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# Descent of Ordinary Differential Equations with Rational General Solutions*

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**Abstract** Let $F$ be an irreducible differential polynomial over $k(t)$ with $k$ being an algebraically closed field of characteristic zero. The authors prove that $F = 0$ has rational general solutions if and only if the differential algebraic function field over $k(t)$ associated to $F$ is generated over $k(t)$ by constants, i.e., the variety defined by $F$ descends to a variety over $k$. As a consequence, the authors prove that if $F$ is of first order and has movable singularities then $F$ has only finitely many rational solutions.

**Keywords** Algebraic ordinary differential equation, differential descent, rational general solution.

## 1 Introduction

Let $K$ be a differential field of characteristic zero. Differential descent theory asks whether a differential algebraic variety over $K$ can descend to an algebraic variety over $C_K$, the field of constants of $K$, and it is viewed as a differential analogue of Shimura-Matsusaka theory of fields of moduli. This theory is initiated by Matsuda in [1, 2] for first order algebraic differential equations and further developed by Buium for higher order and partial differential equations. Let $\mathcal{F}$ be the differential algebraic function field over $K$ associated to some irreducible differential algebraic variety. Then the descent problem is equivalent to the one asks whether $\mathcal{F}$ is generated over $K$ by constants. In the case when $K$ is an ordinary differential field and $\text{tr.deg}(\mathcal{F}/K) = 1$, Matsuda in [1, 2] and Nishioka in [3] proved that $\mathcal{F}$ has no movable singularity if and only if there is an algebraic extension $L$ of $K$ such that $\mathcal{F}(L)$ is generated over $L$ by constants. In [4], Buium proved the higher dimension and partial differential version of the results of Matsuda and Nishioka.

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This paper is mainly concerned about algebraic ordinary differential equations with rational general solutions. Rational solutions are of special interest in the community of symbolic computation. Algorithms have already been well-developed for linear differential equations (e.g., see [5–8] ). However, the situation is quite different in the case of nonlinear differential equations. Although a few algorithms have been developed to deal with the equations of special types, there is no complete algorithm to find all rational solutions so far even if we restrict ourselves to the first order equations. In [9], the authors succeeded in computing all rational (algebraic) solutions of first order autonomous differential equations by introducing an algebro-geometric method. Since then, this method has partially been generalized into the general first order differential equations and partial differential equations (e.g., see [10–13]). The readers are referred to [14] for a survey of the recent developments in this direction. Theoretically, in order to find all rational solutions, one only needs to compute a degree bound for all rational solutions and then reduce the problem to solving algebraic equations via the method of undetermined coefficients. Eremenko proved in [15] that such degree bound exists for a first order differential equation. Recently, Freitag and Moosa in [16] showed that all algebraic solutions of first order differential equations are of bounded height.

In this paper, we shall investigate the differential descent problem for ordinary differential equations with rational general solutions. We prove that a given ordinary differential equation $F = 0$ has rational general solutions if and only if the differential algebraic function field associated to $F$ is generated over the base field by constants. We also prove that $F = 0$ has algebraic general solutions if the corresponding function field contains enough arbitrary constants. The paper is organized as follows. In Section 2, we introduce some basic notations and results of differential algebra. In Section 3, we present our main results. In Section 4, we restrict ourselves to the first order case. We prove that if $F = 0$ has movable singularities then $F = 0$ has only finitely many rational solutions.

## 2 Preliminaries

In this section, we introduce some basic notations and results of differential algebra. The readers are referred to [2, 17, 18] for details.

**Definition 2.1** A derivation on a ring $R$ is a map $\delta : R \to R$ satisfying that for all $a, b \in R$,

$$\delta(a + b) = \delta(a) + \delta(b), \quad \delta(ab) = \delta(a)b + a\delta(b).$$

A ring equipped with a derivation is called a differential ring. A differential ring $R$ is called a differential field if $R$ is a field. An ideal $I \subset R$ is called a differential ideal if $\delta(I) \subset I$. Let $R$ be a differential ring (resp. field) with derivation $\delta$. Then the set $\{c \in R \mid \delta(c) = 0\}$ is a subring (resp. subfield) of $R$, which is called the ring (resp. field) of constants of $R$ and denoted by $C_R$.

Throughout this paper, $k$ stands for an algebraically closed field of characteristic zero and $K = k(t)$ denotes the field of rational functions in $t$. Let $\delta$ denote the usual derivation with respect to $t$. Then $K$ becomes a differential field. We shall use $K\{y\}$ to denote the differential polynomial ring over $K$ in the differential indeterminate $y$, i.e., $K\{y\} = K[y_0, y_1, \cdots]$ with $y_0 = y$ and $\delta(y_i) = y_{i+1}$ for any $i \geq 0$. Let $F \in K\{y\} \setminus K$ be a differential polynomial. Then there is a unique nonnegative integer $d$ such that $F \in K[y_0, y_1, \cdots, y_d] \setminus K[y_0, y_1, \cdots, y_{d-1}]$. This unique integer is called the order of $F$ and denoted by $\mathrm{ord}(F)$. When $\mathrm{ord}(F) = 1$, we call $F$ a first order differential polynomial. We shall use $[F]$ to denote the differential ideal in $K\{y\}$ generated by $F$, i.e. the smallest differential ideal in $K\{y\}$ containing $F$. We use $\langle F \rangle$ to stand for the algebraic ideal in $K[y_0, y_1, \cdots, y_d]$ generated by $F$, where $d = \mathrm{ord}(F)$. For any $\eta$ in some differential extension field of $K$, $K\langle \eta \rangle$ stands for the differential field generated over $K$ by $\eta$, i.e. the smallest differential field containing $K$ and $\eta$.

Let $F$ be a differential polynomial of order $d$ and assume that it is irreducible as an algebraic polynomial in $y_0, y_1, \cdots, y_d$. One may expect to construct a function field associated to $F$ by the standard procedure. However, $[F]$ is not always a prime differential ideal. So $K\{y\}/[F]$ may not be a domain and the standard procedure may not work for the differential case. Ritt in [18] introduced the notion of the general component of $F$ by which one can construct a functional field. Denote

$$\Sigma_F = \{A \in K\{y\} \mid \exists\, m > 0 \text{ s.t. } S^m A^m \in [F]\},$$

where $S = \partial F / \partial y_d$ is called the separant of $F$. $\Sigma_F$ is called the general component of $F = 0$. It was proved on page 31 of [18] that $\Sigma_F$ is a prime differential ideal. Then the quotient ring $K\{y\}/\Sigma_F$ is a differential domain and the field of its fractions is a differential field. We shall view this differential field as a function field associated to $F$. On the other hand, since $F$ is irreducible, $K[y_0, y_1, \cdots, y_d]/\langle F \rangle$ is a domain. In general, the quotient rings $K\{y\}/\Sigma_F$ and $K[y_0, y_1, \cdots, y_d]/\langle F \rangle$ are not isomorphic, because $K\{y\}/\Sigma_F$ is not a finitely generated $K$-algebra. However the following lemma shows that the fields of fractions of $K\{y\}/\Sigma_F$ and $K[y_0, y_1, \cdots, y_d]/\langle F \rangle$ are isomorphic.

**Lemma 2.2** *Assume that $F \in K\{y\}$ is an irreducible differential polynomial of order $d$. Then the fields of fractions of $K\{y\}/\Sigma_F$ and $K[y_0, y_1, \cdots, y_d]/\langle F \rangle$ are isomorphic.*

*Proof* Denote by $\mathcal{F}_1$ and $\mathcal{F}_2$ the fields of fractions of $K\{y\}/\Sigma_F$ and $K[y_0, y_1, \cdots, y_d]/\langle F \rangle$ respectively. Consider the natural homomorphism

$$\begin{aligned} \pi : K[y_0, y_1, \cdots, y_d] &\longrightarrow K\{y\}/\Sigma_F \\ f &\longrightarrow f + \Sigma_F. \end{aligned}$$

Due to Rosenfeld's Lemma in [19], $\ker(\pi) = \Sigma_F \cap K[y_0, y_1, \cdots, y_d] = \langle F \rangle$. Hence, the quotient ring $K[y_0, y_1, \cdots, y_d]/\langle F \rangle$ can be embedded into $K\{y\}/\Sigma_F$, and then $\mathcal{F}_2$ can be considered as a subfield of $\mathcal{F}_1$. One sees easily that as the extension fields of $K(y_0, y_1, \cdots, y_{d-1})$,

$$[\mathcal{F}_1 : K(y_0, y_1, \cdots, y_{d-1})] \leq [\mathcal{F}_2 : K(y_0, y_1, \cdots, y_{d-1})] = \deg(F, y_d)$$

where $\deg(F, y_d)$ denotes the degree of $F$ in $y_d$. Therefore, $\mathcal{F}_1 = \mathcal{F}_2$. ∎

**Definition 2.3** Any field that is isomorphic to the field of fractions of $K\{y\}/\Sigma_F$ is called the differential algebraic function field over $K$ associated to $F$. Throughout this paper, we shall denote by $\mathcal{F}$ the differential algebraic function field over $K$ associated to $F$.

Let $\mathcal{U}$ be a universal extension of $K$ (see Theorem 2 on page 134 of [17]). A zero of $\Sigma_F$ is an element in $\mathcal{U}$ which annihilates every differential polynomial in $\Sigma_F$. A zero $\eta \in \mathcal{U}$ of $\Sigma_F$ is called a generic zero of $\Sigma_F$ if

$$\Sigma_F = \{P \in K\{y\} | P(\eta) = 0\}.$$

It is well-known that given a differential ideal distinct from $K\{y\}$, it has a generic zero if and only if it is prime. Now suppose that $\eta$ is a generic zero of $\Sigma_F$ then the field of fractions of $K\{y\}/\Sigma_F$ is isomorphic to $K\langle\eta\rangle$ as differential fields. Hence, $K\langle\eta\rangle$ is the differential function field associated to $F$.

**Definition 2.4** A generic zero $\eta$ of $\Sigma_F$ is called a rational general solution of $F = 0$ if $\eta$ is rational in $t$, i.e.,

$$\eta = \frac{u_0 + u_1 t + \cdots + u_n t^n}{v_0 + v_1 t + \cdots + t^m},$$

where $u_i, v_j \in \mathcal{U}$ are constants. $\eta$ is called an algebraic general solution of $F = 0$ if $\eta$ is algebraic over $K(C_{\mathcal{U}})(= C_{\mathcal{U}}(t))$.

## 3 Main Results

In this section, we shall prove that $F = 0$ has rational general solutions if and only if $\mathcal{F}$ is generated over $K$ by constants. Let us first introduce some notations from [9]. For nonnegative integers $n, m$, set

$$\mathcal{M}_{n,m}(y) = \begin{pmatrix} \binom{n+1}{0} y_{n+1} & \binom{n+1}{1} y_n & \cdots & \binom{n+1}{m} y_{n+1-m} \\ \binom{n+2}{0} y_{n+2} & \binom{n+2}{1} y_{n+1} & \cdots & \binom{n+2}{m} y_{n+2-m} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{n+m+1}{0} y_{n+m+1} & \binom{n+m+1}{1} y_{n+m} & \cdots & \binom{n+m+1}{m} y_{n+1} \end{pmatrix},$$

where $\binom{n+j}{m} = 0$ if $m > n + j$. Let $\mathcal{D}_{n,m} = \det(\mathcal{M}_{n,m})$. Lemma 3 of [9] implies that the solutions $y$ in $\mathcal{U}$ of $\mathcal{D}_{n,m} = 0$ are of the following form

$$y = \frac{a_n t^n + \cdots + a_0}{b_m t^m + \cdots + b_0},$$

where $a_i, b_j \in \mathcal{U}$ are constants, and conversely the proof of that lemma implies that all rational functions of the above form are solutions of $\mathcal{D}_{n,m} = 0$.

**Lemma 3.1** *Assume that $\eta \in K(C_{\mathcal{U}})$. Then $K\langle\eta\rangle$ is generated over $K$ by constants.*

*Proof* If $\eta = 0$, there is nothing to prove. Assume that $\eta \neq 0$. Write $\eta = P/Q$ where

$$P = t^n + u_{n-1} t^{n-1} + \cdots + u_0, \quad Q = v_m t^m + \cdots + v_0, \quad u_i, v_j \in C_{\mathcal{U}}.$$

Without loss of generality, we may assume that $P$ and $Q$ are coprime. We shall prove that $K\langle\eta\rangle = K(u_0, u_1, \cdots, u_{n-1}, v_0, v_1, \cdots, v_m)$. For this, it suffices to show that $u_i, v_j \in K\langle\eta\rangle$.

Differentiating both sides of $Q\eta = P$ successively yields that

$$(Q\eta)^{(j)} = P^{(j)}, \quad j = 0, 1, \cdots, n, \tag{1}$$

$$(Q\eta)^{(n+j)} = 0, \quad j = 1, 2, \cdots, m, \tag{2}$$

where the superscript $*^{(i)}$ denotes the $i$-th derivative of $*$ with respect to $\delta$. Note that $Q^{(l)} = 0$ when $l > m$ and $\binom{n+j}{l} = 0$ when $l > n + j$. We have that

$$(Q\eta)^{(n+j)} = \sum_{l=0}^{n+j} \binom{n+j}{l} Q^{(l)} \eta^{(n+j-l)} = \sum_{l=0}^{m} \binom{n+j}{l} Q^{(l)} \eta^{(n+j-l)}.$$

Hence, the equations in (2) together with $(Q\eta)^{(n)} = n!$ can be rewritten in the following matrix form:

$$\underbrace{\mathcal{M}_{n-1,m}(\eta) \begin{pmatrix} 1 & t & \cdots & t^m \\ 0 & 1 & \cdots & mt^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m! \end{pmatrix}}_{W} \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_m \end{pmatrix} = \begin{pmatrix} n! \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since the degree of the numerator of $\eta$ in $t$ equals $n$, Lemma 3 of [9] implies that $\mathcal{D}_{n-1,m}(\eta) \neq 0$. Therefore $W$ is invertible. This implies that all $v_i$ belong to $K\langle\eta\rangle$. By the equations (1), one immediately sees that all $u_i$ are in $K\langle\eta\rangle$. ∎

**Proposition 3.2** *$F = 0$ has rational general solutions if and only if $\mathcal{F}$ is generated over $K$ by constants.*

*Proof* Let $\eta$ be a generic zero of $\Sigma_F$, then $\mathcal{F} = K\langle\eta\rangle$. As $\eta$ is rational in $t$, i.e., $\eta \in K(C_{\mathcal{U}})$, due to Lemma 3.1, $K\langle\eta\rangle$ is generated over $K$ by constants. The "if part" is obvious. ∎

**Remark 3.3** Assume that $\mathcal{F} = K(c_1, c_2, \cdots, c_m)$ where the $c_i$ are constants, i.e., $c_i \in C_{\mathcal{F}}$. Denote by $J$ the vanishing ideal of $(c_1, c_2, \cdots, c_m)$ in $K[z_1, z_2, \cdots, z_m]$. We claim that $J$ is generated by $J \cap C_K[z_1, z_2, \cdots, z_m]$. To see this, let $\widetilde{J}$ be the ideal in $K[z_1, z_2, \cdots, z_m]$ generated by $J \cap C_K[z_1, z_2, \cdots, z_m]$. Suppose that $J \setminus \widetilde{J} \neq \emptyset$ and $f \in J \setminus \widetilde{J}$. Without loss of generality, we may assume that $f$ has minimal number of terms among all elements in $J \setminus \widetilde{J}$ and one of the coefficients of $f$ is equal to 1. As $f(c_1, c_2, \cdots, c_m) = 0$, one has that

$$0 = \delta(f(c_1, c_2, \cdots, c_m)) = f_\delta(c_1, c_2, \cdots, c_m),$$

where $f_\delta$ denotes the polynomial obtained by applying $\delta$ to the coefficients of $f$. Then $f_\delta \in J$. Note that $f_\delta$ has less number of terms than $f$. Hence $f_\delta \in \widetilde{J}$. Moreover $f_\delta \neq 0$, because $f \notin \widetilde{J}$. As all monomials in $f_\delta$ already appear in $f$, there is $\lambda \in K$ such that $f - \lambda f_\delta$ has less number of terms than $f$. However, this implies that $f - \lambda f_\delta \in \widetilde{J}$ and thus $f \in \widetilde{J}$, a contradiction. This proves our claim. The claim implies that $\mathcal{F}$ is the algebraic function field associated to a variety defined over $C_K$. Remark that by Lemma 2.2 $\mathcal{F}$ is the algebraic function field associated to the variety defined by $F$ viewed as a polynomial in $y, \delta(y), \cdots, \delta^o(y)$ with $o = \mathrm{ord}(F)$. In the geometric point of view, Proposition 3.2 states that $F=0$ has rational general solutions if and only if the variety defined by $F$ over $K$ is birationally equivalent over $K$ to a variety defined over $C_K$.

If we replace "rational general solutions" by "algebraic general solutions" in the above proposition, the assertion does not hold anymore even if $K$ is replaced by its finite extension.

**Example 3.4** Let $K=\mathbb{C}(t)$ and $F=2yy_1-1$. Then $\eta=\sqrt{t+c}$ is an algebraic general solution of $F=0$, where $c$ is a constant transcendental over $K$. Suppose that there is $\alpha\in\overline{\mathbb{C}(t)}$ such that the differential algebraic function field over $K(\alpha)$ associated to $F$, denoted by $\mathcal{F}'$, is generated over $K(\alpha)$ by constants, where $\overline{\mathbb{C}(t)}$ is the algebraic closure of $\mathbb{C}(t)$. Since the transcendental degree of $\mathcal{F}'$ over $K(\alpha)$ equals one, due to the Primitive Element Theorem, we have that

$$\mathcal{F}'=K(\alpha)\langle\eta\rangle=K(\alpha,\eta)=K(\alpha,c_1,c_2),$$

where $c_i$ are constants in $\mathcal{U}$. Note that $\eta^2-t=c$. Hence, $\eta^2-t$ is a constant that is transcendental over $K(\alpha)$. So one of the $c_i$, say $c_1$, can be chosen to be $\eta^2-t$. Let $P$ be an irreducible polynomial in $K(\alpha)[z_1,z_2]$ satisfying that $P(c_1,c_2)=0$. Since both $c_1$ and $c_2$ are constants, $P$ can be chosen to be a polynomial with coefficients in $C_{K(\alpha)}(=\mathbb{C})$. Furthermore as

$$[\mathcal{F}':K(\alpha,c_1)]=[K(\alpha,\eta):K(\alpha,\eta^2-t)]=2,$$

$\deg(P,z_2)=2$. Under a suitable transformation if necessary, we may assume that

$$c_2^2-\frac{r(c_1)}{s(c_1)}=0,$$

where $r,s\in\mathbb{C}[z]$ are coprime and not all $r,s$ are the square of some polynomial in $\mathbb{C}[z]$. On the other hand, $c_2=g(\eta)/h(\eta)$ where $g,h\in K(\alpha)[z]$ are coprime. Hence, we have that

$$\left(\frac{g(\eta)}{h(\eta)}\right)^2=\frac{r(\eta^2-t)}{s(\eta^2-t)}.$$

Because $r,s$ have coefficients in $\mathbb{C}$, one sees that $r(\eta^2-t),s(\eta^2-t)$ are still coprime as polynomials in $\eta$. Without loss of generality, we may assume that $r,g$ are monic. Then we have that

$$r(\eta^2-t)=g(\eta)^2,\quad s(\eta^2-t)=h(\eta)^2.$$

Again because $r,s$ have coefficients in $\mathbb{C}$, the above equalities imply that both $r$ and $s$ are the squares of some polynomials in $\mathbb{C}[z]$, a contradiction.

On the other hand, we have the following result.

**Proposition 3.5** *If* $\operatorname{tr.deg}(C_{\mathcal{F}}/C_K)=\operatorname{ord}(F)$ *then* $F=0$ *has algebraic general solutions.*

*Proof* Assume that $d=\operatorname{ord}(F)$ and $\{a_1,a_2,\cdots,a_d\}$ is a transcendence basis of $C_{\mathcal{F}}$ over $C_K$. We claim that $a_1,a_2,\cdots,a_d$ are algebraically independent over $K$. Otherwise, there is a nonzero polynomial $P\in K[z_1,z_2,\cdots,z_d]$ such that $P(a_1,a_2,\cdots,a_d)=0$. Write

$$P=\sum_{j=1}^{s}\alpha_j z_1^{l_{1,j}}z_2^{l_{1,j}}\cdots z_d^{l_{d,j}},$$

where $\alpha_j \in K$ and $(l_{1,i}, l_{2,i}, \cdots, l_{d,i}) \neq (l_{1,j}, l_{2,j}, \cdots, l_{d,j})$ if $i \neq j$. Then the equality $P(a_1, a_2, \cdots, a_d) = 0$ implies that the set $\{a_1^{l_{1,j}} a_2^{2_{1,j}} \cdots a_d^{l_{d,j}} | j = 1, 2, \cdots, s\}$ is linearly dependent over $K$. Due to Corollary 1 on page 87 of [17], $C_{\mathcal{F}}$ and $K$ are linearly disjoint over $C_K$. Hence the set $\{a_1^{l_{1,j}} a_2^{l_{1,j}} \cdots a_d^{l_{d,j}} | j = 1, 2, \cdots, s\}$ is linearly dependent over $C_K$. This implies that $a_1, a_2, \cdots, a_d$ are algebraically dependent over $C_K$. This contradicts to the assumption. Note that $\text{tr.deg}(\mathcal{F}/K) = \text{ord}(F) = d$. So

$$[\mathcal{F} : K(a_1, a_2, \cdots, a_d)] < \infty.$$

Therefore, $F$ has a generic zero that is algebraic over $K(a_1, a_2, \cdots, a_d)$, i.e., it has an algebraic general solution. ∎

## 4 First Order Ordinary Differential Equations

Let $F$ be an irreducible first order differential polynomial with coefficients in $K$. In this section, we shall prove that if $F = 0$ has infinitely many rational solutions then the differential function field $\mathcal{F}$ is generated over $K$ by constants and thus $F = 0$ has no movable singularity. As a corollary, if $F = 0$ has infinitely many rational solutions then it is of special form (3). The notion of movable singularities plays an important role in the classification of algebraic differential equations and the differential descent theory. It was first introduced by Fuchs from the analytic viewpoint. Roughly speaking, an integral of a differential equation is said to have movable singularities if this integral has a branch point whose location is "movable", i.e., the location depends on the initial condition. In [2], Matsuda presented an algebraic definition of movable singularities and reproduced many classical results of first order algebraic differential equations via an abstract treatment. The Matsuda's treatment has been extended to higher order and partial differential equations by Buium in [4]. As we will focus on first order algebraic differential equations, we shall only introduce the algebraic definition of movable singularities given by Matsuda in [2]. For brevity, we sometimes use $'$ to denote the differentiation.

Lemma 2.2 implies that $\mathcal{F}$ is not only a differential field but also an algebraic function field of one variable over the field $K$. A $V$-ring of $\mathcal{F}$ over $K$ is a subring $\mathfrak{D}$ of $\mathcal{F}$ satisfying that

1) $K \subset \mathfrak{D}$;

2) $\mathfrak{D} \neq \mathcal{F}$;

3) if $a \in \mathcal{F} \setminus \mathfrak{D}$, then $a^{-1} \in \mathfrak{D}$.

One may wonder whether every $V$-ring is a differential subring of $\mathcal{F}$, i.e., $V' \subset V$. The following example shows that this is not always true.

**Example 4.1** Let $K = \mathbb{C}(t)$ and $F = yy' - 1$. Then $\mathcal{F} = K(y)$ with $y' = 1/y$ and the set

$$\mathfrak{D} = \left\{ \frac{f(y)}{g(y)} \middle| f, g \in K[y], \gcd(f, g) = 1 \text{ and } y \text{ does not divide } g(y) \right\}$$

is a $V$-ring which is not closed under the differentiation $'$, because $y \in \mathfrak{D}$ but $y' = 1/y \notin \mathfrak{D}$.

In [2] (see Theorem 11 on page 70), Matsuda proved that there are only finitely many $V$-rings of $\mathcal{F}$ that are not closed under the differentiation and he introduced the following algebraic definition of movable singularities.

**Definition 4.2** ([2], Definition on page 7) $F = 0$ (or $\mathcal{F}$) is said to have no movable singularity if each $V$-ring of $\mathcal{F}$ is closed under differentiation.

The Fuchs' theorem states that if $\mathcal{F}$ (or $F$) has no movable singularity then $F$ must be of special form.

**Theorem 4.3** ([2], Theorem 2 on page 11) *Let $x$ be an element of $\mathcal{F}$ which is transcendental over $K$. Denote by $f(Y)$ the minimal polynomial of $x'$ over $K(x)$:*

$$f(Y) = Y^n + \sum_{i=1}^{n} a_i(x) Y^{n-i}, \quad a_i(x) \in K(x). \tag{3}$$

*If $\mathcal{F}$ has no movable singularity, then $a_i(x) \in K[x]$ and if $a_i(x) \neq 0$ then* $\deg(a_i(x)) \leq 2i$.

It was proved in [2] on page 29 that if $\mathcal{F}$ is generated over $K$ by constants then $F = 0$ has no movable singularity and therefore $F$ is of the form (3). The following proposition states that for a first order differential polynomial, it has rational general solutions if and only if it has infinitely many rational solutions. Hence, by Proposition 3.2 and Theorem 4.3, if a first order differential polynomial has infinitely many rational solutions then it has the form (3).

**Proposition 4.4** *If $F = 0$ has infinitely many rational solutions then all but finitely many solutions of $F = 0$ are rational in $t$. In particular, the generic zeroes of $\Sigma_F$ are rational in $t$.*

*Proof* Due to Theorem 1 of [15], there is an integer $N$ such that all rational solutions of $F = 0$ are of degree not greater than $N$. By reduction process (see page 6 of [18]), there exist nonnegative integers $\mu, \nu$ such that

$$S_F^{\mu} I_F^{\nu} \mathcal{D}_{N,N} = \sum A_i \delta^i(F) + R,$$

where $I_F$ is the leading coefficient of $F$ in $y_1$ and if $R \neq 0$ then $\deg(R, y_1) < \deg(F, y_1)$. As all rational functions of degree not greater than $N$ are solutions of $\mathcal{D}_{N,N} = 0$, all rational solutions of $F = 0$ are solutions of $\mathcal{D}_{N,N} = 0$ and thus they are solutions of $R = 0$. This implies that $F$ and $R$ have infinitely many common zeroes. However, since $F$ is irreducible, if $R \neq 0$ then there are only finitely many common zeroes of $F$ and $R$, a contradiction. Therefore, $R = 0$. So except for the common solutions of $F = 0$ and $S_F^{\mu} I_F^{\nu} = 0$ which are only finitely many, all solutions of $F = 0$ are also solutions of $\mathcal{D}_{N,N} = 0$ and thus they are rational in $t$. Since the generic zeroes of $\Sigma_F$ do not annihilate $S_F^{\mu} I_F^{\nu}$, they are zeroes of $\mathcal{D}_{N,N}$ and thus the second assertion holds. ∎

By Propositions 4.4 and 3.2, one immediately has the following corollaries.

**Corollary 4.5** *If $F = 0$ has infinitely many rational solutions, then $F$ is of the form* (3). *Moreover, if* $\deg(F, y') = 1$*, then*

$$F = y' + a_2 y^2 + a_1 y + a_0, \quad a_i \in K,$$

*i.e., it is a Riccati equation.*

**Corollary 4.6** *If $F = 0$ has movable singularities then $F = 0$ has only finitely many rational solutions.*

If the number of rational solutions of $F = 0$ is finite, one may ask whether one can give a bound for it. Below are some examples.

**Example 4.7** Let $F = y' + \sum_{i=0}^{n} a_i y^i$ with $a_i \in \mathbb{C}(t), a_n \neq 0$. Suppose that $F = 0$ has only finitely many rational solutions.

1) Case $n = 1$: It is easy to see that $F = 0$ has at most one rational solution.

2) Case $n = 2$: $F = 0$ has at most two rational solutions. Actually, if $F = 0$ has three distinct rational solutions, say $f_1, f_2, f_3$, then any solution of $F = 0$ is of the form

$$\frac{c(f_3 - f_1)f_2 - (f_3 - f_2)f_1}{c(f_3 - f_1) - (f_3 - f_2)},$$

where $c$ is an arbitrary constant.

3) Case $n > 2$: Corollary 4.5 implies that $F = 0$ always has only finitely many rational solutions. One may expect that it has at most $n$ rational solutions. Unfortunately, this is not true. Let

$$F = y' - \frac{1}{(t^n - 1)t}(y^n - 1)y.$$

Then $0, \omega, \omega t$ with $\omega^n = 1$ are $2n + 1$ rational solutions of $F = 0$.

One can produce a naive bound for the number of rational solutions in the following way. By Theorem 1 of [15], one computes an integer $N$ for a bound of the degrees of all rational solutions of $F = 0$. For $0 \leq l \leq N$, set

$$f_l(t) = \frac{a_N t^N + \cdots + a_1 t + a_0}{t^l + b_{l-1}t^{l-1} + \cdots + b_0},$$

where $a_i, b_j$ are indeterminates. Substituting $f_l(t)$ to $F = 0$ and clearing the denominators yield a system $S_l$ of polynomial equations in $a_i, b_j$. Let $r_l$ be the resultant of $a_N t^N + \cdots + a_1 t + a_0$ and $t^l + b_{l-1}t^{l-1} + \cdots + b_0$ with respect to $t$. Let $T_l = S_l \cup \{r_l z - 1\}$, and let $(\overline{\boldsymbol{a}}, \overline{\boldsymbol{b}}, \overline{z})$ be a zero of $T_l$ where $\overline{\boldsymbol{a}} = (\overline{a}_N, \overline{a}_{N-1}, \cdots, \overline{a}_0)$ and $\overline{\boldsymbol{b}} = (\overline{b}_{l-1}, \overline{b}_{l-2}, \cdots, \overline{b}_0)$. Since $r_l(\overline{\boldsymbol{a}}, \overline{\boldsymbol{b}}) \neq 0$, the polynomials $\sum_{i=0}^{N} \overline{a}_i t^i$ and $t^l + \sum_{i=0}^{l-1} \overline{b}_i t^i$ are coprime. This implies that each zero of $T_l$ produces a rational solution of $F = 0$ with $l$ as the degree of its denominator. Conversely, each rational solution of $F = 0$ with $l$ as the degree of its denominator will induce a zero of $T_l$. As $F = 0$ has only finitely many rational solutions, $T_l$ is a system of dimension zero. Elimination theory (see for example Corollary 8.28 on page 347 of [20]) allows us to estimate the number of zeroes of $T_l$. Now let $s_l$ be the number of zeroes of $T_l$ and set

$$\mu = \sum_{0 \leq l \leq N} s_l + 1.$$

Then $\mu$ is a bound for the number of rational solutions of $F = 0$. Usually, the bound $\mu$ generated by the above method depends not only on the degree of $F$ but also on the coefficients of $F$ where $F$ is viewed as a polynomial in $t, y, y'$.

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