(\mathbf{a}_2) = \mathfrak{h}(\alpha) + \mathfrak{h}(\mathbf{a}_2). \tag{8}$$

By Proposition 2.16 again, $\mathfrak{h}(\mathbf{a}_2) = (n - 1)\mathfrak{h}((1 : \alpha : \beta))$. This together with (7) and (8) yields that

$$\mathfrak{h}((1 : \alpha : \beta)) \leq \mathfrak{h}(\alpha) + \mathfrak{h}(Q).$$

The proposition then follows from the fact that $\mathfrak{h}(\beta) \leq \mathfrak{h}((1 : \alpha : \beta))$. □

As a corollary, we have the following quasi-equivalence of heights for points on an algebraic curve of special type.

Corollary 4.2. *Suppose that $Q = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{i,j}X^iY^j$ with $a_{i,j} \in K$, $m = \deg(Q, X)$ and $n = \deg(Q, Y)$. Assume that for all $0 \leq i \leq m$ and $0 \leq j \leq n$ if $a_{i,j} \neq 0$ then $mj + ni \leq mn$. Then for each $(\alpha, \beta) \in K^2$ with $Q(\alpha, \beta) = 0$,*

$$n\mathfrak{h}(\beta) - mn\mathfrak{h}(Q) \leq m\mathfrak{h}(\alpha) \leq n\mathfrak{h}(\beta) + mn\mathfrak{h}(Q).$$

Proof. Set $\tilde{Q} = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{i,j}X^{ni}Y^{mj}$. Then $\deg(\tilde{Q}, Y) = \deg(\tilde{Q}, X) = \mathrm{tdeg}(\tilde{Q})$ and $\mathfrak{h}(\tilde{Q}) = \mathfrak{h}(Q)$. Suppose that $(\alpha, \beta) \in K^2$ satisfies $Q(\alpha, \beta) = 0$. Then $\tilde{Q}(\alpha^{1/n}, \beta^{1/m}) = 0$. By Proposition 4.1, $\mathfrak{h}(\beta^{1/m}) \leq \mathfrak{h}(\alpha^{1/n}) + \mathfrak{h}(\tilde{Q})$. By Proposition 2.7, one has that $n\mathfrak{h}(\beta) \leq m\mathfrak{h}(\alpha) + mn\mathfrak{h}(Q)$. Similarly, one has that $m\mathfrak{h}(\alpha) \leq n\mathfrak{h}(\beta) + mn\mathfrak{h}(Q)$. □

The polynomial Q usually does not satisfy the assumption of Proposition 4.1, i.e. $\deg(Q,Y) = \mathrm{tdeg}(Q)$. In order to apply Proposition 4.1, Eremenko proved in [9] that if $\mathrm{div}(y)^- \leq \mathrm{div}(x)^-$ then the irreducible polynomial Q with $Q(x,y) = 0$ satisfies $\deg(Q,Y) = \mathrm{tdeg}(Q)$. The following lemma is a generalization of Lemma 1 in [9].

Lemma 4.3. *Assume that $Q \in K[X,Y]$ is a nonzero polynomial irreducible over K and $f,g \in L \setminus K$ satisfying $Q(f,g) = 0$. Suppose that*

$$m_1\mathrm{div}\left(\tau_1(f)\right)^- \leq m_2\mathrm{div}\left(\tau_2(g)\right)^-$$

where m_1, m_2 are positive integers and τ_1, τ_2 are two linear fractional transformations with coefficients in $\mathbb{Q}$. Then for every place $\mathfrak{p}$ of L over K,

$$m_1\mathfrak{h}(\pi_\mathfrak{p}(f)) \leq m_2\mathfrak{h}(\pi_\mathfrak{p}(g)) + m_1m_2\mathfrak{h}(Q).$$

Proof. Write $\tau_1(X) = \frac{a_1X+b_1}{c_1X+d_1}$, $\tau_2(Y) = \frac{a_2Y+b_2}{c_2Y+d_2}$ with $a_i, b_i, c_i, d_i \in \mathbb{Q}$ and $a_id_i - b_ic_i \neq 0$. Denote $\bar{f} = \tau_1(f)^{m_1}$ and $\bar{g} = \tau_2(g)^{m_2}$. Let $\bar{Q} \in K[Z_1, Z_2]$ be a nonzero irreducible polynomial such that $\bar{Q}(\bar{f}, \bar{g}) = 0$. Set

$$\begin{aligned}
&H_1 = (c_1X + d_1)^{m_1}Z_1 - (a_1X + b_1)^{m_1},\\
&H_2 = (c_2Y + d_2)^{m_2}Z_2 - (a_2Y + b_2)^{m_2},\\
&R_1(Z_1, Y) = \mathrm{res}_X(H_1, Q(X,Y)), R_2(Z_1, Z_2) = \mathrm{res}_Y(H_2, R_1(Z_1, Y)).
\end{aligned}$$

As Q does not divide H_1, $R_1 \neq 0$. Similarly, $R_2 \neq 0$. Moreover, one can easily check that $R_2(\bar{f}, \bar{g}) = 0$. Hence $\bar{Q}$ divides R_2. By Proposition 2.15 and Corollary 2.12, one has that

$$\begin{aligned}
\mathfrak{h}(\bar{Q}) \leq \mathfrak{h}(R_2) &\leq \deg(R_1, Y)\mathfrak{h}(H_2) + \deg(H_2, Y)\mathfrak{h}(R_1)\\
&\leq \deg(H_2, Y)(\deg(H_1, X)\mathfrak{h}(Q) + \deg(Q, X)\mathfrak{h}(H_1))\\
&= \deg(H_2, Y)\deg(H_1, X)\mathfrak{h}(Q) = m_1m_2\mathfrak{h}(Q).
\end{aligned}$$

Since $\mathrm{div}(\bar{f})^- \leq \mathrm{div}(\bar{g})^-$, $\deg(\bar{Q}, X) = \mathrm{tdeg}(\bar{Q})$ by the Proposition 2 in [9]. If a place $\mathfrak{p}$ is not a pole of $\bar{g}$ then it is not a pole of $\bar{f}$ too. For such places, the lemma follows from Propositions 4.1, 2.7 and Corollary 2.10. We are left to consider the case in which $\mathfrak{p}$ is a pole of $\bar{g}$. Suppose that $\mathfrak{p}$ is a pole of $\bar{g}$. If $\mathfrak{p}$ is also a pole of f then $\mathfrak{h}(\pi_\mathfrak{p}(f)) = 0$ and there is nothing to prove. Assume that $\mathfrak{p}$ is not a pole of f. If $\mathfrak{p}$ is a pole of g then $\pi_\mathfrak{p}(f)$ is a zero of $\tilde{Q}(X, 0)$, where $\tilde{Q} = Y^rQ(X, 1/Y)$ and r is the smallest integer such that $\tilde{Q} \in K[X,Y]$, and thus $\mathfrak{h}(\pi_\mathfrak{p}(f)) \leq \mathfrak{h}(\tilde{Q}) = \mathfrak{h}(Q)$ and we are done. Now suppose that $\mathfrak{p}$ is not a pole of g. Since $\mathfrak{p}$ is a pole of $\bar{g}$, $c_2\pi_\mathfrak{p}(g) + d_2 = 0$, i.e. $\pi_\mathfrak{p}(g) = -d_2/c_2$. Applying $\pi_\mathfrak{p}$ to $Q(f,g) = 0$ yields that $\pi_\mathfrak{p}(f)$ is a solution of $Q(X, -d_2/c_2) = 0$. Write $Q = \sum_{i=0}^{\ell} A_i(Y)X^i$ where $A_i(Y) = \sum_{j=0}^{s} a_{i,j}Y^j \in K[Y]$. Note that $A_i(-d_2/c_2)$ viewed as a polynomial in the $a_{i,j}$ is either 0 or homogeneous in the $a_{i,j}$ of degree 1 with coefficients in $\mathbb{Q}$. By Proposition 2.9,

$$\mathfrak{h}(Q(X,-d_2/c_2)) = \mathfrak{h}((A_0(-d_2/c_2) : \cdots : A_\ell(-d_2/c_2)))$$
$$\leq \mathfrak{h}((\cdots : a_{i,j} : \cdots)) = \mathfrak{h}(Q).$$

Hence $\mathfrak{h}(\pi_\mathfrak{p}(f)) \leq \mathfrak{h}(Q(X,-d_2/c_2)) \leq \mathfrak{h}(Q)$ and the lemma holds. □

Lemma 4.4. *Assume S is a finite set of places of L over K and $f \in L$. Then there are $c_1, c_2 \in \mathbb{Q}$ with $c_2 \neq 0$ such that*

$$\operatorname{supp}\left(\operatorname{div}\left(\frac{f}{c_1 f + c_2}\right)^-\right) \cap S = \emptyset.$$

Proof. Set

$$M = \{\pi_\mathfrak{p}(f) \mid \forall\, \mathfrak{p} \in S \text{ with } \operatorname{ord}_\mathfrak{p}(f) \geq 0\}.$$

Then M is a finite subset of K. Let $c_1, c_2 \in \mathbb{Q}$ satisfy that $c_2 \neq 0$ and $c_1 a + c_2 \neq 0$ for all $a \in M$. For $\mathfrak{p} \in S$ with $\operatorname{ord}_\mathfrak{p}(f) \geq 0$, one has that

$$\pi_\mathfrak{p}(c_1 f + c_2) = c_1 \pi_\mathfrak{p}(f) + c_2 \neq 0, \text{ i.}e\text{. } \operatorname{ord}_\mathfrak{p}(c_1 f + c_2) = 0.$$

This implies that $\operatorname{ord}_\mathfrak{p}(f/(c_1 f + c_2)) = \operatorname{ord}_\mathfrak{p}(f) \geq 0$ for all $\mathfrak{p} \in S$ with $\operatorname{ord}_\mathfrak{p}(f) \geq 0$. On the other hand, for $\mathfrak{p} \in S$ with $\operatorname{ord}_\mathfrak{p}(f) < 0$, one has that

$$\operatorname{ord}_\mathfrak{p}(f/(c_1 f + c_2)) = \operatorname{ord}_\mathfrak{p}(f) - \operatorname{ord}_\mathfrak{p}(c_1 f + c_2) = \operatorname{ord}_\mathfrak{p}(f) - \operatorname{ord}_\mathfrak{p}(f) = 0.$$

In either case, $\mathfrak{p}$ is not a pole of $f/(c_1 f + c_2)$. Thus c_1, c_2 have the desired property. □

Lemma 4.5. *Suppose that P is a nonzero irreducible polynomial of total degree ρ in $K[X,Y]$ satisfying $P(x,y) = 0$ and $\mathbf{d}(X)$ is the discriminant of P with respect to Y. Let $\bar{x} = x/(c_1 x + c_2)$ where $c_1, c_2 \in \mathbb{Q}$ with $c_2 \neq 0$. Then*

$$\mathfrak{h}(\pi_\mathfrak{p}(x)) \leq 2\rho\mathfrak{h}(P)$$

for each $\mathfrak{p} \in \operatorname{supp}(\operatorname{div}(x)^-) \cup \operatorname{supp}(\operatorname{div}(\bar{x})^-) \cup \operatorname{supp}(\operatorname{div}(y)^-) \cup \operatorname{supp}(\operatorname{div}(\mathbf{d}(x)))$.

Proof. Suppose that $\pi_\mathfrak{p}(x) = \infty$. Then $\mathfrak{h}(\pi_\mathfrak{p}(x)) = 0$ and the lemma is clear. In the following suppose that $\pi_\mathfrak{p}(x) \neq \infty$. Suppose that $\mathfrak{p} \in \operatorname{supp}(\operatorname{div}(\bar{x})^-)$. Then $\pi_\mathfrak{p}(c_1 x + c_2) = c_1 \pi_\mathfrak{p}(x) + c_2 = 0$ and so $\pi_\mathfrak{p}(x) = -c_2/c_1$. Hence $\mathfrak{h}(\pi_\mathfrak{p}(x)) = 0$ and thus the lemma holds. Suppose that $\mathfrak{p} \in \operatorname{supp}(\operatorname{div}(y)^-)$. Then $\pi_\mathfrak{p}(y) = \infty$ and $(\pi_\mathfrak{p}(x), 0)$ is a zero of $\bar{P} = Y^r P(X, 1/Y)$ where r is the smallest integer such that $Y^r P(X,1/Y) \in K[X,Y]$. In other word, $\pi_\mathfrak{p}(x)$ is a zero of $\bar{P}(X,0)$. Note that $\mathfrak{h}(\bar{P}) = \mathfrak{h}(P)$. Hence $\mathfrak{h}(\pi_\mathfrak{p}(x)) \leq \mathfrak{h}(\bar{P}(X,0)) \leq \mathfrak{h}(P) \leq 2\rho\mathfrak{h}(P)$. Finally, suppose that $\operatorname{supp}(\operatorname{div}(\mathbf{d}(x))) \neq \emptyset$, i.e. $\mathbf{d}(x) \notin K$ and $\mathfrak{p} \in \operatorname{supp}(\operatorname{div}(\mathbf{d}(x)))$. If $\mathfrak{p}$ is a pole of $\mathbf{d}(x)$ then it is a pole of x and we are already done. Suppose that $\mathfrak{p}$ is a zero of $\mathbf{d}(x)$. Then $\pi_\mathfrak{p}(\mathbf{d}(x)) = 0$ which implies that $\mathbf{d}(\pi_\mathfrak{p}(x)) = 0$. Hence $\mathfrak{h}(\pi_\mathfrak{p}(x)) \leq \mathfrak{h}(\mathbf{d}(X)) \leq 2\rho\mathfrak{h}(P)$. □

Now we are ready to prove the main result of this paper.

Theorem 4.6. *Let P be an irreducible polynomial in $K[X,Y]$ of degree m with respect to X and of degree n with respect to Y. Suppose that $\rho = \mathrm{tdeg}(P)$ and $0 < \epsilon < 1$. Then for every $a, b \in K$ with $P(a,b) = 0$, one has that*

$$(1-\epsilon)n\mathfrak{h}(b) - C \leq m\mathfrak{h}(a) \leq (1+\epsilon)n\mathfrak{h}(b) + C$$

where

$$C = 75 \cdot 2^{13} \cdot (1/\epsilon)^6 (\rho+1)^{\frac{40(\rho+1)^9}{\epsilon^3}} \mathfrak{h}(P).$$

Proof. Let L be the field of fractions of $K[X,Y]/(P)$. Then L is an algebraic function field of one variable over K. Set $x = X + (P)$ and $y = Y + (P)$. Then $P(x,y) = 0$. Choose $c_1, c_2 \in \mathbb{Q}$ with $c_2 \neq 0$ such that

$$\mathrm{supp}(\mathrm{div}(x/(c_1x + c_2))^-) \cap \mathrm{supp}(\mathrm{div}(y)^-) = \emptyset.$$

Such c_1, c_2 exist because of Lemma 4.4. Let λ_2 be the smallest integer not less than $\frac{\rho}{2\epsilon}$ and let λ_1 be the largest integer not greater than $\lambda_2 + \rho/2$. Then $\frac{\rho}{2\epsilon} \leq \lambda_2 < \frac{\rho}{2\epsilon} + 1$ and $\lambda_2 + \rho/2 - 1 < \lambda_1 \leq \lambda_2 + \rho/2$. These imply that

$$\lambda_1 - \lambda_2 \geq \frac{\rho}{2} - 1, \ \frac{\lambda_1}{\lambda_2} \leq 1 + \epsilon, \ \lambda_1 + \lambda_2 \leq \frac{2(\rho+1)}{\epsilon}. \tag{9}$$

Set $\bar{x} = x/(c_1x + c_2)$ and

$$D = \lambda_1 n\mathrm{div}(y)^- - \lambda_2 m\mathrm{div}(\bar{x})^-.$$

Note that $\deg(\mathrm{div}(y)^-) = [L : K(y)] = m$ and $\deg(\mathrm{div}(\bar{x})^-) = [L : K(\bar{x})] = n$. One sees that

$$\deg(D) = \lambda_1 nm - \lambda_2 nm = (\lambda_1 - \lambda_2)nm.$$

As $nm \geq n + m - 1 \geq \rho - 1$ and $\lambda_1 - \lambda_2 \geq \rho/2 - 1$, $\deg(D) \geq (\rho-1)(\rho-2)/2$ is not less than the genus of $P(X,Y) = 0$. Consequently, $\mathcal{L}_K(D) \neq \{0\}$. Let $\delta_D, h(D)$ be as in Notation 3.4. Then $h(D) \leq 2\rho\mathfrak{h}(P)$ by Lemma 4.5 and by (9), $\delta_D = (\lambda_1 + \lambda_2)nm < 2(1+\rho)\rho^2/\epsilon$.

Due to Proposition 3.11, $\mathcal{L}_K(D)$ contains a nonzero element $z = g(\mathbf{a})/q(x)$, i.e. $\mathrm{div}(z) + D \geq 0$, where $g(\mathbf{a})$ is of the form (5). As $\mathrm{supp}(\mathrm{div}(\bar{x})^-) \cap \mathrm{supp}(\mathrm{div}(y)^-) = \emptyset$, one sees that

$$\mathrm{div}(z)^- \leq \lambda_1 n \mathrm{div}(y)^-, \ \lambda_2 m \mathrm{div}(\bar{x})^- \leq \mathrm{div}(z)^+ = \mathrm{div}(1/z)^-.$$

Suppose that $Q_1 \in K[X,Z], Q_2 \in K[Y,Z]$ are nonzero irreducible polynomials such that $Q_1(x,z) = 0$ and $Q_2(y,z) = 0$. By Proposition 3.12, one has that

$$\mathfrak{h}(Q_1), \mathfrak{h}(Q_2) \leq 1600(\rho + \delta_D)^6 (\rho+1)^{5(\rho+\delta_D)^3 - 9} h(D) \triangleq T.$$

Let $\mathfrak{p}$ be a place of L over K such that $\pi_{\mathfrak{p}}(x) = a$ and $\pi_{\mathfrak{p}}(y) = b$. By Lemma 4.3,

$$\begin{aligned} \mathfrak{h}(\pi_{\mathfrak{p}}(z)) &\leq \lambda_1 n \mathfrak{h}(b) + \lambda_1 n \mathfrak{h}(Q_2), \\ \lambda_2 m \mathfrak{h}(a) &\leq \mathfrak{h}(\pi_{\mathfrak{p}}(z)) + \lambda_2 m \mathfrak{h}(Q_1). \end{aligned}$$

The above two inequalities imply that

$$\lambda_2 m \mathfrak{h}(a) \leq \lambda_1 n \mathfrak{h}(b) + \lambda_1 n \mathfrak{h}(Q_2) + \lambda_2 m \mathfrak{h}(Q_1).$$

In other words, $m\mathfrak{h}(a) \leq (\lambda_1/\lambda_2) n \mathfrak{h}(b) + (m + n\lambda_1/\lambda_2)T$. Note that by (9) $\lambda_1/\lambda_2 \leq 1 + \epsilon \leq 2$. One has that

$$m\mathfrak{h}(a) \leq (1+\epsilon) n \mathfrak{h}(b) + 3\rho T. \tag{10}$$

Note that $\rho + \delta_D \leq \rho + 2(\rho+1)\rho^2/\epsilon < 2(\rho+1)^3/\epsilon$. One sees that

$$\begin{aligned} 3\rho T &= 3\rho \times 1600(\rho+\delta_D)^6 (\rho+1)^{5(\rho+\delta_D)^3 - 9} h(D) \\ &\leq 9600\rho^2 (2(\rho+1)^3/\epsilon)^6 (\rho+1)^{5(\rho + 2(\rho+1)\rho^2/\epsilon)^3 - 9} \mathfrak{h}(P) \\ &\leq 9600 \cdot 2^6 \cdot (1/\epsilon)^6 (\rho+1)^{5(\rho+2(\rho+1)\rho^2/\epsilon)^3 + 11} \mathfrak{h}(P) \\ &\leq 75 \cdot 2^{13} \cdot (1/\epsilon)^6 (\rho+1)^{\frac{40(\rho+1)^9}{\epsilon^3}} \mathfrak{h}(P) \triangleq C. \end{aligned}$$

The last inequality holds because

$$5(\rho + 2(\rho+1)\rho^2/\epsilon)^3 + 11 < 5(\rho + 2(\rho+1)\rho^2/\epsilon + 2)^3 < 40(\rho+1)^9 (1/\epsilon)^3.$$

Set $\tilde{D} = \lambda_2 m \mathrm{div}(\bar{x})^- - (2\lambda_2 - \lambda_1) n \mathrm{div}(y)^-$. Then

$$\begin{aligned} \lambda_2 - (2\lambda_2 - \lambda_1) &= \lambda_1 - \lambda_2 \geq \frac{\rho}{2} - 1, \\ \frac{2\lambda_2 - \lambda_1}{\lambda_2} &= 2 - \frac{\lambda_1}{\lambda_2} \geq 1 - \epsilon, \\ \lambda_2 + 2\lambda_2 - \lambda_1 &= 3\lambda_2 - \lambda_1 \leq 2\lambda_2 - \frac{\rho}{2} + 1 < \frac{2(\rho+1)}{\epsilon}. \end{aligned}$$

Using a similar argument, one has that

$$(1-\epsilon)n\mathfrak{h}(b) \leq m\mathfrak{h}(a) + C. \tag{11}$$

Combining (11) with (10) yields the conclusion. □

References

[1] Simon Abelard, Alain Couvreur, Grégoire Lecerf, Sub-quadratic time for Riemann-Roch spaces: case of smooth divisors over nodal plane projective curves, in: ISSAC'20—Proceedings of the 45th International Symposium on Symbolic and Algebraic Computation, ACM, New York, 2020, pp. 14–21.

[2] Mourad Abouzaid, Heights and logarithmic gcd on algebraic curves, Int. J. Number Theory 4 (2) (2008) 177–197.

[3] Boris Bartolome, The Skolem-Abouzaïd theorem in the singular case, Atti Accad. Naz. Lincei, Rend. Lincei, Mat. Appl. 26 (3) (2015) 263–289.

[4] Enrico Bombieri, On Weil's "théorème de décomposition", Am. J. Math. 105 (2) (1983) 295–308.

[5] Claude Chevalley, Introduction to the Theory of Algebraic Functions of One Variable, Mathematical Surveys, vol. VI, American Mathematical Society, Providence, R.I., 1963.

[6] J. Coates, Construction of rational functions on a curve, Proc. Camb. Philos. Soc. 68 (1970) 105–123.

[7] David Cox, John Little, Donal O'Shea, Ideals, Varieties, and Algorithms: an Introduction to Computational Algebraic Geometry and Commutative Algebra, Springer Science & Business Media, 2013.

[8] James Harold Davenport, On the Integration of Algebraic Functions, Lecture Notes in Computer Science, vol. 102, Springer-Verlag, Berlin-New York, 1981.

[9] A. Eremenko, Rational solutions of first-order differential equations, Ann. Acad. Sci. Fenn., Math. 23 (1) (1998) 181–190.

[10] Philipp Habegger, Heights and multiplicative relations on algebraic varieties, Thesis (Ph.D.), University of Basel, 2007.

[11] Philipp Habegger, Quasi-equivalence of heights and Runge's theorem, in: Number Theory—Diophantine Problems, Uniform Distribution and Applications, Springer, Cham, 2017, pp. 257–280.

[12] F. Hess, Computing Riemann-Roch spaces in algebraic function fields and related topics, J. Symb. Comput. 33 (4) (2002) 425–445.

[13] Ming-Deh Huang, Doug Ierardi, Efficient algorithms for the Riemann-Roch problem and for addition in the Jacobian of a curve, J. Symb. Comput. 18 (6) (1994) 519–539.

[14] Serge Lang, Fundamentals of Diophantine Geometry, Springer-Verlag, New York, 1983.

[15] Aude Le Gluher, Pierre-Jean Spaenlehauer, A fast randomized geometric algorithm for computing Riemann-Roch spaces, Math. Comput. 89 (325) (2020) 2399–2433.

[16] A. Néron, Quasi-fonctions et hauteurs sur les variétés abéliennes, Ann. Math. 82 (2) (1965) 249–331.

[17] Wolfgang M. Schmidt, Construction and estimation of bases in function fields, J. Number Theory 39 (2) (1991) 181–224.

[18] Jean Pierre Serre, Lectures on the Mordell-Weil Theorem, Springer Fachmedien Wiesbaden, 1997.

[19] Emil J. Volcheck, Computing in the Jacobian of a plane algebraic curve, in: Algorithmic Number Theory, Ithaca, NY, 1994, in: Lecture Notes in Comput. Sci., vol. 877, Springer, Berlin, 1994, pp. 221–233.

[20] Robert J. Walker, Algebraic Curves, Princeton University Press, 1950.