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Galois groups of linear difference-differential equations ☆

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ABSTRACT

We study the relation between the Galois group G of a linear difference-differential system and two classes $\mathcal{C}_1$ and $\mathcal{C}_2$ of groups that are the Galois groups of the specializations of the linear difference equation and the linear differential equation in this system respectively. We show that almost all groups in $\mathcal{C}_1 \cup \mathcal{C}_2$ are algebraic subgroups of G, and there is a nonempty subset of $\mathcal{C}_1$ and a nonempty subset of $\mathcal{C}_2$ such that G is the product of any pair of groups from these two subsets. These results have potential application to the computation of the Galois group of a linear difference-differential system. We also give a criterion for testing linear dependence of elements in a simple difference-differential ring, which generalizes Kolchin's criterion for partial differential fields.

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1. Introduction

The study of parametrized differential/difference Galois theories was initiated in [7,24]. In particular, in [24], a $\Sigma\Delta\Pi$-Galois theory was developed for $\Sigma\Delta\Pi$-linear systems and was applied to giving a group-theoretic proof of Hölder's Theorem that the Gamma function is hypertrancendental. Here Σ is a set of automorphisms, Δ is a set of derivations and Π is a set of other derivations that endow a differential structure on parameters. Inspired by the work in [7,24], various Galois theories were developed for other kind of systems, for instance, σ-Galois theory for linear difference equations in [30], difference Galois theory for linear differential equations in [12] and so on. These theories provide powerful tools to study differential/difference algebraic properties of solutions defined by the corresponding linear systems. In particular, using these Galois theories, one is able to prove the hypertranscendence of functions arising in combinatorics, number theory etc, see [1,2,6,14,15,23,24,31] and references therein. These applications essentially rely on the fact that Galois groups measure differential/difference algebraic relations among solutions, *i.e.* the group is larger, the differential/difference algebraic relations are fewer. On the other hand, the problem of determining Galois groups is one of fundamental problems in Galois theories, which leads to many questions in other areas of mathematics. So far, except for linear differential/difference equations, there is no complete algorithm for computing the Galois groups of other kinds of equations. The readers are referred to [3,4,10,13,16,18,20,23,25] and references therein for methods to calculate Galois groups.

In this paper, we shall focus on $\sigma\delta$-linear systems, *i.e.* systems of linear difference-differential equations with one single automorphism σ, one single derivative δ and without parameters. Moreover, we assume that σ and δ commute. Many higher transcendental functions such as Hermite polynomials, Bessel polynomials, Tchebychev polynomials etc satisfy $\sigma\delta$-linear systems (see Chapters 7 and 10 of [17]). For simplicity, we take the Tchebychev polynomials as an example to state the main results of this paper. Let C be an algebraically closed field of characteristic zero and let $C(m,t)$ be the $\sigma\delta$-field with $\sigma(m) = m+1$ and $\delta = d/dt$. Consider the Tchebychev polynomials

$$T_m(t) = \frac{m}{2}\sum_{\ell=0}^{[\frac{m}{2}]} \frac{(-1)^\ell (m-\ell-1)!}{\ell!(m-2\ell)!}(2t)^{m-2\ell}.$$

Denote $Y = (T_m(t), T_{m+1}(t))^t$. Then Y satisfies the following $\sigma\delta$-linear system:

$$\sigma(Y) = A(m,t)Y,\ \ \delta(Y) = B(m,t)Y \tag{1}$$

where

$$A(m,t) = \begin{pmatrix} 0 & 1 \\ -1 & 2t \end{pmatrix},\ B(m,t) = \begin{pmatrix} \frac{(m-1)t}{1-t^2} & -\frac{m-1}{1-t^2} \\ \frac{m}{1-t^2} & -\frac{mt}{1-t^2} \end{pmatrix}.$$

Identified with an algebraic subgroup of $\mathrm{GL}_n(C)$, the $\sigma\delta$-Galois group G of (1) over $C(m,t)$ is

$$G = \left\{ \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \mid \xi \in C^\times \right\} \cup \left\{ \begin{pmatrix} 0 & \xi \\ \xi^{-1} & 0 \end{pmatrix} \mid \xi \in C^\times \right\}$$

(See Example 5.7). Consider $\sigma(Y) = A(m,t)Y$ as a family of linear difference equations with parameter t. Setting t to be a $c_1 \in C \setminus \{\pm 1\}$ yields a linear difference equation $\sigma(Y) = A(m,c_1)Y$ over $C(m)$ whose σ-Galois group is

$$G_{\sigma,c_1} = \begin{cases} \left\{ \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \mid \xi^q = 1 \right\} & c_1 + \sqrt{c_1^2 - 1} \text{ is a } q\text{-th root of unity} \\ \left\{ \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \mid \xi \in C^\times \right\} & c_1 + \sqrt{c_1^2 - 1} \text{ is not a root of unity} \end{cases}$$

(see Example 6.7). Note that the condition $c_1 \neq \pm 1$ is necessary to guarantee that the matrix $B(m,c_1)$ is well-defined but the above equality for G_{σ,c_1} is still true even when $c_1 = \pm 1$. Similarly, setting m to be a $c_2 \in C$ yields a linear differential equation $\delta(Y) = B(c_2,t)Y$ over $C(t)$ whose δ-Galois group is

$$G_{\delta,c_2} = \begin{cases} G & c_2 \notin \mathbb{Q} \\ \left\{ \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}, \begin{pmatrix} 0 & \xi \\ \xi^{-1} & 0 \end{pmatrix} \mid \xi^q = 1 \right\} & c_2 = \frac{p}{q} \in \mathbb{Q} \end{cases}$$

(see Example 6.8). One sees that both G_{σ,c_1} and G_{δ,c_2} are algebraic subgroups of G. Moreover setting $U_1 = \{c_1 \in C \mid \ c_1 + \sqrt{c_1^2 - 1} \text{ is not a root of unity}\}$ and $U_2 = C$, one sees that both U_1 and U_2 are Zariski dense and $G = G_{\sigma,c_1} G_{\delta,c_2}$ for any $(c_1,c_2) \in U_1 \times U_2$. The goal of this paper is to show that these relations still hold true for general $\sigma\delta$-linear systems (see Theorems 6.3, 6.6 and 6.18, where we use $\mathrm{stab}(\mathbf{m})$ and $\mathrm{stab}(\mathbf{n})$ instead of G_{σ,c_1} and G_{δ,c_2} respectively). We hope these results will be helpful for computing the $\sigma\delta$-Galois groups of linear difference-differential equations. Notice that the relation between the δ-Galois group of $\delta(Y) = B(m,t)Y$ over $\overline{C(m)}(t)$ and the δ-Galois group of $\delta(Y) = B(c_2,t)Y$ over $C(t)$ has been well investigated in [22,26], and its difference analogue is presented in [19].

It is well-known that a finite number of elements in a δ-field (resp., σ-field) are linearly dependent over the field of constants if and only if their Wronskian (resp., Casoration) equals zero (see page 271 of [9] and page 9 of [35]). For a Δ-field, Kolchin (see page 86 of [27]) proved the following: a finite number of elements in a Δ-field are linearly dependent over the field of constants if and only if all Wronskian-like determinants vanish. For elements in a $\Sigma\Delta$-ring, the above criteria are not valid in general (see Example 3.5). However, under certain assumptions one can still have similar criteria. For instance, in [29], a criterion for hyperexponential elements in a $\Sigma\Delta$-ring is presented. In this paper,

we shall present a criterion for elements in a simple $\Sigma\Delta$-ring, which generalizes Kolchin's result.

The paper is organized as follows. In Section 2, we introduce some basic definitions about $\Sigma\Delta$-linear systems. In Section 3, we present a criterion for testing linear dependence of elements in a simple $\Sigma\Delta$-ring. In Section 4, we present some properties of the specializations of $\Sigma\Delta$-Picard-Vessiot rings. In Section 5, we focus on $\sigma\delta$-Picard-Vessiot rings. We provides a sufficient condition for a $\sigma\delta$-Picard-Vessiot ring to be a σ-Picard-Vessiot ring. The main results of this paper are presented in Section 6.

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2. $\Sigma\Delta$-linear systems

In this section, we shall recall some basic concepts of $\Sigma\Delta$-linear systems. The readers are referred to the references [24] for details. All fields in this paper are of characteristic zero.

A $\Sigma\Delta$-ring is a ring R with a set of automorphisms Σ and a set of derivations Δ such that for any $\mu, \tau \in \Sigma \cup \Delta$, $\tau(\mu(r)) = \mu(\tau(r))$ for all $r \in R$. The notations of $\Sigma\Delta$-field, $\Sigma\Delta$-ideal, $\Sigma\Delta$-homomorphism, etc. are defined similarly. The $\Sigma\Delta$-constants $R^{\Sigma\Delta}$ of a $\Sigma\Delta$-ring R is the set

$$R^{\Sigma\Delta} = \{r \in R \mid \sigma(r) = r \ \forall \sigma \in \Sigma, \text{and } \delta(r) = 0 \ \forall \delta \in \Delta\}.$$

A simple $\Sigma\Delta$-ring is a $\Sigma\Delta$-ring whose only $\Sigma\Delta$-ideals are (0) and R. Given a $\Sigma\Delta$-field k, a $\Sigma\Delta$-linear system is a system of equations of the form

$$\begin{cases} \sigma_i(Y) = A_i Y, & A_i \in \mathrm{GL}_n(k), \ \forall \sigma_i \in \Sigma \\ \delta_i(Y) = B_i Y, & B_i \in \mathrm{gl}_n(k), \ \forall \delta_i \in \Delta \end{cases} \tag{2}$$

where the A_i, B_j satisfy the integrability condition:

$$\begin{aligned} \sigma_i(A_j)A_i &= \sigma_j(A_i)A_j, \\ \sigma_i(B_j)A_i &= \delta_j(A_i) + A_i B_j, \\ \delta_i(B_j) + B_j B_i &= \delta_j(B_i) + B_i B_j \end{aligned}$$

for all $\sigma_i, \sigma_j \in \Sigma$ and all $\delta_i, \delta_j \in \Delta$. Assume that k is a $\Sigma\Delta$-field.

Definition 2.1 *(Definition 6.10 of [24] with $\Pi = \emptyset$).* A $\Sigma\Delta$-ring R is a $\Sigma\Delta$-Picard-Vessiot ring over k for (2) if

1. R is a simple $\Sigma\Delta$-ring, and
2. $R = k[\mathcal{X}, \frac{1}{\det(\mathcal{X})}]$ where $\mathcal{X} \in \mathrm{GL}_n(R)$ satisfies $\sigma_i(\mathcal{X}) = A_i\mathcal{X}$ $\forall \sigma_i \in \Sigma$ and $\delta_i(\mathcal{X}) = B_i\mathcal{X}$ $\forall \delta_i \in \Delta$.

The invertible matrix $\mathcal{X}$ in Definition 2.1 is usually called a fundamental solution matrix of the corresponding $\Sigma\Delta$-linear system. Suppose that $\mathcal{X}$ is a fundamental solution matrix of (2). Then the above integrability condition can be naturally derived from the equality $\tau(\mu(\mathcal{X})) = \mu(\tau(\mathcal{X})$ for any $\tau, \mu \in \Sigma \cup \Delta$.

Definition 2.2. Suppose that R is a $\Sigma\Delta$-Picard-Vessiot ring over k for (2). The $\Sigma\Delta$-Galois group of (2) over k (or R over k) is defined to be the set of $\Sigma\Delta$-k-automorphisms of R over k, denoted by $\Sigma\Delta\text{-Gal}(R/k)$.

A $\Sigma\Delta$-Picard-Vessiot ring R over k for (2) alway exists. When $k^{\Sigma\Delta}$ is algebraically closed, R is unique up to $\Sigma\Delta$-k-isomorphisms (see Proposition 6.16 of [24] for a proof), and $\Sigma\Delta\text{-Gal}(R/k)$ can be identified with an algebraic subgroup of $\mathrm{GL}_n(k^{\Sigma\Delta})$ defined over $k^{\Sigma\Delta}$ (see Proposition 6.18 of [24] for a proof). The second assertion is still true if the condition that $k^{\Sigma\Delta}$ is algebraically closed is replaced with $R^{\Sigma\Delta} = k^{\Sigma\Delta}$. For the difference case, *i.e.* $\Sigma = \{\sigma\}$ and $\Delta = \emptyset$, this has already been proved in [8,36]. Using an argument similar to the proof of Theorem A.2 in [33], one will see that the general case is also true. Note that by Proposition 6.14 of [24] with $\Pi = \emptyset$, if $k^{\Sigma\Delta}$ is algebraically closed then $R^{\Sigma\Delta} = k^{\Sigma\Delta}$. We begin with the following lemma that is a generalization of Lemma 1.7 on page 6 of [34].

Lemma 2.3. *The $\Sigma\Delta$-constants $R^{\Sigma\Delta}$ of a simple $\Sigma\Delta$-ring R is a field.*

Proof. It is clear that $R^{\Sigma\Delta}$ is a ring and it is not the zero ring. It remains to show that every nonzero element of $R^{\Sigma\Delta}$ is invertible. Suppose that $a \in R^{\Sigma\Delta} \setminus \{0\}$. One can verify that the set $\{ba \mid b \in R\}$ is a nonzero $\Sigma\Delta$-ideal. Since R is a simple $\Sigma\Delta$-ring, $\{ba \mid b \in R\} = R$. Hence there is $b \in R$ such that $ba = 1$, *i.e.* a is invertible. □

Lemma 2.4. *Suppose that $S \subseteq T$ are two $\Sigma\Delta$-rings and S is $\Sigma\Delta$-simple. Then S and $T^{\Sigma\Delta}$ are linearly disjoint over $S^{\Sigma\Delta}$.*

Proof. Use an argument similar to the proof of Lemma 1.1.6 of [36]. □

Given two $n \times n$ matrices $(a_{i,j}), (b_{i,j})$, we shall use $(a_{i,j}) \otimes (b_{i,j})$ to stand for the matrix $(\sum_{l=1}^{n} a_{i,l} \otimes b_{l,j})$. Suppose that R is a $\Sigma\Delta$-Picard-Vessiot ring over k for (2) and $R^{\Sigma\Delta} = k^{\Sigma\Delta}$. Then R can be viewed as a subring of $R \otimes_k R$ and $R \otimes_k R$ can be endowed with a $\Sigma\Delta$-structure naturally. Moreover, $R \otimes_k R$ is generated by $(R \otimes_k R)^{\Sigma\Delta}$ as an R-module. Due to Lemma 2.4, R and $(R \otimes_k R)^{\Sigma\Delta}$ are linearly disjoint over $R^{\Sigma\Delta}$ and then over $k^{\Sigma\Delta}$. These imply that the R-homomorphism $R \otimes_{k^{\Sigma\Delta}} (R \otimes_k R)^{\Sigma\Delta} \to R \otimes_k R$ given by $a \otimes b \mapsto (a \otimes 1)b$ is isomorphic (see, for instance, Lemma 2.4 of [33] for a proof of the differential case). The inverse map φ of this isomorphism can be given explicitly as follows:

$$\begin{aligned} \varphi: \ R \otimes_k R &\longrightarrow R \otimes_{k^{\Sigma\Delta}} (R \otimes_k R)^{\Sigma\Delta} \\ a \otimes b &\longmapsto (a \otimes 1) b (\mathcal{X} \otimes \mathcal{Z}) \end{aligned} \tag{3}$$

where $\mathcal{X}$ is a fundamental solution matrix in $\mathrm{GL}_n(R)$ and $\mathcal{Z} = \mathcal{X}^{-1} \otimes \mathcal{X}$. Using an argument similar to the proof of Theorem A.2 in [33], one has the following proposition.

Proposition 2.5. *Suppose that R is a $\Sigma\Delta$-Picard-Vessiot ring over k for (2) and $R^{\Sigma\Delta} = k^{\Sigma\Delta}$. Then $\Sigma\Delta$-$\mathrm{Gal}(R/k)$ can be identified with the set of $k^{\Sigma\Delta}$-points of an affine algebraic group defined over $k^{\Sigma\Delta}$ with $(R \otimes_k R)^{\Sigma\Delta}$ as its coordinate ring.*

For the remainder of this paper, when we speak of the $\Sigma\Delta$-Galois group of R over k, we usually refer to $\mathrm{Hom}_{k^{\Sigma\Delta}}((R \otimes_k R)^{\Sigma\Delta}, k^{\Sigma\Delta})$. Let $\mathcal{K}$ be the total ring of fractions of R. Then each $\tau \in \Sigma\Delta$-$\mathrm{Gal}(R/k)$ can be uniquely extended into an automorphism of $\mathcal{K}$ over k. We still use τ to denote this extended automorphism and we have that $\Sigma\Delta$-$\mathrm{Gal}(R/k) = \Sigma\Delta$-$\mathrm{Gal}(\mathcal{K}/k)$.

3. Linear dependence of elements in a simple $\Sigma\Delta$-ring

In this section, we shall give a criterion for testing linear dependence of elements in a simple $\Sigma\Delta$-ring. This criterion will be used in the later sections and it may be of independent interest. The following notation will be used frequently.

Notation 3.1. Θ is the semigroup generated by $\Sigma \cup \Delta$.

Suppose that R is a simple $\Sigma\Delta$-ring. Then we have an action of Θ on R. We define an action of Θ on the polynomial ring $R[X_1, \ldots, X_m]$ by setting $\theta(f) = \sum_{i=1}^{\ell} \theta(a_i)\mathfrak{m}_i$ for $\theta \in \Theta$ and $f = \sum_{i=1}^{\ell} a_i \mathfrak{m}_i$, where $a_i \in R$ and $\mathfrak{m}_i$ is a monomial in the indeterminates $X_1, \ldots, X_m$. Let W be a subset of $R[X_1, \ldots, X_m]$. We say W is closed under the action of Θ if $\theta(f) \in W$ for all $\theta \in \Theta, f \in W$. A zero $\mathbf{u}$ in R^m of W is said to be nontrivial if $\mathbf{u} \neq (0, 0, \ldots, 0)$.

Lemma 3.2. *Suppose that $W \subset R[X_1, \ldots, X_m]$ is a set of linear homogeneous polynomials and W is closed under the action of Θ. Then W has a nontrivial zero in R^m if and only if for any $f_1, \ldots, f_m \in W$, $\det(M_{f_1,\ldots,f_m}) = 0$, where $M_{f_1,\ldots,f_m}$ stands for the coefficient matrix of $f_1, \ldots, f_m$.*

Proof. We prove the sufficiency by induction on m. Suppose that $m = 1$. Then each $f \in W$ must be of the form aX_1 for some $a \in R$ because f is linear and homogeneous in X_1. Furthermore $a = \det(M_f) = 0$ by the assumption. Thus $f = 0$ and then $W = \{0\}$. In this case, 1 is a nontrivial zero. Suppose that $m > 1$. Let $\tilde{W}$ be the R-module generated by W. Then $W \subset \tilde{W}$ and so every zero of $\tilde{W}$ is also a zero of W. Hence it suffices to

show that $\tilde{W}$ has a nontrivial zero in R^m. Since W is closed under the action of Θ, so is $\tilde{W}$. Set

$$\tilde{W}_m = \tilde{W} \cap R[X_1, \dots, X_{m-1}].$$

Then $\tilde{W}_m$ is also an R-module that is closed under the action of Θ. Note that all elements of $\tilde{W}_m$ are linear and homogeneous in $X_1, \dots, X_{m-1}$. Thus $(0, \dots, 0, 1) \in R^m$ is a nontrivial zero of $\tilde{W}_m$ viewed as a set of polynomials in $R[X_1, \dots, X_m]$. If $\tilde{W}_m = \tilde{W}$ then $(0, \dots, 0, 1)$ is a nontrivial zero of $\tilde{W}$ and also a nontrivial zero of W. So the assertion holds in this case. Suppose that $\tilde{W}_m \neq \tilde{W}$. We claim that for any $h_1, \dots, h_{m-1} \in \tilde{W}_m$, $\det(M_{h_1,\dots,h_{m-1}}) = 0$. Let $f = a_1X_1 + \cdots + a_mX_m \in \tilde{W} \setminus \tilde{W}_m$. Then $a_m \neq 0$. Since R is $\Sigma\Delta$-simple, there are $b_1, \dots, b_\ell \in R$ and $\theta_1, \dots, \theta_\ell \in \Theta$ such that $\sum_{i=1}^{\ell} b_i\theta_i(a_m) = 1$. Set $h_m = \sum_{i=1}^{\ell} b_i\theta_i(f)$. Then

$$h_m = \tilde{a}_1X_1 + \tilde{a}_2X_2 + \cdots + X_m \in \tilde{W} \tag{4}$$

where $\tilde{a}_1, \dots, \tilde{a}_{m-1} \in R$. Suppose that $h_1, \dots, h_{m-1} \in \tilde{W}_m$. One sees that

$$M_{h_1,\dots,h_m} = \begin{pmatrix} & M_{h_1,\dots,h_{m-1}} & & 0 \\ \tilde{a}_1 & \cdots & \tilde{a}_{m-1} & 1 \end{pmatrix}$$

and so $\det(M_{h_1,\dots,h_{m-1}}) = \det(M_{h_1,\dots,h_m})$. Write $h_i = \sum_{j=1}^{s} c_{ij}f_j$ with $c_{ij} \in R$, where $i = 1, 2, \dots, m$ and $f_j \in W$. Without loss of generality, we may assume that $s \geq m$. Let $D = (c_{ij})_{1\leq i\leq m, 1\leq j\leq s}$ and $T = M_{f_1,\dots,f_s}$. Then $M_{h_1,\dots,h_m} = DT$ and by Cauchy-Binet formula,

$$\begin{aligned} \det(M_{h_1,\dots,h_m}) &= \det(DT) \\ &= \sum_{1\leq l_1<\cdots<l_m\leq s} D\begin{pmatrix} l_1 & \dots & l_m \\ 1 & \dots & m \end{pmatrix} T\begin{pmatrix} 1 & \dots & m \\ l_1 & \dots & l_m \end{pmatrix} = 0 \end{aligned}$$

where $D(\cdot), T(\cdot)$ denotes the m order minors of D and T respectively. The last equality holds because $T\begin{pmatrix} 1 & \dots & m \\ l_1 & \dots & l_m \end{pmatrix} = \det(M_{f_{l_1},\dots,f_{l_m}}) = 0$. Hence $\det(M_{h_1,\dots,h_{m-1}}) = 0$. This proves our claim. By induction hypothesis, viewed as a set of polynomials in $R[X_1, \dots, X_{m-1}]$, $\tilde{W}_m$ has a nontrivial zero in R^{m-1}, say $(c_1, \dots, c_{m-1})$. Set $c_m = -\sum_{i=1}^{m-1} \tilde{a}_ic_i$, where $\tilde{a}_i$ is given in (4). Then $h_m(c_1, \dots, c_m) = 0$. For each $g \in \tilde{W}$, one has that $g - b_gh_m \in \tilde{W}_m$ where b_g is the coefficient of X_m in g. This implies that $g(c_1, \dots, c_m) - b_gh_m(c_1, \dots, c_m) = 0$. Consequently, $g(c_1, \dots, c_m) = 0$ and thus $(c_1, \dots, c_m)$ is a nontrivial zero of $\tilde{W}$ and then also a nontrivial zero of W.

Assume that W has a nontrivial zero $(a_1, \dots, a_m)$ in R^m. Then there exists $a_{i_0} \neq 0$ for some $1 \leq i_0 \leq m$. Note that $(g_1, \dots, g_m)^t = M_{g_1,\dots,g_m}(X_1, \dots, X_m)^t$ for any $g_1, \dots, g_m \in W$. Multiplying both sides by the adjoin matrix $M^*_{g_1,\dots,g_m}$ of $M_{g_1,\dots,g_m}$ yields that

$$M^*_{g_1,\dots,g_m}(g_1,\dots,g_m)^t = \det(M_{g_1,\dots,g_m})(X_1,\dots,X_m)^t.$$

Substituting a_i for X_i in the above equality yields $\det(M_{g_1,\dots,g_m})a_i = 0$, for all $\{g_1,\dots,g_m\} \subset W$ and all $i = 1,\dots,m$. In particular

$$\det(M_{g_1,\dots,g_m})a_{i_0} = 0, \forall\ g_1,\dots,g_m \in W. \tag{5}$$

We shall show that $\det(M_{g_1,\dots,g_m}) = 0$ for any $g_1,\dots,g_m \in W$. Suppose on the contrary that there exist $f_1,\dots,f_m \in W$ such that $d = \det(M_{f_1,\dots,f_m}) \neq 0$. Under this assumption, we shall show that $a_{i_0} = 0$. This will contradict with the fact that $a_{i_0} \neq 0$. We first claim that $\theta(d)a_{i_0} = 0$ for all $\theta \in \Theta$. For each $\sigma \in \Sigma$, since $\sigma(f_i) \in W$ and $M_{\sigma(f_1),\dots,\sigma(f_m)} = \sigma(M_{f_1,\dots,f_m})$, $\sigma(d)a_{i_0} = \det(M_{\sigma(f_1),\dots,\sigma(f_m)})a_{i_0} = 0$ by (5). For each $\delta \in \Delta$, since $\delta(f_i) \in W$ and

$$\delta(d) = \sum_{i=1}^{m} \det(M_{f_1,\dots,\delta(f_i),\dots,f_m}),$$

one has that $\delta(d)a_{i_0} = 0$. Repeating the above process yields that $\theta(d)a_{i_0} = 0$ for all $\theta \in \Theta$. This proves our claim. Next, we show that $a_{i_0} = 0$. Since R is $\Sigma\Delta$-simple and $d \neq 0$, there are $\theta_1,\dots,\theta_s$ and $u_1,\dots,u_s \in R$ such that $\sum_{i=1}^{s} u_i\theta_i(d) = 1$. So $a_{i_0} = (\sum_{i=1}^{s} u_i\theta_i(d))a_{i_0} = 0$ by the previous claim. Hence we obtain a contradiction as required. The contradiction implies that for any $g_1,\dots,g_m \in W$, $\det(M_{g_1,\dots,g_m}) = 0$. □

Denote by Θ^{-1} the semigroup generated by $\{\sigma^{-1} \mid \sigma \in \Sigma\} \cup \Delta$. Then we also have an action of Θ^{-1} on R. More generally, for each $m > 0$, we have an action of Θ^{-1} on R^m by setting $\theta((a_1,\dots,a_m)) = (\theta(a_1),\dots,\theta(a_m))$ for all $\theta \in \Theta^{-1}$.

Lemma 3.3. *Suppose that $W \subset R[X_1,\dots,X_m]$ is a set of linear homogeneous polynomials and W is closed under the action of Θ. Let U be the set of all zeroes of W in R^m. Then U is closed under the action of Θ^{-1}, i.e. $\theta(U) \subset U$ for all $\theta \in \Theta^{-1}$.*

Proof. It suffices to show that $\theta(U) \subset U$ for each $\theta \in \{\sigma^{-1} \mid \sigma \in \Sigma\} \cup \Delta$. Suppose $\mathbf{b} \in U$ and $f \in W$. Then $f(\mathbf{b}) = 0$. Since W is closed under the action of Θ, for every $\sigma \in \Sigma, \delta \in \Delta$, one has that $\sigma(f), \delta(f) \in W$ and thus $\sigma(f)(\mathbf{b}) = 0 = \delta(f)(\mathbf{b})$. Furthermore, one has that $\sigma(f(\sigma^{-1}(\mathbf{b}))) = \sigma(f)(\mathbf{b}) = 0$. As σ is an automorphism, $f(\sigma^{-1}(\mathbf{b})) = 0$. This implies that $\sigma^{-1}(\mathbf{b}) \in U$ and then $\sigma^{-1}(U) \subset U$. It remains to show that $\delta(U) \subset U$ for all $\delta \in \Delta$. Note that for each $f \in W$, f is linear and homogeneous. From this, one sees that $0 = \delta(f(\mathbf{b})) = \delta(f)(\mathbf{b}) + f(\delta(\mathbf{b}))$. As $\delta(f)(\mathbf{b}) = 0$, $f(\delta(\mathbf{b})) = 0$. Therefore $\delta(\mathbf{b}) \in U$ and then $\delta(U) \subset U$. □

Proposition 3.4. *Suppose that R is a simple $\Sigma\Delta$-ring and $a_1,\dots,a_m \in R$. Then $a_1,\dots,a_m$ are linearly dependent over $R^{\Sigma\Delta}$ if and only if for all $\theta_1,\dots,\theta_m \in \Theta$, one has that $\det((\theta_i(a_j))_{1\le i,j\le m}) = 0$.*

Proof. Suppose that $c_1a_1 + \cdots + c_ma_m = 0$ for some $c_1, \ldots, c_m \in R^{\Sigma\Delta}$, not all zero. Then for any $\theta_1, \ldots, \theta_m \in \Theta$, $c_1\theta_i(a_1)+\cdots+c_m\theta_i(a_m) = 0$ for all $i = 1, \ldots, m$. In matrix form, $(\theta_i(a_j))\mathbf{c} = 0$ where $\mathbf{c} = (c_1, \ldots, c_m)^t$. Multiplying the adjoint matrix of $(\theta_i(a_j))$, we obtain $\det((\theta_i(a_j)))\mathbf{c} = 0$. Since not all c_i are zero and $R^{\Sigma\Delta}$ is a field by Lemma 2.3, $\det((\theta_i(a_j))) = 0$.

Conversely, suppose that for any $\theta_1, \ldots, \theta_m \in \Theta$, $\det((\theta_i(a_j))) = 0$. We shall show that $a_1, \ldots, a_m$ are linearly dependent over $R^{\Sigma\Delta}$ by induction on m. If $m = 1$ then by taking $\theta_1 = 1$ one has that $a_1 = \det((\theta_1(a_1))) = 0$ and thus the case $m = 1$ holds. Suppose that $m > 1$. Set $W = \{\sum_{i=1}^m \theta(a_i)X_i \mid \forall\, \theta \in \Theta\}$. Then W is closed under the action of Θ. By Lemma 3.2, W has a nontrivial zero in R^m. Let U be the set of all zeroes of W in R^m. Then $U \neq \{(0, \ldots, 0)\}$. We shall show that U contains an element with 1 as some coordinate. Suppose $(b_1, \ldots, b_m) \in U \setminus \{(0, \ldots, 0)\}$. Then there exists b_{i_0} that is not zero, where $1 \leq i_0 \leq m$. For convenience, we assume that $i_0 = m$. The cases $i_0 \neq m$ can be proved similarly. Since R is $\Sigma\Delta$-simple, there are $\theta_1, \ldots, \theta_s \in \Theta$ and $u_1, \ldots, u_s \in R$ such that $\sum_{i=1}^s u_i\theta_i(b_m) = 1$. For each $i = 1, \ldots, s$, write

$$\theta_i = \prod_{\sigma\in\Sigma} \sigma^{d_{i,\sigma}} \prod_{\delta\in\Delta} \delta^{e_{i,\delta}}$$

where $d_{i,\sigma}, e_{i,\delta}$ are nonnegative integers. Let $\tau = \prod_{\sigma\in\Sigma} \sigma^{-\max\{d_{1,\sigma},\ldots,d_{s,\sigma}\}}$. Then $\tau\theta_i \in \Theta^{-1}$ for all $i = 1, \ldots, s$. Set $\tilde{b}_j = \sum_{i=1}^s \tau(u_i)\tau\theta_i(b_j)$ and $\tilde{\mathbf{b}} = (\tilde{b}_1, \ldots, \tilde{b}_m)$. Then one sees that

$$\tilde{b}_m = \sum_{i=1}^s \tau(u_i)\tau\theta_i(b_m) = \tau\left(\sum_{i=1}^s u_i\theta_i(b_m)\right) = \tau(1) = 1.$$

Since $\theta(U) \subseteq U$ for any $\theta \in \Theta^-$ due to Lemma 3.3, $\tau\theta_i((b_1, \ldots, b_m)) \in U$ for all $i = 1, \ldots, s$. Moreover, because W consists of linear homogeneous polynomials, U is an R-module and so $\tilde{\mathbf{b}} = \sum_{i=1}^s \tau(u_i)\tau\theta_i((b_1, \ldots, b_m)) \in U$. Thus $\tilde{\mathbf{b}}$ is an element of U that has the required property. In particular, $\tilde{\mathbf{b}} \neq (0, \ldots, 0)$. If $\tilde{\mathbf{b}} \in (R^{\Sigma\Delta})^m$ then $a_1, \ldots, a_m$ are linearly dependent over $R^{\Sigma\Delta}$ because $\sum_{i=1}^m a_i\tilde{b}_i = 0$. Otherwise there exists a $\sigma \in \Sigma$ such that $\sigma^{-1}(\tilde{\mathbf{b}}) - \tilde{\mathbf{b}} \neq 0$ or there exists a $\delta \in \Delta$ such that $\delta(\tilde{\mathbf{b}}) \neq 0$. Set $(c_1, \ldots, c_m)$ to be $\sigma^{-1}(\tilde{\mathbf{b}}) - \tilde{\mathbf{b}}$ in the first case or to be $\delta(\tilde{\mathbf{b}})$ in the second case. One then has that $c_m = 0$ because $\tilde{b}_m = 1$. Since $(c_1, \ldots, c_m) \neq (0, \ldots, 0)$, $(c_1, \ldots, c_{m-1}) \neq (0, \ldots, 0)$. Again, since U is an R-module and is closed under the action of Θ^{-1}, $(c_1, \ldots, c_m) \in U$, namely that $(c_1, \ldots, c_m)$ is a zero of W. As $c_m = 0$, $(c_1, \ldots, c_{m-1})$ is a zero of $W_{m-1} = \{\sum_{i=1}^{m-1} \theta(a_i)X_i \mid \forall\theta \in \Theta\}$. Moreover it is a nontrivial zero. Due to Lemma 3.2, $\det(M_{g_1,\ldots,g_{m-1}}) = 0$ for any $g_1, \ldots, g_{m-1} \in W_{m-1}$, where $M_{g_1,\ldots,g_{m-1}}$ is the coefficient matrix of $g_1, \ldots, g_{m-1}$. In particular, for any $\theta_1, \ldots, \theta_{m-1} \in \Theta$, setting $f_{\theta_i} = \sum_{j=1}^{m-1} \theta_i(a_j)X_j \in W_{m-1}$, one sees that

$$\det((\theta_i(a_j))_{1\leq i,j\leq m-1}) = \det(M_{f_{\theta_1},\ldots,f_{\theta_{m-1}}}) = 0.$$

By induction hypothesis, one has that $a_1, \ldots, a_{m-1}$ are linearly dependent over $R^{\Sigma\Delta}$ and so are $a_1, \ldots, a_m$. $\square$

The example below shows that the above proposition is not true if R is not $\Sigma\Delta$-simple.

Example 3.5. Let $R = \mathbb{Q}[y, z]$ where y, z are two indeterminates. Define $\sigma : R \to R$ as follows: $\sigma(c) = c \; \forall \, c \in \mathbb{Q}, \sigma(y) = 2y, \sigma(z) = 2z$. Then R is a σ-ring but not σ-simple. One can verify that $R^\sigma = \mathbb{Q}$ and for any $i, j \geq 0$, $\sigma^i(y)\sigma^j(z) - \sigma^i(z)\sigma^j(y) = 0$. However, y, z are linearly independent over $\mathbb{Q}$.

Corollary 3.6. *Suppose that $\Sigma \subseteq \{\sigma\}$, R is a simple $\Sigma\Delta$-ring and $a_1, \ldots, a_m \in R$. If $a_1, \ldots, a_m$ are linearly independent over $R^{\Sigma\Delta}$ then there are $\theta_1, \ldots, \theta_m$ with $\theta_1 = 1$ such that* $\det((\theta_i(a_j))_{1 \leq i,j \leq m}) \neq 0$.

Proof. For any $\theta_1, \ldots, \theta_m \in \Theta$, set

$$d(\theta_1, \ldots, \theta_m) = \det((\theta_i(a_j))_{1 \leq i,j \leq m}).$$

We shall prove the corollary by contraposition. Suppose that $d(1, \theta_2, \ldots, \theta_m) = 0$ for any $\theta_2, \ldots, \theta_m \in \Theta$. We need to show that $a_1, \ldots, a_m$ are linearly dependent over $R^{\Sigma\Delta}$. Due to Proposition 3.4, we only need to show that $d(\theta_1, \ldots, \theta_m) = 0$ for any $\theta_1, \ldots, \theta_m \in \Theta$. To this end, note that for any $\delta \in \Delta$,

$$\delta(d(\theta_1, \theta_2, \ldots, \theta_m)) = \sum_{i=1}^{m} d(\theta_1, \theta_2, \ldots, \delta\theta_i, \ldots, \theta_m).$$

The above equality with $\theta_1 = 1$ implies that $d(\delta, \theta_2, \ldots, \theta_m) = 0$ for any $\theta_2, \ldots, \theta_m \in \Theta$ and any $\delta \in \Delta$. Using the above equality repeatedly, we have that $d(\theta_1, \theta_2, \ldots, \theta_m) = 0$ for any $\theta_2, \ldots, \theta_m \in \Theta$ and any $\theta_1 \in \Theta_\Delta$, the semigroup generated by Δ. If $\Sigma = \emptyset$ then we are done. Suppose $\Sigma = \{\sigma\}$ and $\theta_1, \ldots, \theta_m \in \Theta$. Then there is $j \geq 0$ such that $\theta_i = \sigma^j \bar{\theta}_i$ where $\bar{\theta}_i \in \Theta$ and at least one of them is in Θ_Δ, say $\bar{\theta}_\ell$. The previous discussion implies that $d(\bar{\theta}_1, \ldots, \bar{\theta}_m) = 0$ as $\bar{\theta}_\ell \in \Theta_\Delta$. So $d(\theta_1, \ldots, \theta_m) = \sigma^j(d(\bar{\theta}_1, \ldots, \bar{\theta}_m)) = 0$. $\square$

The following proposition shows that in some special case $\det(\theta_i(a_j))$ can be not only nonzero but also invertible in R.

Proposition 3.7. *Suppose that $\Sigma \subseteq \{\sigma\}, \Delta \subseteq \{\delta\}$, $\Sigma \cup \Delta \neq \emptyset$ and k is a $\Sigma\Delta$-field with algebraically closed $k^{\Sigma\Delta}$. Assume that R is a $\Sigma\Delta$-Picard-Vessiot ring over k for some $\Sigma\Delta$-linear system. Let $V \subset R$ be a $\Sigma\Delta$-Gal(R/k)-invariant $k^{\Sigma\Delta}$-vector space of finite dimension and $\{a_1, \ldots, a_m\}$ a basis of V. Then there exist $\theta_1, \ldots, \theta_m \in \Theta$ with $\theta_1 = 1$ such that* $\det((\theta_i(a_j))_{1 \leq i,j \leq m})$ *is invertible in R.*

Proof. By Corollary 3.6, there are $\theta_1, \ldots, \theta_m \in \Theta$ with $\theta_1 = 1$ such that $d = \det((\theta_i(a_j))_{1 \leq i,j \leq m}) \neq 0$. For each $g \in \Sigma\Delta$-Gal(R/k), one has that $g((a_1, \ldots, a_m)) =$

$(a_1, \ldots, a_m)M_g$ for some $M_g \in \mathrm{GL}_n(k^{\Sigma\Delta})$, because V is $\Sigma\Delta$-Gal(R/k)-invariant. Furthermore, by the commutativity of g and θ_i, one sees that $g(d) = c_g d$ where $c_g = \det(M_g)$. We shall show that d is invertible in R. We first claim that d is not a zero divisor of R. If $\Sigma = \emptyset$ then $\Delta = \{\delta\}$ because $\Sigma \cup \Delta \neq \emptyset$. In this case R is an integral domain and the claim holds. Suppose that $\Sigma = \{\sigma\}$. By Lemma 6.8 of [24] with $\Sigma = \{\sigma\}, \Delta \subseteq \{\delta\}$ and $\Pi = \emptyset$, there exist idempotents $e_0, \ldots, e_{s-1}$ such that

1. $R = Re_0 \oplus \cdots \oplus Re_{s-1}$,
2. each Re_i is an integral domain and $\sigma^s\delta$-simple.

Moreover, due to Corollary 1.17 of [34] and Lemma 6 of [21], there is an $h \in \Sigma\Delta$-Gal(R/k) such that $h(e_i) = e_{i+1 \mod s}$ for all $0 \leq i \leq s-1$. Since $d \neq 0$, $de_i \neq 0$ for all $0 \leq i \leq s-1$. Otherwise, suppose that $de_{i_0} = 0$ for some i_0. Applying h to de_{i_0} repeatedly yields that

$$de_{j+i_0 \mod s} = \frac{1}{c_h^j} h^j(de_{i_0}) = 0$$

for all j and thus $d = d(e_0 + \cdots + e_{s-1}) = 0$, a contradiction. Now suppose that $ud = 0$ where $u \in R$. Then $ude_i = (ue_i)(de_i) = 0$ for all i. Since Re_i is an integral domain and $de_i \neq 0$, $ue_i = 0$ for all i. Hence $u = 0$ and then d is not a zero divisor of R. This proves our claim. Next, we shall show that d is invertible in R. Let $\mathcal{F}$ be the total ring of fractions of R. Since d is not a zero divisor of R, d is invertible in $\mathcal{F}$. Since $g(\theta(d)) = \theta(g(d)) = c_g\theta(d)$ for all $\theta \in \Sigma \cup \Delta$ and all $g \in \Sigma\Delta$-Gal(R/k), $g(\theta(d)/d) = \theta(d)/d$. The Galois correspondence (see Lemma 6.19 of [24]) implies that $\theta(d) = r_\theta d$ for all $\theta \in \Sigma \cup \Delta$, where $r_\theta \in k$. This means that the set $\{ud \mid u \in R\}$ is a nonzero $\Sigma\Delta$-ideal. Because R is $\Sigma\Delta$-simple, $1 \in \{ud \mid u \in R\}$, *i.e.* there exists an $u \in R$ such that $ud = 1$. Therefore d is invertible in R. □

As a corollary, we have the following result that has already appeared in Corollary 1.15 of [34] for the difference case.

Corollary 3.8. *Suppose that $\Sigma \subseteq \{\sigma\}, \Delta \subseteq \{\delta\}$, $\Sigma \cup \Delta \neq \emptyset$ and k is a $\Sigma\Delta$-field with algebraically closed $k^{\Sigma\Delta}$. Assume that R is a $\Sigma\Delta$-Picard-Vessiot ring over k for some $\Sigma\Delta$-linear system. Then R has no nontrivial $\Sigma\Delta$-Gal(R/k)-invariant ideal.*

Proof. Suppose that I is a $\Sigma\Delta$-Gal(R/k)-invariant ideal and $I \neq (0)$. We shall show that $I = R$. Let $a \in I \setminus \{0\}$ and let $\{a_1, \ldots, a_m\}$ be a basis of the $k^{\Sigma\Delta}$-vector space spanned by $\{g(a) \mid g \in \Sigma\Delta\text{-Gal}(R/k)\}$. Since I is $\Sigma\Delta$-Gal(R/k)-invariant, $\{a_1, \ldots, a_m\} \subset I$. By Proposition 3.7, there are $\theta_1, \ldots, \theta_m \in \Theta$ with $\theta_1 = 1$ such that $d = \det((\theta_i(a_j))_{1 \leq i,j \leq m})$ is invertible in R. Expanding d by the first row, one sees that $d \in I$. This implies that $I = R$. □

4. Specializations of ΣΔ-Picard-Vessiot rings

Throughout this section, k is a $\Sigma\Delta$-field with algebraically closed field of constants $C = k^{\Sigma\Delta}$, R is a $\Sigma\Delta$-Picard-Vessiot ring for (2) over k and $G = \mathrm{Hom}_C((R \otimes_k R)^{\Sigma\Delta}, C)$. As shown in Proposition 2.5, we may identify the $\Sigma\Delta$-Galois group $\Sigma\Delta\text{-Gal}(R/k)$ with G. We shall fix a fundamental solution matrix $\mathcal{X}$ of (2) in $\mathrm{GL}_n(R)$, and set $\mathcal{Z} = \mathcal{X}^{-1} \otimes_k \mathcal{X}$. Then $C[G] = (R \otimes_k R)^{\Sigma\Delta} = C[\mathcal{Z}, 1/\det(\mathcal{Z})]$.

We shall investigate the specializations of R. These specializations play an important role to connect G to the Galois groups of the specializations of the linear difference equation and the differential equation in (7) respectively. To construct the specializations of R, we need to introduce a simple-$\Sigma\Delta$ subring of R. We assume that

- D is simple $\Sigma\Delta$-ring such that k is the field of fractions of D,
- $\mathcal{R} = D[\mathcal{X}, 1/\det(\mathcal{X})]$.

Note that D in the above assumptions always exists, for instance, we may simply set $D = k$. Due to [36], one has that $D^{\Sigma\Delta} = k^{\Sigma\Delta} = C$. We shall use $\mathcal{R}$ to construct the Picard-Vessiot rings corresponding to the specializations of the equations in (7). Let us start with a lemma that has already appeared in the literature (see for example [24], Lemma 1.11 of [34], Lemma 1.23 of [35] and Proposition 1.4.15 of [36]) for special cases. Remark that if J is an ideal of the ring $T^{\Sigma\Delta}$ of the constants of a $\Sigma\Delta$-ring T then the algebraic ideal generated by J in T is a $\Sigma\Delta$-ideal. Here speaking of algebraic ideals of T, we mean that T is not viewed as a $\Sigma\Delta$-ring but just a ring. We shall denote by (J) the algebraic ideal of T generated by J.

Lemma 4.1. *Suppose that $S \subseteq T$ are two $\Sigma\Delta$-rings and S is $\Sigma\Delta$-simple. Assume further that T is generated by $T^{\Sigma\Delta}$ as an S-module. Then the map $J \mapsto (J)$ is a bijective correspondence from the set of ideals of $T^{\Sigma\Delta}$ to the set of $\Sigma\Delta$-ideals of T, where J is an ideal of $T^{\Sigma\Delta}$.*

Proof. Suppose that I is a $\Sigma\Delta$-ideal of T. We first show that I is generated by $I \cap T^{\Sigma\Delta}$ as an algebraic ideal. This will imply that the map is surjective. Since T is generated by $T^{\Sigma\Delta}$ as an S-module, any $f \in I$ can be written as $f = \sum_{i=1}^{s} a_i b_i$ where $a_i \in S, b_i \in T^{\Sigma\Delta}$ and $a_1, \ldots, a_s$ are linearly independent over $S^{\Sigma\Delta}$. By Proposition 3.4, there are $\theta_1, \ldots, \theta_s \in \Theta$ such that $d = \det((\theta_i(a_j))) \neq 0$. We have that $(\theta_1(f), \ldots, \theta_s(f))^t = (\theta_i(a_j))(b_1, \ldots, b_s)^t$. Multiplying both sides of the previous linear equations by the adjoint matrix of $(\theta_i(a_j))$ yields that $db_i \in I$ for any $i = 1, \ldots, s$. Hence $\Gamma_i = \{a \in S \mid ab_i \in I\}$ is a nonzero ideal. As b_i is a constant, Γ_i is a nonzero $\Sigma\Delta$-ideal. Since S is $\Sigma\Delta$-simple, $1 \in \Gamma_i$, *i.e.* $b_i \in I \cap T^{\Sigma\Delta}$. Consequently, f belongs to the ideal generated by $I \cap T^{\Sigma\Delta}$. It remains to show that the map is injective. Suppose that J is an ideal of $T^{\Sigma\Delta}$. We shall show that $J = (J) \cap T^{\Sigma\Delta}$. From this, one sees that the map is injective. It is obvious that $J \subset (J) \cap T^{\Sigma\Delta}$. Let $f \in (J) \cap T^{\Sigma\Delta}$. Write $f = \sum_{i=1}^{s} a_i b_i$ where $a_i \in T, b_i \in J$ and

$b_1, \ldots, b_s$ are linearly independent over $S^{\Sigma\Delta}$. Since $f \in T^{\Sigma\Delta}$, $\delta(f) = \sum_{i=1}^{s} \delta(a_i)b_i = 0$ for all $\delta \in \Delta$ and $\sigma(f) - f = \sum_{i=1}^{s}(\sigma(a_i) - a_i)b_i = 0$ for all $\sigma \in \Sigma$. Because $T^{\Sigma\Delta}$ and S are linearly disjoint over $S^{\Sigma\Delta}$ due to Lemma 2.4, one has that $\sigma(a_i) - a_i = 0$ and $\delta(a_i) = 0$ for all $\sigma \in \Sigma, \delta \in \Delta$ and all $i \in \{1, \ldots, s\}$. Therefore $a_i \in T^{\Sigma\Delta}$ for all $i \in \{1, \ldots, s\}$. This implies that $f \in J$ and thus $J = (J) \cap T^{\Sigma\Delta}$. $\square$

Suppose that F is a field extension of C and $c \in \mathrm{Hom}_C(D, F)$. Then F can be viewed as a D-algebra and one can consider $F \otimes_D \mathcal{R}$, where the tensor product is formed using c. We have already known that $R \otimes_k R$ is isomorphic to $R \otimes_C C[G]$ via the map φ given in (3). In what follows, we shall show that $(F \otimes_D \mathcal{R}) \otimes_F (F \otimes_D \mathcal{R})$ is isomorphic to $(F \otimes_D \mathcal{R}) \otimes_C C[G]$, namely that $F \otimes_D \mathcal{R}$ is a G-torsor over F. To prove this, we will first show that $\mathcal{R}$ is a simple $\Sigma\Delta$-ring and then show that $\mathcal{R} \otimes_D \mathcal{R}$ can be embedded into $R \otimes_k R$ (see Lemma 4.3). This embedding map together with φ induces an isomorphism from $(F \otimes_D \mathcal{R}) \otimes_F (F \otimes_D \mathcal{R})$ to $(F \otimes_D \mathcal{R}) \otimes_C C[G]$.

Lemma 4.2. *The ring $\mathcal{R}$ is $\Sigma\Delta$-simple.*

Proof. Suppose that I is a nonzero $\Sigma\Delta$-ideal of $\mathcal{R}$. It suffices to show that $1 \in I$. Let $a \in I \setminus \{0\}$. Since R is $\Sigma\Delta$-simple, there are $b_1, \ldots, b_s \in R$ and $\theta_1, \ldots, \theta_s \in \Theta$ such that $\sum_{i=1}^{s} b_i\theta_i(a) = 1$. Let $p \in D$ be nonzero such that $pb_i \in \mathcal{R}$ for all i. Then $p = \sum_{i=1}^{s} pb_i\theta_i(a) \in I \cap D$. Hence $I \cap D$ is a nonzero $\Sigma\Delta$-ideal of D and so $1 \in I \cap D$ because D is $\Sigma\Delta$-simple. Consequently, $1 \in I$. $\square$

Since D is a $\Sigma\Delta$-ring, like $R \otimes_k R$, $\mathcal{R} \otimes_D \mathcal{R}$ can be endowed with a $\Sigma\Delta$-structure and the natural homomorphism $\mathfrak{i} : \mathcal{R} \otimes_D \mathcal{R} \to R \otimes_k R$ given by $a \otimes_D b \mapsto a \otimes_k b$ is a $\Sigma\Delta$-homomorphism. Furthermore, we have the following lemma.

Lemma 4.3. *The natural homomorphism $\mathfrak{i}$ is injective.*

Proof. Since $\mathcal{R}$ is $\Sigma\Delta$-simple due to Lemma 4.2, the map $\mathcal{R} \to \mathcal{R} \otimes_D \mathcal{R}, a \mapsto a \otimes 1$ is injective and thus $\mathcal{R}$ can be viewed as a $\Sigma\Delta$-subring of $\mathcal{R} \otimes_D \mathcal{R}$. We have that $(\mathcal{R} \otimes_D \mathcal{R})^{\Sigma\Delta} = C[\mathcal{Z}, 1/\det(\mathcal{Z})]$ and $\mathcal{R} \otimes_D \mathcal{R}$ is generated by $C[\mathcal{Z}, 1/\det(\mathcal{Z})]$ as a $\mathcal{R}$-module. By Lemma 4.1, in order to show that $\ker(\mathfrak{i}) = \{0\}$, it suffices to show that $\ker(\mathfrak{i}) \cap C[\mathcal{Z}, 1/\det(\mathcal{Z})] = \{0\}$. Suppose that $a \in \ker(\mathfrak{i}) \cap C[\mathcal{Z}, 1/\det(\mathcal{Z})]$. Write $a = \sum_{i=1}^{m} a_i \otimes_D b_i$. Without loss of generality, we may assume that $\{b_1, \ldots, b_s\}$ is a k-basis of the vector space spanned by $b_1, \ldots, b_m$. Let $d \in D$ be nonzero such that $db_{s+j} = \sum_{i=1}^{s} c_{ij}b_i$ for some $c_{ij} \in D$ where $j = 1, \ldots, m-s$. Then we have that $da = \sum_{i=1}^{s}(da_i + \sum_{j=1}^{m-s} c_{ij}a_j) \otimes_D b_i$. We still have that $\mathfrak{i}(da) = 0$, i.e. $\sum_{i=1}^{s}(da_i + \sum_{j=1}^{m-s} c_{ij}a_j) \otimes_k b_i = 0$. This implies that $da_i + \sum_{j=1}^{m-s} c_{ij}a_j = 0$ for all $i = 1, \ldots, s$, because $b_1, \ldots, b_s$ are linearly independent over k. Hence $da = 0$. Since $a \in C[\mathcal{Z}, 1/\det(\mathcal{Z})] \subseteq (\mathcal{R} \otimes_D \mathcal{R})^{\Sigma\Delta}$, the set $J = \{b \in \mathcal{R} \mid ba = 0\}$ is a $\Sigma\Delta$-ideal, and it is a nonzero ideal because it contains d. As $\mathcal{R}$ is $\Sigma\Delta$-simple, $1 \in J$. In other words, $a = 0$ and thus $\ker(\mathfrak{i}) \cap C[\mathcal{Z}, 1/\det(\mathcal{Z})] = \{0\}$. $\square$

Proposition 4.4. *Suppose that $T = F \otimes_D \mathcal{R}$ is not the zero ring. Then*

$$\begin{aligned} \varphi_T : T \otimes_F T &\longrightarrow T \otimes_F F[G] = T \otimes_C C[G] \\ (a_1 \otimes_D b_1) \otimes_F (a_2 \otimes_D b_2) &\longmapsto (a_1 a_2 \otimes_D b_1 \otimes_C 1) b_2 (1 \otimes_D \mathcal{X} \otimes_C \mathcal{Z}) \end{aligned}$$

is T-isomorphic.

Proof. Let $\varphi : R \otimes_k R \to R \otimes_C C[G]$ be the isomorphism given in (3) with $k^{\Sigma\Delta} = C$ and $C[G] = (R \otimes_k R)^{\Sigma\Delta}$. Consider the composition of φ and $\mathfrak{i}$:

$$\begin{aligned} \varphi \circ \mathfrak{i} : \mathcal{R} \otimes_D \mathcal{R} &\longrightarrow R \otimes_k k[G] = R \otimes_C C[G] \\ a \otimes_D b &\longmapsto (a \otimes_C 1) b (\mathcal{X} \otimes_C \mathcal{Z}), \end{aligned}$$

where $\mathfrak{i}$ is given in Lemma 4.3. It is easy to verify that the image of $\varphi \circ \mathfrak{i}$ is included in $\mathcal{R} \otimes_C C[G]$. Suppose that $a \otimes_C b \in \mathcal{R} \otimes_C C[G]$ where $a \in \mathcal{R}$ and $b \in C[G]$. Set $w = (a \otimes_D 1) b (\mathcal{X}^{-1} \otimes_D \mathcal{X})$. Then $w \in \mathcal{R} \otimes_D \mathcal{R}$ and

$$\begin{aligned} \varphi \circ \mathfrak{i}(w) &= (a \otimes_C 1) b ((\mathcal{X}^{-1} \otimes_C 1)(\mathcal{X}^{-1} \otimes_C \mathcal{Z})) = (a \otimes_C 1) b (1 \otimes_C \mathcal{Z}) \\ &= (a \otimes_C 1)(1 \otimes_C b(\mathcal{Z})) = a \otimes_C b. \end{aligned}$$

Therefore the image of $\varphi \circ \mathfrak{i}$ is $\mathcal{R} \otimes_C C[G]$ and $\varphi \circ \mathfrak{i}$ induces an isomorphism from $\mathcal{R} \otimes_D \mathcal{R}$ to $\mathcal{R} \otimes_C C[G]$. For the sake of notation, we shall still use $\varphi \circ \mathfrak{i}$ to denote this isomorphism. Suppose that $F \otimes_D \mathcal{R}$ is not the zero ring. Then we have the following isomorphism

$$\begin{aligned} 1 \otimes \varphi \circ \mathfrak{i} : F \otimes_D \mathcal{R} \otimes_D \mathcal{R} &\longrightarrow F \otimes_D \mathcal{R} \otimes_C C[G] \\ \alpha \otimes_D a \otimes_D b &\longmapsto (\alpha \otimes_D a \otimes_C 1)(1 \otimes_D b(\mathcal{X} \otimes_C \mathcal{Z})) \\ &\qquad = (\alpha \otimes_D a \otimes_C 1) b (1 \otimes_D \mathcal{X} \otimes_C \mathcal{Z}). \end{aligned}$$

On the other hand, from page 624 of [28], one has the following natural isomorphism:

$$\begin{aligned} \pi : (F \otimes_D \mathcal{R}) \otimes_F (F \otimes_D \mathcal{R}) &\longrightarrow F \otimes_D \mathcal{R} \otimes_D \mathcal{R} \\ (a_1 \otimes_D b_1) \otimes_F (a_2 \otimes_D b_2) &\longmapsto a_1 a_2 \otimes_D b_1 \otimes_D b_2. \end{aligned}$$

One can verify that $\varphi_T = (1 \otimes \varphi \circ \mathfrak{i}) \circ \pi$. Hence φ_T is an isomorphism. Finally, since

$$\begin{aligned} \varphi_T((a_1 \otimes_D b_1) \otimes_F (a_2 \otimes_D b_2)) &= (a_1 a_2 \otimes_D b_1 \otimes_C 1) b_2 (1 \otimes_D \mathcal{X} \otimes_C \mathcal{Z}) \\ &= (a_1 \otimes_D b_1 \otimes_C 1)(a_2 \otimes_D 1 \otimes_C 1) b_2 (1 \otimes_D \mathcal{X} \otimes_C \mathcal{Z}) \\ &= (a_1 \otimes_D b_1 \otimes_C 1) \varphi_T((1 \otimes_D 1) \otimes_F (a_2 \otimes_D b_2)), \end{aligned}$$

φ_T is T-isomorphic. $\square$

The following lemma implies that if D is finitely generated over C then the set of $c \in \mathrm{Hom}_C(D, F)$ such that $F \otimes_D \mathcal{R}$ is not the zero ring is a nonempty Zariski open subset of $\mathrm{Hom}_C(D, F)$. The proof follows from that of Lemma 2.15 of [22].

Lemma 4.5. *Suppose that F is a field extension of C. There is a nonzero $a \in D$ such that for any $c \in \mathrm{Hom}_C(D, F)$ with $c(a) \neq 0$, $F \otimes_D \mathcal{R}$ is not the zero ring.*

Proof. Consider c as a homomorphism from D to $\bar{F}$, the algebraic closure of F. By Corollary 3 in Section 3.1, Chapter V of [5], there exists a nonzero $a \in D$ such that if $c(a) \neq 0$ then there exists a homomorphism h from $\mathcal{R}$ to $\bar{F}$ such that $c = h|_D$. Now suppose that $c(a) \neq 0$ and h is the extension of c to $\mathcal{R}$. Then we have the homomorphism $F \otimes_D \mathcal{R} \to \bar{F}$ given by $b_1 \otimes b_2 \mapsto b_1 h(b_2)$. Since $\bar{F}$ is not the zero ring, so is $F \otimes_D \mathcal{R}$. $\square$

Remark 4.6. Lemma 4.5 does not provide an explicit $a \in D$. We may find the required homomorphisms c as follows. Write $R = k[X, 1/\det(X)]/\mathfrak{q}$ where $\mathfrak{q}$ is a maximal $\Sigma\Delta$-ideal. Let $\tilde{\mathfrak{q}} = \mathfrak{q} \cap D[X, 1/\det(X)]$. Then $\mathcal{R} \cong D[X, 1/\det(X)]/\tilde{\mathfrak{q}}$ and

$$F \otimes_D \mathcal{R} \cong F[X, 1/\det(X)]/\langle \tilde{\mathfrak{q}}^c \rangle$$

where $\tilde{\mathfrak{q}}^c = \{P^c \mid \forall\, P \in \tilde{\mathfrak{q}}\}$ and $\langle \tilde{\mathfrak{q}}^c \rangle$ denotes the ideal in $F[X, 1/\det(X)]$ generated by $\tilde{\mathfrak{q}}^c$. Therefore $F \otimes_D \mathcal{R}$ is not the zero ring if and only if $\langle \tilde{\mathfrak{q}}^c \rangle \neq \langle 1 \rangle$.

Remark that $T = F \otimes_D \mathcal{R}$ inherits the structure of F, *i.e.* if we endow F with a differential or difference structure then T will become a differential or difference ring respectively. Precisely, assume that F is a $\tilde{\Sigma}\tilde{\Delta}$-field where $\tilde{\Sigma} \subseteq \Sigma$ and $\tilde{\Delta} \subseteq \Delta$. Then T can be endowed with a $\tilde{\Sigma}\tilde{\Delta}$-structure by setting $\sigma(a \otimes_D b) = \sigma(a) \otimes_D \sigma(b)$ for all $\sigma \in \tilde{\Sigma}$ and $\delta(a \otimes_D b) = \delta(a) \otimes_D b + a \otimes_D \delta(b)$ for all $\delta \in \tilde{\Delta}$. Furthermore, the map φ_T given in Proposition 4.4 is T-$\tilde{\Sigma}\tilde{\Delta}$-isomorphic in this case. In the following, we assume that $\tilde{\Sigma} \subseteq \Sigma, \tilde{\Delta} \subseteq \Delta$ and F is a $\tilde{\Sigma}\tilde{\Delta}$-field with algebraically closed field of constants $C = F^{\tilde{\Sigma}\tilde{\Delta}}$. Let $c : D \to F$ be a C-$\tilde{\Sigma}\tilde{\Delta}$-homomorphism such that T is not the zero ring. Then T is a $\tilde{\Sigma}\tilde{\Delta}$-ring. Let $\mathbf{m}$ be a maximal $\tilde{\Sigma}\tilde{\Delta}$-ideal of T. Then $T/\mathbf{m}$ is a simple $\tilde{\Sigma}\tilde{\Delta}$-ring. Due to Proposition 4.4, for every $g \in G = \mathrm{Hom}_C(C[G], C)$, the map

$$\begin{aligned} \rho_g : \quad & T \longrightarrow T \otimes_k T \longrightarrow T \otimes_C C[G] \longrightarrow T \\ & a(\tilde{\mathcal{X}}) \longmapsto 1 \otimes_k a(\tilde{\mathcal{X}}) \longmapsto a(\tilde{\mathcal{X}} \otimes_C \mathcal{Z}) \longmapsto a(\tilde{\mathcal{X}} g(\mathcal{Z})) \end{aligned} \tag{6}$$

is an F-automorphism, where $\tilde{\mathcal{X}} = 1 \otimes_D \mathcal{X}$. Let I be a $\tilde{\Sigma}\tilde{\Delta}$-ideal of T. Denote

$$\mathrm{stab}(\mathbf{m}, I) = \{g \in G \mid \rho_g(I) \subseteq \mathbf{m}\}.$$

If $I = \mathbf{m}$ then we abbreviate $\mathrm{stab}(\mathbf{m}, I)$ as $\mathrm{stab}(\mathbf{m})$. It is clear that $\mathrm{stab}(\mathbf{m})$ is a subgroup of G.

Proposition 4.7. *Let the notation be as above.*

1. *Suppose I is a $\tilde{\Sigma}\tilde{\Delta}$-ideal of T. Then there is an ideal $\Phi_{\mathbf{m},I}$ of $C[G]$ such that the following map is $T/\mathbf{m}$-$\tilde{\Sigma}\tilde{\Delta}$-isomorphic:*

$$\begin{aligned} \bar{\varphi}_T : \quad T/\mathbf{m} \otimes T/I &\longrightarrow T/\mathbf{m} \otimes C[G]/\Phi_{\mathbf{m},I} \\ a(\bar{\mathcal{X}}_{\mathbf{m}}) \otimes b(\bar{\mathcal{X}}_I) &\longmapsto (a(\bar{\mathcal{X}}_{\mathbf{m}}) \otimes 1) b(\bar{\mathcal{X}}_{\mathbf{m}} \otimes \bar{\mathcal{Z}}) \end{aligned}$$

   where $\bar{\mathcal{X}}_{\mathbf{m}} = 1 \otimes \mathcal{X} \mod \mathbf{m}, \bar{\mathcal{X}}_I = 1 \otimes \mathcal{X} \mod I$ and $\bar{\mathcal{Z}} = \mathcal{Z} \mod \Phi_{\mathbf{m},I}$.
2. $\mathrm{stab}(\mathbf{m}, I) = \{g \in G \mid g(P) = 0 \ \forall\ P \in \Phi_{\mathbf{m},I}\}$. *Consequently,* $\mathrm{stab}(\mathbf{m})$ *is an algebraic subgroup of G.*

Proof. 1. Let φ_T be the isomorphism given in Proposition 4.4. Since $\varphi_T(\mathbf{m} \otimes T) = \mathbf{m} \otimes C[G]$, φ_T induces the $T/\mathbf{m}$-$\tilde{\Sigma}\tilde{\Delta}$-isomorphism $\tilde{\varphi}_T : T/\mathbf{m} \otimes T \to T/\mathbf{m} \otimes C[G]$ which sends $a(\bar{\mathcal{X}}_{\mathbf{m}}) \otimes b(\mathcal{X})$ to $(a(\bar{\mathcal{X}}_{\mathbf{m}}) \otimes 1) b(\bar{\mathcal{X}}_{\mathbf{m}} \otimes \mathcal{Z})$. Therefore, it suffices to show that there is an ideal $\Phi_{\mathbf{m},I}$ of $C[G]$ such that $\tilde{\varphi}_T(T/\mathbf{m} \otimes I) = T/\mathbf{m} \otimes \Phi_{\mathbf{m},I}$. By Proposition 6.14 of [24] with $\Pi = \emptyset$, one has that $(T/\mathbf{m})^{\tilde{\Sigma}\tilde{\Delta}} = F^{\tilde{\Sigma}\tilde{\Delta}} = C$. Hence $(T/\mathbf{m} \otimes C[G])^{\tilde{\Sigma}\tilde{\Delta}} = 1 \otimes C[G] = C[G]$. Note that $T/\mathbf{m}$ can be viewed as a subring of $T/\mathbf{m} \otimes C[G]$ and moreover $T/\mathbf{m} \otimes C[G]$ is generated by $(T/\mathbf{m} \otimes C[G])^{\tilde{\Sigma}\tilde{\Delta}}$ as a $T/\mathbf{m}$-module. Set

$$\Phi_{\mathbf{m},I} = \tilde{\varphi}_T(T/\mathbf{m} \otimes I) \cap C[G].$$

By Lemma 4.1, the $\tilde{\Sigma}\tilde{\Delta}$-ideal $\tilde{\varphi}_T(T/\mathbf{m} \otimes I)$ is generated by $\Phi_{\mathbf{m},I}$. It is clear that the ideal in $T/\mathbf{m} \otimes C[G]$ generated by $\Phi_{\mathbf{m},I}$ is $T/\mathbf{m} \otimes \Phi_{\mathbf{m},I}$. Hence we have that $\tilde{\varphi}_T(T/\mathbf{m} \otimes I) = T/\mathbf{m} \otimes \Phi_{\mathbf{m},I}$ as desired.

2. Set $H = \{g \in G \mid g(P) = 0 \ \forall\ P \in \Phi_{\mathbf{m},I}\}$. Let $\{a_i \mid i \in \mathcal{I}_1\}$ be a C-basis of $\mathbf{m}$ and let $\{a_i \mid i \in \mathcal{I}_1 \cup \mathcal{I}_2\}$ be a C-basis of T. Suppose that $g \in H$ and $b \in I$. From the statement 1, one sees that $\varphi_T(\mathbf{m} \otimes T + T \otimes I) = \mathbf{m} \otimes C[G] + T \otimes \Phi_{\mathbf{m},I}$. Hence we may write

$$\varphi_T(1 \otimes b) = b(\tilde{\mathcal{X}} \otimes \mathcal{Z}) = \sum_{i \in \mathcal{I}_1 \cup \mathcal{I}_2} a_i \otimes \beta_i$$

where $\beta_i \in C[G]$ and moreover $\beta_i \in \Phi_{\mathbf{m},I}$ if $i \in \mathcal{I}_2$. Using (6), one sees that

$$\rho_g(b) = b(\tilde{\mathcal{X}} g(\mathcal{Z})) = \sum_{i \in \mathcal{I}_1 \cup \mathcal{I}_2} a_i \beta_i(g(\mathcal{Z})) = \sum_{i \in \mathcal{I}_1 \cup \mathcal{I}_2} a_i g(\beta_i) = \sum_{i \in \mathcal{I}_1} a_i g(\beta_i) \in \mathbf{m}.$$

Thus $\rho_g(I) \subseteq \mathbf{m}$. In other words, $g \in \mathrm{stab}(\mathbf{m}, I)$. On the other hand, suppose $g \in \mathrm{stab}(\mathbf{m}, I)$. Let $\beta \in \Phi_{\mathbf{m},I}$. Then there is $b \in \mathbf{m} \otimes T + T \otimes I$ such that $\varphi_T(b) = 1 \otimes \beta$. Write $b = \tilde{b} + \sum_{i \in \mathcal{I}_2} a_i \otimes b_i$ where $\tilde{b} \in \mathbf{m} \otimes T$ and $b_i \in I$. Since $\varphi_T(\tilde{b}) \in \mathbf{m} \otimes C[G]$, one sees that

$$\varphi_T\left(\sum_{i\in\mathcal{I}_2} a_i\otimes b_i\right)=\varphi_T(b-\tilde{b})=1\otimes\beta+\sum_{i\in\mathcal{I}_1} a_i\otimes\beta_i$$

for some $\beta_i\in C[G]$. Using (6) again, one has that

$$\sum_{i\in\mathcal{I}_2} a_i\rho_g(b_i)=g(\beta)+\sum_{i\in\mathcal{I}_1} a_i g(\beta_i).$$

As $\rho_g(b_i)\in\mathbf{m}$, $g(\beta)+\sum_{i\in\mathcal{I}_1} a_i g(\beta_i)\in\mathbf{m}$. Hence $g(\beta)=0$. In other words, $g\in H$. Thus $\mathrm{stab}(\mathbf{m},I)=H$. $\square$

As a corollary, we have the following result.

Corollary 4.8. *Suppose that $\mathbf{m}'$ is another maximal $\tilde{\Sigma}\tilde{\Delta}$-ideal of T. Then there exists $g\in G$ such that $\rho_g(\mathbf{m}')=\mathbf{m}$. In this case* $\mathrm{stab}(\mathbf{m})$ *is conjugate to* $\mathrm{stab}(\mathbf{m}')$ *by g.*

5. A condition for a $\sigma\delta$-Picard-Vessiot ring to be σ-Picard-Vessiot

In the remainder of this paper, we will focus on $\Sigma\Delta$-rings with at most one single automorphism σ and at most one single derivative δ. When Σ and Δ are specified, we shall use the prefixes σ-, δ-, $\sigma\delta$- and the superscripts $(\cdot)^\sigma,(\cdot)^\delta,(\cdot)^{\sigma\delta}$ instead of $\Sigma\Delta$- or $(\cdot)^{\Sigma\Delta}$.

Throughout this section, let k_0 be a δ-field with algebraically closed field of constants $C=k_0^\delta$ and let $k_0(x)$ is the $\sigma\delta$-field with $\sigma(x)=x+1$. We consider the following $\sigma\delta$-linear system over $k_0(x)$:

$$\sigma(Y)=AY,\ \delta(Y)=BY \tag{7}$$

where $A\in\mathrm{GL}_n(k_0(x)), B\in\mathrm{gl}_n(k_0(x))$ and A,B satisfy the integrability condition: $\sigma(B)A=\delta(A)+AB$.

Notation 5.1. Throughout this section, we further assume

- R is a $\sigma\delta$-Picard-Vessiot ring over $k_0(x)$ for (7).
- $\mathcal{X}$ is a fixed fundamental solution matrix in $\mathrm{GL}_n(R)$.
- K is a δ-Picard-Vessiot extension field of k_0 for $\delta(Y)=B(c)Y$ for some $c\in C$, where $B(c)$ denotes replacing x with c in B.
- $\hat{R}$ is a $\sigma\delta$-Picard-Vessiot ring over $K(x)$ for (7) containing R.
- $\mathcal{K},\hat{\mathcal{K}}$ are the total rings of fractions of $R,\hat{R}$ respectively.
- $\breve{R}$ is the composite of R and $\mathcal{K}^\sigma(x)$ inside $\mathcal{K}$.

Note that $\hat{R}$ always exists. For instance, let $\mathbf{m}$ be a maximal $\sigma\delta$-ideal of $K(x)\otimes_{k_0(x)}R$. Then $(K(x)\otimes_{k_0(x)}R)/\mathbf{m}$ is a $\sigma\delta$-Picard-Vessiot ring over $K(x)$ for (7). Since R is $\sigma\delta$-

simple, the $\sigma\delta$-homomorphism $R \to (K(x) \otimes_{k_0(x)} R)/\mathbf{m}, a \mapsto \overline{1 \otimes a}$ is injective and thus we may consider R as a subring of $\hat{R}$.

If R is a simple σ-ring then it will be a σ-Picard-Vessiot ring for $\sigma(Y) = AY$. However R is generally not σ-simple. In this section, we shall show that $\breve{R}$ is a simple σ-ring and thus it is a σ-Picard-Vessiot ring for $\sigma(Y) = AY$ over $\mathcal{K}^\sigma(x)$. We start with the following lemma.

Lemma 5.2. *$\mathcal{K}^\sigma$ is a δ-field.*

Proof. We first show that any nonzero element of $\mathcal{K}^\sigma$ is not a zero divisor of $\mathcal{K}$. Suppose that $a \in \mathcal{K}^\sigma \setminus \{0\}$ and a is a zero divisor of $\mathcal{K}$, *i.e.* there is a nonzero $b \in \mathcal{K}$ such that $ab = 0$. Write $a = p_1/q_1, b = p_2/q_2$ where $p_i, q_i \in R$ and neither of q_1, q_2 is a zero divisor. Then p_1 is a zero divisor. Lemma 19 of [21] implies that there is a positive integer s such that $\prod_{i=1}^s \sigma^i(p_1) = 0$. This implies that $\prod_{i=1}^s \sigma^i(a) = 0$. Since $\sigma(a) = a$, $a^{s+1} = 0$. In other words, $p_1^s = 0$. Since R is reduced, $p_1 = 0$ and thus $a = 0$, a contradiction. Now for each $a \in \mathcal{K}^\sigma \setminus \{0\}$, since a is not a zero divisor of $\mathcal{K}$, there is $b \in \mathcal{K}$ such that $ab = 1$. It is clear that $\sigma(b) = b$, *i.e.* $b \in \mathcal{K}^\sigma$. So $\mathcal{K}^\sigma$ is a field. $\square$

It was shown in [34] as well as [37] that every σ-Picard-Vessiot ring over $F(x)$ can be embedded into the ring of sequences Seq_F, where F is a field. We shall first show that the $\sigma\delta$-Picard-Vessiot ring R can be embedded into the ring of sequences Seq_K.

In the following, we fix a $c \in C$ such that for each $i \in \mathbb{Z}$, $A(c+i), B(c+i)$ are well-defined and $\det(A(c+i)) \neq 0$. Let K be a δ-Picard-Vessiot extension field of k_0 for $\delta(Y) = B(c)Y$.

Remark 5.3. For each $i \in \mathbb{Z}$, K is also a δ-Picard-Vessiot extension field of k_0 for $\delta(Y) = B(c+i)Y$. Since $\sigma(B)A = \delta(A) + AB$, one has that

$$B(c+1)A(c) = \delta(A(c)) + A(c)B(c).$$

By induction, one can verify that for each $s > 0$,

$$B(c+s)\prod_{i=1}^{s} A(c+s-i) = \delta\left(\prod_{i=1}^{s} A(c+s-i)\right) + \left(\prod_{i=1}^{s} A(c+s-i)\right) B(c).$$

Similarly, since $BA^{-1} = \delta(A^{-1}) + A^{-1}\sigma(B)$, one has that for each $s < 0$,

$$B(c+s)\prod_{i=s}^{1} A^{-1}(c+i) = \delta\left(\prod_{i=s}^{1} A^{-1}(c+i)\right) + \left(\prod_{i=s}^{1} A^{-1}(c+i)\right) B(c).$$

As $\det(\prod_{i=1}^s A(c+s-i)) \neq 0$ and $\det(\prod_{i=s}^1 A^{-1}(c+i)) \neq 0$, the above two equalities imply that the systems $\delta(Y) = B(c)Y$ and $\delta(Y) = B(c+s)Y$ are equivalent over k_0. Hence they have the same δ-Picard-Vessiot extension fields.

The ring of sequences Seq_K is defined to be the set

$$\mathrm{Seq}_K = \{(c_0, c_1, \dots) \mid c_i \in K\}/\sim$$

where $(b_0, b_1, \dots) \sim (c_0, c_1, \dots)$ if there is a nonnegative integer d such that $b_i = c_i$ for all $i \geq d$. We may endow Seq_K with a $\sigma\delta$-ring structure by setting

$$\delta((c_0, c_1, \dots)) = (\delta(c_0), \delta(c_1), \dots),\ \sigma((c_0, c_1, \dots)) = (c_1, c_2, \dots).$$

By sending $a \in K$ into $(a, a, \dots)$, we can embed K into Seq_K and moreover this embedding map is a δ-homomorphism. Furthermore, we can embed $K(x)$ into Seq_K by sending $f(x) \in K(x)$ to

$$(0, \dots, 0, f(c + \nu_f), f(c + \nu_f + 1), \dots)$$

where ν_f is a nonnegative integer such that $f(x)$ is well-defined at $x = c+i$ for all $i \geq \nu_f$. One can verify that this embedding map is a $\sigma\delta$-homomorphism. Under this embedding map, we may consider $K(x)$ as a subring of Seq_K.

Proposition 5.4. *R can be embedded over $k_0(x)$ into Seq_K.*

Proof. It suffices to show that $\hat{R}$ can be $K(x)$-embedded into Seq_K. To this end, we only need to show that there is a $\sigma\delta$-Picard-Vessiot ring over $K(x)$ for (7) inside Seq_K, because all $\sigma\delta$-Picard-Vessiot rings over $K(x)$ for (7) are isomorphic. By Proposition 2.4 of [37], there is a σ-Picard-Vessiot ring over $K(x)$ for $\sigma(Y) = AY$ inside Seq_K. Precisely, set $W = (W_0, W_1, \dots)$ with

$$W_0 = I_n, W_s = A(c + s - 1)W_{s-1}, \forall s \geq 1.$$

Then $K(x)[W, \frac{1}{\det(W)}]$ is a σ-Picard-Vessiot ring over $K(x)$ for $\sigma(Y) = AY$. Let $U \in \mathrm{GL}_n(K)$ be a fundamental matrix of $\delta(Y) = B(c)Y$. We then have that $\sigma(WU) = \sigma(W)U = AWU$. Moreover, for each $s \geq 0$,

$$\begin{aligned}\delta(W_sU) &= \delta(W_s)U + W_s\delta(U)\\ &= (\delta(W_s)W_s^{-1} + W_sB(c)W_s^{-1})W_sU = B(c + s)W_sU.\end{aligned}$$

The last equality holds because of Remark 5.3. Hence $\delta(WU) = BWU$. In other words, WU is a fundamental solution matrix of (7). Note that $K(x)[W, \frac{1}{\det(W)}] = K(x)[WU, \frac{1}{\det(WU)}]$. Thus $K(x)[WU, \frac{1}{\det(WU)}]$ is a $\sigma\delta$-ring and moreover since it is σ-simple, it is $\sigma\delta$-simple. Consequently, $K(x)[WU, \frac{1}{\det(WU)}]$ is a $\sigma\delta$-Picard-Vessiot ring over $K(x)$ for (7). $\square$

Remark 5.5. The proof of Proposition 5.4 relies on Proposition 2.4 of [37], where the base σ-field is supposed to be of the form $F(x)$ with F being a field of characteristic zero. This is why we only take the $\sigma\delta$-field $k_0(x)$ for consideration in the last two sections. Note that it was also proved in Proposition 4.1 of [34] that inside $\mathrm{Seq}_{\bar{F}}$ there always exists a σ-Picard-Vessiot ring for a σ-system over a σ-field k satisfying that $\bar{F} \subseteq k \subset \mathrm{Seq}_{\bar{F}}$ and $\bar{k}$ also lies in $\mathrm{Seq}_{\bar{F}}$, where the bar denotes the algebraic closure of a field. Since k_0 might not be algebraically closed, Proposition 4.1 of [34] can not be applied directly in our situation.

Example 5.6. We shall construct a $\sigma\delta$-Picard-Vessiot ring over $C(m,t)$ for (1). Consider the following linear differential equation

$$\delta(Y) = \begin{pmatrix} -\frac{t}{1-t^2} & \frac{1}{1-t^2} \\ 0 & 0 \end{pmatrix} Y.$$

We have that

$$U = \begin{pmatrix} t+\sqrt{t^2-1} & t-\sqrt{t^2-1} \\ 1 & 1 \end{pmatrix}$$

is a fundamental solution matrix and thus $K = C(t, \sqrt{t^2-1})$ is a δ-Picard-Vessiot extension of $C(t)$ for the above equation. Set $A = \begin{pmatrix} 0 & 1 \\ -1 & 2t \end{pmatrix}$ and

$$W = \left(U, AU, A^2U, A^3U, \dots\right).$$

Then $C(m,t)[W, 1/\det(W)]$ is a $\sigma\delta$-Picard-Vessiot ring over $C(m,t)$ for (1). We claim that $C(m,t)[W, 1/\det(W)]$ is an integral domain. An easy calculation yields that $d = U^{-1}AU = \mathrm{diag}(t-\sqrt{t^2-1}, t+\sqrt{t^2-1})$. Therefore

$$W = Ud^m = \begin{pmatrix} (t-\sqrt{t^2-1})^{m-1} & (t+\sqrt{t^2-1})^{m-1} \\ (t-\sqrt{t^2-1})^{m} & (t+\sqrt{t^2-1})^{m} \end{pmatrix}.$$

Let $\eta = (t+\sqrt{t^2-1})^{m-1}$. Then $C(m,t)[W, 1/\det(W)] = K(m)[\eta, \frac{1}{\eta}]$. Due to Corollary 2.4 of [29], η is transcendental over $K(m)$. This implies that $K(m)[\eta, \frac{1}{\eta}]$ is an integral domain.

Example 5.7. Let us compute the corresponding $\sigma\delta$-Galois group of the system (1). In this example as well as the examples in the remainder of this paper, we always embed the Galois group into $\mathrm{GL}_2(C)$. Consider the $C(m,t)$-$\sigma\delta$-homomorphism

$$\begin{aligned} \varphi : C(m,t)[X, 1/\det(X)] &\longrightarrow C(m,t)[W, 1/\det(W)] \\ f(X) &\longmapsto f(W) \end{aligned}$$

where W is given as in Example 5.6. Let us calculate $\ker(\varphi)$. Set $I = \langle f_1, f_2, f_3 \rangle$, where

$$f_1 = X_{11}X_{12} - 1, f_2 = X_{21}X_{22} - 1, f_3 = (X_{11}X_{22})^2 - 2tX_{11}X_{22} + 1. \tag{8}$$

We claim that $\ker(\varphi) = I$. It is easy to verify that $I \subset \ker(\varphi)$. Suppose that $f \in \ker(\varphi) \cap C(m,t)[X]$. There are positive integers d_1, d_2 such that

$$f \equiv X_{11}^{d_1} X_{22}^{d_2} f \equiv \sum_{i=0}^{s} (a_{i,1}(X_{11}X_{22}) + a_{i,0}) X_{11}^i \mod I,$$

where $a_{i,1}, a_{i,0} \in C(m,t)$. Since $f \in \ker(\varphi)$, $\sum_{i=0}^{s}(a_{i,1}(X_{11}X_{22}) + a_{i,0})X_{11}^i \in \ker(\varphi)$. In other words,

$$\sum_{i=0}^{s} (a_{i,1}(t + \sqrt{t^2-1}) + a_{i,0})\eta^i = 0$$

where $\eta = (t + \sqrt{t^2-1})^{m-1}$. As η is transcendental over K, $a_{i,1}(t + \sqrt{t^2-1}) + a_{i,0} = 0$ for all i. So $a_{i,1} = a_{i,0} = 0$ for all i. Consequently, $f \in I$ and then $I = \ker(\varphi)$. From this, we have

$$\begin{aligned} G &= \{g \in \mathrm{GL}_2(C) \mid \rho_g(P) = P(Xg) \in I, \forall\ P \in I\} \\ &= \{(g_{ij}) \in \mathrm{GL}_2(C) \mid g_{11}g_{12} = 0, g_{21}g_{22} = 0, g_{11}g_{22} + g_{12}g_{21} = 1\}. \end{aligned}$$

From the proof of Proposition 5.4, one can easily see that

Corollary 5.8. *$\hat{R}$ is a σ-Picard-Vessiot ring over $K(x)$ for $\sigma(Y) = AY$ and $\hat{R}^\sigma = K$.*

Proof. For the first assertion, it suffices to show that $\hat{R}$ is σ-simple. The proof of Proposition 5.4 implies that $\hat{R}$ is $\sigma\delta$-isomorphic to $K(x)[WU, 1/\det(WU)] = K(x)[W, 1/\det(W)]$. Since $K(x)[W, 1/\det(W)]$ as a σ-Picard-Vessiot ring for $\delta(Y) = AY$ is σ-simple, $\hat{R}$ is σ-simple. The second assertion follows from the fact that $(\mathrm{Seq}_K)^\sigma = K$. $\square$

The following example implies that not all $\sigma\delta$-Picard-Vessiot rings are σ-simple.

Example 5.9. Consider the $\sigma\delta$-system

$$\sigma(y) = y, \delta(y) = 2ty.$$

Then $C(x,t)[y, 1/y]$ is a $\sigma\delta$-Picard-Vessiot ring over $C(x,t)$ for this system. While $C(x,t)[y, 1/y]$ is not a σ-Picard-Vessiot ring over $C(x,t)$ for $\sigma(y) = y$, because the σ-ideal generated by $y + 1$ is nontrivial. However, set $K = C(t, e^{t^2})$ which is a δ-Picard-Vessiot extension field of $C(t)$ for $\delta(y) = 2ty$. Then $K(x)$ is a $\sigma\delta$-Picard-Vessiot ring over $K(x)$ for the above $\sigma\delta$-system and it is also a σ-Picard-Vessiot ring over $K(x)$ for $\sigma(y) = y$.

In what follows, we shall show that not only $\hat{R}$ but also $\breve{R}$ are σ-simple.

Lemma 5.10. $\mathcal{K} \cap K(x) = \mathcal{K}^\sigma(x)$.

Proof. It suffices to show that $\mathcal{K} \cap K(x) \subset \mathcal{K}^\sigma(x)$. Suppose $f = p/q \in \mathcal{K} \cap K(x)$ where $p, q \in K[x], \gcd(p, q) = 1$ and q is monic. Write $q = x^d + \sum_{i=0}^{d-1} a_i x^i$ and $p = \sum_{j=0}^{s} a_{j+d} x^i$, where $a_i \in K$. Then we only need to show that $a_i \in \mathcal{K}^\sigma$. First of all, we have that $1, x, \ldots, x^s, fx^{d-1}, \ldots, f$ are linearly independent over K. Otherwise, there are $\alpha_0, \ldots, \alpha_s, \beta_0, \ldots, \beta_{d-1} \in K$, not all zero, such that $\sum_{i=0}^{s} \alpha_i x^i - f \sum_{j=0}^{d-1} \beta_j x^j = 0$. Replacing f with p/q yields that $q(\sum_{i=0}^{s} \alpha_i x^i) = p(\sum_{j=0}^{d-1} \beta_j x^j) = 0$. Since $\gcd(p, q) = 1$, q divides $\sum_{j=0}^{d-1} \beta_j x^j$. So $\sum_{j=0}^{d-1} \beta_j x^j = 0$ and then $\sum_{i=0}^{s} \alpha_i x^i = 0$. In other words, all α_i and β_j are zero, a contradiction. Secondly, let M be the matrix formed by $(x + l)^s, \ldots, 1, \sigma^l(fx)^{d-1}, \ldots, \sigma^l(f)$ with $l = 0, 1, \ldots, s+d$. Then $\det(M) \neq 0$. Applying σ^l to $\sum_{i=0}^{s} a_{d+i} x^i - \sum_{j=0}^{d-1} a_j (fx^j) = x^d f, l = 0, \ldots, s + d$ yields that

$$M(a_{s+d}, \ldots, a_0)^t = (x^d f, \ldots, \sigma^{s+d}(x^d f))^t.$$

Multiplying the adjoint matrix of M on both sides, we have that $\det(M) a_i \in \mathcal{K}$ for all i. Note that both $\mathcal{K}$ and $K(x)$ are in $\hat{\mathcal{K}}$. As $\det(M)$ is invertible in $K(x)$, it is invertible in $\hat{\mathcal{K}}$ and so it is not a zero divisor in $\mathcal{K}$. This implies that $a_i \in \mathcal{K}$ and then $a_i \in \mathcal{K}^\sigma$ as desired. $\square$

Proposition 5.11. *Let the notation be as in Notation 5.1. Then $\breve{R}$ is a σ-Picard-Vessiot ring over $\mathcal{K}^\sigma(x)$ for $\sigma(Y) = AY$ and $\breve{R}^\sigma = \mathcal{K}^\sigma$.*

Proof. Note that $\breve{R} = \mathcal{K}^\sigma(x)[\mathcal{X}, 1/\det(\mathcal{X})]$. It suffices to show that $\breve{R}$ is σ-simple. We first prove that $\breve{R}$ is $\sigma\delta$-simple. Suppose that I is a $\sigma\delta$-ideal of $\breve{R}$ and $I \neq (0)$. Then $I \cap R \neq (0)$ and it is a $\sigma\delta$-ideal of R. Since R is $\sigma\delta$-simple, $1 \in I \cap R \subset I$ and thus $I = \breve{R}$.

Now suppose $a \in \breve{R} \setminus \{0\}$. We shall show that the σ-ideal of $\breve{R}$ generated by a is trivial. Let $\{a_1 = a, \ldots, a_m\}$ be a basis of K-vector space spanned by $\{g(a) \mid \forall\, g \in \sigma\delta\text{-Gal}(\hat{R}/K(x))\}$. Note that $\breve{R}$ is invariant under the action of $\sigma\delta$-$\mathrm{Gal}(\hat{R}/K(x))$. All a_i can be chosen to be in $\breve{R}$. By Corollary 5.8, $\hat{R}$ is σ-simple and $\hat{R}^\sigma = K$. By Proposition 3.4 with $\Sigma = \{\sigma\}$ and $\Delta = \emptyset$, there are $s_1, \ldots, s_m$ with $s_1 = 0$ such that $d = \det((\sigma^{s_i}(a_j))_{1 \le i,j \le m}) \neq 0$. For each $g \in \sigma\delta$-$\mathrm{Gal}(\hat{R}/K(x))$, one has that $g(d) = c_g d$ with $c_g \in K$. In other words, the ideal (d) of $\hat{R}$ generated by d is a $\sigma\delta$-$\mathrm{Gal}(\hat{R}/K(x))$-ideal. Since $\hat{R}$ is a $\sigma\delta$-Picard-Vessiot ring over $K(x)$ for (7), Corollary 3.8 implies that d is invertible in $\hat{R}$. Now one has that both $\sigma(d)d^{-1}$ and $\delta(d)d^{-1}$ are invariant under the action of $\sigma\delta$-$\mathrm{Gal}(\hat{R}/K(x))$. The Galois correspondence implies that $\sigma(d)d^{-1}, \delta(d)d^{-1} \in K(x)$. Set $b_1 = \sigma(d)d^{-1}, b_2 = \delta(d)d^{-1}$. Since d is not a zero divisor in $\hat{R}$ and $d \in \breve{R}$, d is not a zero divisor in $\breve{R}$. Therefore $b_1, b_2 \in \mathcal{K}$ because $\mathcal{K}$ is also the total ring of fractions of $\breve{R}$. This implies that $b_1, b_2 \in \mathcal{K} \cap K(x)$. By Lemma 5.10, $\mathcal{K} \cap K(x) = \mathcal{K}^\sigma(x)$. Thus the ideal (d) of $\breve{R}$ generated by d is a $\sigma\delta$-ideal. As $\breve{R}$ is $\sigma\delta$-simple, d is invertible in $\breve{R}$. Expanding

d by the first column, one sees that d belongs to the σ-ideal of $\breve{R}$ generated by a and so this ideal is trivial. The second assertion is obvious. $\square$

Since $\breve{R}^\sigma = \mathcal{K}^\sigma = (\mathcal{K}^\sigma(x))^\sigma$, by Proposition 2.5 or [8], σ-$\mathrm{Gal}(\breve{R}/\mathcal{K}^\sigma(x))$ can be identified with $\mathrm{Hom}_{\mathcal{K}^\sigma}((\breve{R}\otimes_{\mathcal{K}^\sigma(x)}\breve{R})^\sigma, \mathcal{K}^\sigma)$. As usual, for $H = \mathrm{Hom}_C(D, C)$ and a C-algebra S, denote by $H(S)$ the set of S-points of H, *i.e.* $H(S) = \mathrm{Hom}_C(D, S) = \mathrm{Hom}_S(S \otimes_C D, S)$.

Lemma 5.12. *Let $H = \mathrm{Hom}_C((\breve{R}\otimes_{\mathcal{K}^\sigma(x)}\breve{R})^{\sigma\delta}, C)$. Then σ-$\mathrm{Gal}(\breve{R}/\mathcal{K}^\sigma(x))$ can be identified with $H(\mathcal{K}^\sigma)$.*

Proof. By Proposition 5.11, one has that $\breve{R}^\sigma = \mathcal{K}^\sigma$. Since $\breve{R}$ is σ-simple, one sees that $(\breve{R} \otimes_{\mathcal{K}^\sigma(x)} \breve{R})^\sigma = \mathcal{K}^\sigma[\mathcal{Z}, 1/\det(\mathcal{Z})]$, where $\mathcal{Z} = \mathcal{X}^{-1} \otimes_{\mathcal{K}^\sigma(x)} \mathcal{X}$. As the total ring of fractions of a $\sigma\delta$-simple ring, it is easy to see that $\mathcal{K}$ is $\sigma\delta$-simple. So $\mathcal{K}^\sigma$ is δ-simple. By Lemma 2.4 with $S = \mathcal{K}^\sigma, T = \mathcal{K}^\sigma[\mathcal{Z}, 1/\det(\mathcal{Z})]$ and $\Sigma = \emptyset, \Delta = \{\delta\}$, $\mathcal{K}^\sigma$ and $C[\mathcal{Z}, 1/\det(\mathcal{Z})]$ are linearly disjoint over $(\mathcal{K}^\sigma)^\delta = C$. Hence the natural homomorphism $\mathcal{K}^\sigma \otimes_C C[\mathcal{Z}, 1/\det(\mathcal{Z})] \to \mathcal{K}^\sigma[\mathcal{Z}, 1/\det(\mathcal{Z})], a \otimes b \mapsto ab$ is isomorphic. Since $(\breve{R} \otimes_{\mathcal{K}^\sigma(x)} \breve{R})^{\sigma\delta} = C[\mathcal{Z}, 1/\det(\mathcal{Z})]$, one sees that

$$\begin{aligned} H(\mathcal{K}^\sigma) &= \mathrm{Hom}_C((\breve{R} \otimes_{\mathcal{K}^\sigma(x)} \breve{R})^{\sigma\delta}, \mathcal{K}^\sigma) = \mathrm{Hom}_C(C[\mathcal{Z}, 1/\det(\mathcal{Z})], \mathcal{K}^\sigma) \\ &= \mathrm{Hom}_{\mathcal{K}^\sigma}(\mathcal{K}^\sigma \otimes_C C[\mathcal{Z}, 1/\det(\mathcal{Z})], \mathcal{K}^\sigma) \\ &= \mathrm{Hom}_{\mathcal{K}^\sigma}(\mathcal{K}^\sigma[\mathcal{Z}, 1/\det(\mathcal{Z})], \mathcal{K}^\sigma) = \mathrm{Hom}_{\mathcal{K}^\sigma}((\breve{R} \otimes_{\mathcal{K}^\sigma(x)} \breve{R})^\sigma, \mathcal{K}^\sigma). \end{aligned}$$

The lemma then follows from Proposition 2.5. $\square$

6. Main results

In this section, we shall present the main results of this paper. We assume that

- $\mathcal{D}$ is a simple δ-domain that is finitely generated over $C = \mathcal{D}^\delta$. For example, $\mathcal{D} = C[t]$,
- k_0 is the field of fractions of $\mathcal{D}$ (thus $k_0^\delta = \mathcal{D}^\delta = C$),
- R is a $\sigma\delta$-Picard-Vessiot ring over $k_0(x)$ for (7),
- $\mathcal{X}$ is a fundamental solution matrix of (7) in $\mathrm{GL}_n(R)$,
- $\mathfrak{h} \in \mathcal{D}[x] \setminus \{0\}$ satisfies that all entries of A and B are in $\mathcal{D}[x, 1/\mathfrak{h}]$,
- $D = \mathcal{D}[x][\{\frac{1}{\sigma^i(\mathfrak{h})} \mid \forall\, i \in \mathbb{Z}\}]$,
- $\mathcal{R} = D[\mathcal{X}, 1/\det(\mathcal{X})]$.

It is clear that D is a $\sigma\delta$-ring. Furthermore, we shall show that D is actually a simple $\sigma\delta$-ring and so the results presented in Section 4 can be applied.

Lemma 6.1. *D is a simple $\sigma\delta$-ring.*

Proof. Let I be a nonzero $\sigma\delta$-ideal of D. Then $\mathcal{D} \cap I$ is a δ-ideal. We shall show that it is a nonzero ideal. As I is nonzero, let f be a nonzero element of I. One may write

$f = p/q$ for some $p, q \in \mathcal{D}[x]$. Such p, q exist because $\mathcal{D}[x]$ and D have the same field of fractions. Since $q \in \mathcal{D}[x] \subset D$, one has that $p = qf \in I \cap \mathcal{D}[x]$. Let s be a positive integer such that p and $\sigma^s(p)$ viewed as polynomials in x have no common roots. Then there are $a, b \in \mathcal{D}[x]$ such that $ap + b\sigma^s(p) \in (\mathcal{D} \cap I) \setminus \{0\}$. Therefore $\mathcal{D} \cap I$ is a nonzero δ-ideal. Since $\mathcal{D}$ is δ-simple, $1 \in \mathcal{D} \cap I$ and thus $I = D$. $\square$

6.1. Galois groups of the specializations

In this subsection, we shall show that the Galois groups of the specializations of the linear difference equation and the linear differential equation in (7) are algebraic subgroups of G.

Let $c_1 \in \mathrm{Hom}_C(\mathcal{D}, C)$. Then c_1 lifts to a unique element in $\mathrm{Hom}_{C[x]}(D, C(x))$ whose restrict on $\mathcal{D}$ is equal to c_1. As before, we still use c_1 to denote its lifting. Suppose that $c_1(\mathfrak{h}) \neq 0$. As c_1 is a σ-homomorphism, $C(x) \otimes_D \mathcal{R}$ is a σ-ring if it is not the zero ring. Suppose that $C(x) \otimes_D \mathcal{R}$ is not the zero ring. Let $\mathbf{m}$ be a maximal σ-ideal of $C(x) \otimes_D \mathcal{R}$. Further assume that $\det(A^{c_1}) \neq 0$, where $(\cdot)^{c_1}$ denotes the application of c_1 to the entries of the corresponding matrix. Then $(C(x) \otimes_D \mathcal{R})/\mathbf{m}$ is a σ-Picard-Vessiot ring over $C(x)$ for the following system

$$\sigma(Y) = A^{c_1} Y.$$

Notation 6.2. Set $\mathcal{S}_{\sigma,c_1} = (C(x) \otimes_D \mathcal{R})/\mathbf{m}$.

Due to Proposition 4.7 with $F = C(x), \tilde{\Sigma} = \{\sigma\}$ and $\tilde{\Delta} = \emptyset$, one has that

$$\mathrm{stab}(\mathbf{m}) = \{g \in G \mid g(P) = 0 \ \forall\, P \in \Phi_{\mathbf{m},\mathbf{m}}\} = \mathrm{Hom}_C(C[G]/\Phi_{\mathbf{m},\mathbf{m}}, C),$$

where $\Phi_{\mathbf{m},\mathbf{m}}$ is given as in Proposition 4.7. As $\mathrm{stab}(\mathbf{m})$ is an algebraic subgroup of G by Proposition 4.7, $C[G]/\Phi_{\mathbf{m},\mathbf{m}}$ is a Hopf algebra. Using Proposition 2.5, we immediately have the following theorem.

Theorem 6.3. *Let $c_1 \in \mathrm{Hom}_C(\mathcal{D}, C)$ be such that $c_1(\mathfrak{h}) \neq 0$, $\det(A^{c_1}) \neq 0$ and $C(x) \otimes_D \mathcal{R}$ is not the zero ring. Suppose that $\mathbf{m}$ is a maximal σ-ideal of $C(x) \otimes_D \mathcal{R}$. Then $\mathrm{stab}(\mathbf{m})$ is the σ-Galois group of $\mathcal{S}_{\sigma,c_1}$ over $C(x)$.*

Proof. Due to Proposition 4.7 with $F = C(x), \tilde{\Sigma} = \{\sigma\}$, $\tilde{\Delta} = \emptyset$ and $I = \mathbf{m}$, one sees that $(\mathcal{S}_{\sigma,c_1} \otimes_{C(x)} \mathcal{S}_{\sigma,c_1})^\sigma$ is isomorphic to $C[G]/\Phi_{\mathbf{m},\mathbf{m}}$. Furthermore, one can verify that they are isomorphic as Hopf algebras. By Proposition 2.5, σ-$\mathrm{Gal}(\mathcal{S}_{\sigma,c_1}/C(x))$ can be identified with $\mathrm{Hom}_C(\mathcal{S}_{\sigma,c_1} \otimes_{C(x)} \mathcal{S}_{\sigma,c_1})^\sigma, C)$ and thus with $\mathrm{Hom}_C(C[G]/\Phi_{\mathbf{m},\mathbf{m}}, C) = \mathrm{stab}(\mathbf{m})$. $\square$

Remark 6.4. Note that different choices of maximal σ-ideals $\mathbf{m}$ may lead to different algebraic groups *stab*$(\mathbf{m})$ of G. Corollary 4.8 implies that these $\mathrm{stab}(\mathbf{m})$ are conjugate by elements of G.

Similarly, let $c_2 \in \mathrm{Hom}_C(C[x], C)$ be such that $\mathfrak{h}(c_2(x)+i) \neq 0$ for all $i \in \mathbb{Z}$. We have that c_2 lifts to a unique $\tilde{c}_2 \in \mathrm{Hom}_{\mathcal{D}}(D, k_0)$ such that $\tilde{c}_2|_{C[x]} = c_2$. Again, for the sake of notation, we still use c_2 to denote $\tilde{c}_2$. Since $c_2(a) = a$ for any $a \in \mathcal{D}$, one sees that c_2 is a δ-homomorphism. Then $k_0 \otimes_D \mathcal{R}$ is a δ-ring if it is not the zero ring. Suppose that $k_0 \otimes_D \mathcal{R}$ is not the zero ring. Let $\mathbf{n}$ be a maximal δ-ideal of $k_0 \otimes_D \mathcal{R}$. Then $(k_0 \otimes_D \mathcal{R})/\mathbf{n}$ is a δ-Picard-Vessiot ring over k_0 for the following system

$$\delta(Y) = B^{c_2} Y.$$

Notation 6.5. Set $\mathcal{S}_{\delta,c_2} = (k_0 \otimes_D \mathcal{R})/\mathbf{n}$.

Theorem 6.6. *Let $c_2 \in \mathrm{Hom}_C(C[x], C)$ be such that $\mathfrak{h}(c_2(x)+i) \neq 0$ for any $i \in \mathbb{Z}$ and $k_0 \otimes_D \mathcal{R}$ is not the zero ring. Suppose that $\mathbf{n}$ is a maximal δ-ideal of $k_0 \otimes_D \mathcal{R}$. Then* $\mathrm{stab}(\mathbf{n})$ *is the δ-Galois group of $\mathcal{S}_{\delta,c_2}$ over k_0.*

Following Example 5.7, let us compute $\mathrm{stab}(\mathbf{m})$ and $\mathrm{stab}(\mathbf{n})$ in the following two examples respectively. Note that in Section 6.1 of [32], there is a simpler method to compute the σ-Galois group of $\sigma(Y) = A(m, c_1)Y$ over $C(m)$, *i.e.* $\mathrm{stab}(\mathbf{m})$. Here, to be consistent with the general case, we shall first compute $\mathbf{m}$ and then $\mathrm{stab}(\mathbf{m})$.

Example 6.7. From Example 5.7, we have that

$$R = C(m,t)[X, 1/\det(X)]/I$$

where $I = \langle f_1, f_2, f_2 \rangle$ with f_i is given in (8). Set $\mathcal{D} = C[t], \mathfrak{h} = t^2 - 1, D = C[m, t, \frac{1}{\mathfrak{h}}]$ and $\mathcal{R} = D[\mathcal{X}, 1/\det(\mathcal{X})]$ where $\mathcal{X} = X \mod I$. Then one has that

$$\mathcal{X}_{11}\mathcal{X}_{12} = 1,\ \mathcal{X}_{21}\mathcal{X}_{22} = 1,\ (\mathcal{X}_{11}\mathcal{X}_{22})^2 - 2t\mathcal{X}_{11}\mathcal{X}_{22} + 1 = 0. \tag{9}$$

Suppose that $c \in C = \mathrm{Hom}_C(C[t], t)$ such that $\mathfrak{h}^c = c^2 - 1 \neq 0$ and set $\alpha = c + \sqrt{c^2 - 1}$. Then $\alpha^2 \neq 1$. As $P = \begin{pmatrix} 1 & 1 \\ \alpha^{-1} & \alpha \end{pmatrix}$ is a zero of I^c in $\mathrm{GL}_2(C)$, $\langle I^c \rangle \neq C(m)[X, 1/\det(X)]$. So $T = C(m) \otimes_D \mathcal{R}$ is not the zero ring. In the following, we separate two cases to compute a maximal σ-ideal $\mathbf{m}$ and $\mathrm{stab}(\mathbf{m})$. For the sake of notation, we still use $\mathcal{X}_{ij}$ to denote $1 \otimes_D \mathcal{X}_{ij}$.

Case 1: α is not a root of unity. Let $\mathbf{m} = (\mathcal{X}_{11}\mathcal{X}_{22} - \alpha)$. Since

$$\sigma(\mathcal{X}_{11}\mathcal{X}_{22} - \alpha) = -\mathcal{X}_{12}\mathcal{X}_{21} + 2c - \alpha = \frac{1}{\alpha\mathcal{X}_{11}\mathcal{X}_{22}}(\mathcal{X}_{11}\mathcal{X}_{22} - \alpha).$$

Thus $\mathbf{m}$ is a σ-ideal. Furthermore, $\mathbf{m} \neq T$, as P is also a zero of $\mathcal{X}_{11}\mathcal{X}_{22} - \alpha$. Suppose that $f \in T \setminus \mathbf{m}$ and $\tilde{\mathbf{m}}$ is the σ-ideal generated by $\mathbf{m}$ and f. Using the relations (9), there are positive integers ν_1, ν_2 such that

$$\mathcal{X}_{22}^{\nu_2}\mathcal{X}_{11}^{\nu_1}f = b(\mathcal{X}_{11}\mathcal{X}_{22}-\alpha)+\tilde{f}(\mathcal{X}_{11})$$

for some $b \in T, \tilde{f} \in C(m)[\mathcal{X}_{11}]$. Since $f \notin \mathbf{m}$, $\tilde{f} \in \tilde{\mathbf{m}} \setminus \{0\}$. Let $\tilde{g} \in \tilde{\mathbf{m}} \setminus \{0\}$ be of the form $\tilde{g} = \mathcal{X}_{11}^s + \sum_{i=0}^{s-1} a_i\mathcal{X}_{11}^i$, where $a_i \in C(m)$ and s is minimal. Then $\sigma(\tilde{g}) = \mathcal{X}_{21}^s + \sum_{i=0}^{s-1}\sigma(a_i)\mathcal{X}_{21}^i \in \tilde{\mathbf{m}}$. Using the relations (9) again, one sees that

$$(\mathcal{X}_{11}\mathcal{X}_{22})^s\sigma(\tilde{g}) = p(\mathcal{X}_{11}\mathcal{X}_{21}-\alpha)+\mathcal{X}_{11}^s+\sum_{i=0}^{s-1}\sigma(a_i)\alpha^{s-i}\mathcal{X}_{11}^i$$

for some $p \in T$ and thus $\sum_{i=0}^{s-1}(a_i-\alpha^{s-i}\sigma(a_i))\mathcal{X}_{11}^i \in \tilde{\mathbf{m}}$. Since s is minimal, we see that $a_i - \alpha^{s-i}\sigma(a_i) = 0$ for all $i = 0, \dots, s-1$. If $a_i = 0$ for all i then $\mathcal{X}_{11} \in \tilde{\mathbf{m}}$ and thus $\tilde{\mathbf{m}} = T$, because $\mathcal{X}_{11}$ is invertible. Now suppose that there is some a_i that is not zero. Then from $\sigma(a_i) = \alpha^{s-i}a_i$, $\alpha^{s-i} = 1$, a contradiction. Hence $\tilde{\mathbf{m}} = T$ and so $\mathbf{m}$ is a maximal σ-ideal.

Now suppose $g = (g_{ij}) \in G$. Then

$$\rho_g(\mathcal{X}_{11}\mathcal{X}_{22}-\alpha) = \begin{cases} \frac{g_{11}}{\mathcal{X}_{11}}(\mathcal{X}_{11}\mathcal{X}_{22}-\alpha) & g_{11}g_{22}=1 \\ -\frac{\alpha g_{12}}{\mathcal{X}_{22}}(\mathcal{X}_{11}\mathcal{X}_{22}-\alpha^{-1}) & g_{12}g_{21}=1 \end{cases}.$$

Since $\alpha^2 \neq 1$, $\mathcal{X}_{11}\mathcal{X}_{22}-\alpha^{-1} \notin \mathbf{m}$. So the condition $\rho_g(\mathbf{m}) \subset \mathbf{m}$ implies that $g_{11}g_{22} = 1$. In other words,

$$\mathrm{stab}(\mathbf{m}) = \left\{ \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \mid \xi \in C^\times \right\}.$$

Case 2: α is a q-th root of unity. Set $\mathbf{m} = (\mathcal{X}_{11}\mathcal{X}_{22}-\alpha, \mathcal{X}_{11}^q - 1)$. Then $\mathbf{m}$ is a nontrivial σ-ideal, as P is a common zero of $\mathcal{X}_{11}\mathcal{X}_{22}-\alpha$ and $\mathcal{X}_{11}^q - 1$. Replacing $\mathbf{m}$ with $(\mathcal{X}_{11}\mathcal{X}_{22}-\alpha, \mathcal{X}_{11}^q-1)$ in **Case 1**, we have that $a_i - \alpha^{s-i}\sigma(a_i) = 0$ for all $i = 0, \dots, s-1$ and $s < q$. If all $a_i = 0$ then $\mathcal{X}_{11} \in \tilde{\mathbf{m}}$ and so $T = \tilde{\mathbf{m}}$. Otherwise there is some a_i that is not zero. Then from $\sigma(a_i) - \alpha^{s-i}a_i = 0$, $\alpha^{s-i} = 1$, a contradiction with the fact that α is a q-th root of unity. Hence $\mathbf{m}$ is a maximal σ-ideal. Now suppose $g = (g_{ij}) \in G$ such that $\rho_g(\mathbf{m}) \subset \mathbf{m}$. Then $g_{11}g_{22} = 1$ by the argument as in **Case 1**. So $g_{12} = 0$ and then $\rho_g(\mathcal{X}_{11}^q - 1) = g_{11}^q\mathcal{X}_{11}^q - 1 \in \mathbf{m}$. This implies that $g_{11}^q = 1$. Consequently, one has that

$$\mathrm{stab}(\mathbf{m}) = \left\{ \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} \mid \xi^q = 1 \right\}.$$

Example 6.8. Let $D, \mathfrak{h}, \mathcal{R}$ be as in Example 6.7. Let $c \in C = \mathrm{Hom}_C(C[m], C)$. As $f_i^c = f_i$ for all $i = 1, 2, 3$, $\langle f_1, f_2, f_3\rangle \neq \langle 1\rangle$. By Remark 4.6, $T = C(t) \otimes_D \mathcal{R}$ is not the zero ring. Let $\mathbf{n}$ be a maximal δ-ideal of T. Denote $(\bar{\mathcal{X}}_{ij}) = 1 \otimes \mathcal{X} \mod \mathbf{n}$. Suppose $\mathbf{n} \neq (0)$. Let $f \in \mathbf{n} \setminus \{0\}$. Using the relations (9), there are positive integers ν_1, ν_2 such that

$$\bar{\mathcal{X}}_{11}^{\nu_1}\bar{\mathcal{X}}_{22}^{\nu_2}\bar{f} = \sum_{i=0}^{s}(a_{i,1}\bar{\mathcal{X}}_{11}\bar{\mathcal{X}}_{22}+a_{i,0})\bar{\mathcal{X}}_{11}^i = 0,$$

where $a_{i,0}, a_{i,1} \in C(t)$. As $\bar{\mathcal{X}}_{11}\bar{\mathcal{X}}_{22}$ is algebraic over $C(t)$, $\bar{\mathcal{X}}_{11}$ is algebraic over $C(t)$. An easy calculation yields that

$$\delta(\bar{\mathcal{X}}_{11}) = \frac{c-1}{1-t^2}(t\bar{\mathcal{X}}_{11} - \bar{\mathcal{X}}_{21}) = \frac{c-1}{1-t^2}\left(t - \frac{1}{\bar{\mathcal{X}}_{11}\bar{\mathcal{X}}_{22}}\right)\bar{\mathcal{X}}_{11}.$$

On the other hand, one has that

$$\delta(\bar{\mathcal{X}}_{11}\bar{\mathcal{X}}_{22}) = \frac{-1}{1-t^2}\left(t - \frac{1}{\bar{\mathcal{X}}_{11}\bar{\mathcal{X}}_{22}}\right)\bar{\mathcal{X}}_{11}\bar{\mathcal{X}}_{22}.$$

Therefore

$$\frac{\delta(\bar{\mathcal{X}}_{11})}{\bar{\mathcal{X}}_{11}} = (1-c)\frac{\delta(\bar{\mathcal{X}}_{11}\bar{\mathcal{X}}_{22})}{\bar{\mathcal{X}}_{11}\bar{\mathcal{X}}_{22}}.$$

Since $\bar{\mathcal{X}}_{11}\bar{\mathcal{X}}_{22}$ is algebraic over $C(t)$, using the Puiseux series expansion of $\bar{\mathcal{X}}_{11}\bar{\mathcal{X}}_{22}$ at some pole, one sees that $\bar{\mathcal{X}}_{11}$ is algebraic over $C(t)$ if and only if $c \in \mathbb{Q}$. This implies that $\mathbf{n} = (0)$ if and only if $c \notin \mathbb{Q}$. Now suppose $c = p/q$ with $p, q \in \mathbb{Z}, \gcd(p,q) = 1$ and $q > 0$. Then $\bar{\mathcal{X}}_{11}^q = \beta(\bar{\mathcal{X}}_{11}\bar{\mathcal{X}}_{22})^{q-p}$ for some nonzero $\beta \in C$. Hence $\mathbf{n} = (\mathcal{X}_{11}^q - \beta(\mathcal{X}_{11}\mathcal{X}_{22})^{q-p})$. In fact, $z^q - \beta(\mathcal{X}_{11}\mathcal{X}_{22})^{q-p}$ is the minimal polynomial of $\bar{\mathcal{X}}_{11}$ over $C(t, \mathcal{X}_{11}\mathcal{X}_{22})$. As $\mathcal{X}_{12}\mathcal{X}_{11} = 1$ and $\mathcal{X}_{21}\mathcal{X}_{22} = 1$, one also has that $\mathbf{n} = (\mathcal{X}_{12}^q - \beta(\mathcal{X}_{12}\mathcal{X}_{21})^{p-q})$. Therefore, for any $(g_{ij}) \in G$,

$$\rho_g(\mathcal{X}_{11}^q - \beta(\mathcal{X}_{11}\mathcal{X}_{22})^{q-p}) = \begin{cases} g_{11}^q\mathcal{X}_{11}^q - \beta(\mathcal{X}_{11}\mathcal{X}_{22})^{q-p} & g_{12} = g_{21} = 0 \\ g_{21}^q\mathcal{X}_{12}^q - \beta(\mathcal{X}_{12}\mathcal{X}_{21})^{q-p} & g_{11} = g_{22} = 0 \end{cases}$$

and the condition $\rho_g(\mathcal{X}_{11}^q - \beta(\mathcal{X}_{11}\mathcal{X}_{22})^{p-q}) \in \mathbf{n}$ implies that either $g_{11}^q = 1$ or $g_{21}^q = 1$. These imply that

$$\mathrm{stab}(\mathbf{n}) = \begin{cases} G & c \notin \mathbb{Q} \\ \left\{\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}, \begin{pmatrix} 0 & \xi \\ \xi^{-1} & 0 \end{pmatrix} \mid \xi^q = 1\right\} & c = \frac{p}{q} \in \mathbb{Q}. \end{cases}$$

6.2. G is the product of two suitable algebraic subgroups

In this subsection, we shall show that for suitable c_1 and c_2 we have that G is the product of $\mathrm{stab}(\mathbf{m})$ and $\mathrm{stab}(\mathbf{n})$, where $\mathbf{m}$ is any maximal σ-ideal of $\mathcal{S}_{\sigma,c_1}$, and $\mathbf{n}$ is any maximal δ-ideal of $\mathcal{S}_{\delta,c_2}$.

Throughout this subsection, let $\breve{R}$ and $\mathcal{K}$ be as in Notation 5.1. Proposition 5.11 implies that $\breve{R}$ is a σ-Picard Vessiot ring over $\mathcal{K}^\sigma(x)$ for $\sigma(Y) = AY$. Remark that for each $\tau \in \sigma\delta\text{-Gal}(\mathcal{K}/k_0(x))$ and $a \in \mathcal{K}^\sigma$ one has that $\tau(a) \in \mathcal{K}^\sigma$. Therefore $\mathcal{K}^\sigma(x)$ is invariant under the action of $\sigma\delta\text{-Gal}(\mathcal{K}/k_0(x))$. From this, $\sigma\delta\text{-Gal}(\mathcal{K}/\mathcal{K}^\sigma(x))$ is a normal

subgroup of $\sigma\delta$-Gal$(\mathcal{K}/k_0(x))$. Using the normality of $\sigma\delta$-Gal$(\mathcal{K}/\mathcal{K}^\sigma(x))$, we shall show that $\mathcal{K}^\sigma(x)$ is the field of fractions of $\mathcal{R}^{\sigma\delta\text{-Gal}(\mathcal{K}/\mathcal{K}^\sigma(x))}$. To this end, we need the following lemma which was shown in the proof of Proposition 6.3.5 on page 157 of [11]. Suppose H is an algebraic subgroup of $\mathrm{GL}_n(C)$. Then $C[H]$ can be endowed with an H-module structure by setting $h(a) = a(\mathcal{Z}h)$ for any $a \in C[H]$ and any $h \in H$.

Lemma 6.9. *Let H be an affine algebraic group over C. Let N be a normal algebraic subgroup of H and χ a character of N. Then there is a nonzero $a \in C[H]$ such that $h(a) = \chi(h)a$ for any $h \in N$.*

Proof. See the proof of Proposition 6.3.5 on page 157 of [11]. □

Lemma 6.10. *Let $P, Q \in \mathcal{K}^\sigma[x]$ be such that $\gcd(P, Q) = 1$ and Q is monic. Suppose that $P/Q \in \mathcal{R}$. Then there is a nonzero $r \in \mathcal{K}^\sigma[x]$ such that $rQ \in \mathcal{D}[x]$ and all coefficients of rP are in $\mathcal{R}^\sigma$.*

Proof. We first show that there is a nonzero $r \in \mathcal{K}^\sigma[x]$ such that $rQ \in \mathcal{D}[x]$. Suppose $Q = \prod_{i=1}^{\ell} Q_i$ where Q_i is monic and irreducible over $\mathcal{K}^\sigma[x]$. Let s_i be the largest integer such that $Q_i(x + s_i)$ divides Q. Since $P/Q \in \mathcal{R}$, the set $\{\sigma^i(P/Q) \mid \forall i \geq 0\}$ generates a $k_0(x)$-vector space of finite dimension. Hence there are $a_0, \ldots, a_m \in \mathcal{D}[x]$ with $a_m \neq 0$ such that $\sum_{i=0}^{m} a_i\sigma^i(P/Q) = 0$. Multiplying both sides by $\prod_{i=0}^{m} \sigma^i(Q)$ yields that

$$\left(\prod_{i=0}^{m-1} \sigma^i(Q)\right) \sigma^m(P)a_m = \sigma^m(Q)N,\ \ N \in \mathcal{K}^\sigma[x].$$

If $Q_i(x+s_i+m)$ divides $\sigma^j(Q)$ for some $0 \leq j \leq m-1$ then $Q_i(x+s_i+m-j)$ divides Q. This contradicts with the choice of s_i. Hence $Q_i(x+s_i+m)$ does not divide $\prod_{j=0}^{m-1} \sigma^j(Q)$. It is clear that $Q_i(x + s_i + m)$ does not divide $\sigma^m(P)$ too. While $Q_i(x + s_i + m)$ divides $\sigma^m(Q)$. This implies that $Q_i(x + s_i + m)$ divides a_m. Write $a_m = Q_i(x + s_i + m)r_i(x + m + s_i)$ for some $r_i \in \mathcal{K}^\sigma[x]$. Then $r_iQ_i = a_m(x - s_i - m) \in \mathcal{D}[x]$. Set $r = \prod_{i=1}^{\ell} r_i$. Then $rQ = \prod_{i=1}^{\ell} a_m(x - s_i - m) \in \mathcal{D}[x]$.

It is clear that $rP \in \mathcal{R}$. Write $rP = \sum_{i=0}^{\ell} p_ix^i$. Applying $\sigma^i, i = 0, \ldots, \ell$ to both sides yields that $M(p_0, \ldots, p_\ell)^t = (rP, \ldots, \sigma^\ell(rP))$. Here M is the Vandermonde matrix formed by $x, x + 1, \ldots, x + \ell$, whose inverse has entries in $\mathcal{R}$. Therefore $(p_0, \ldots, p_\ell)^t = M^{-1}(rP, \ldots, \sigma^\ell(rP))^t \in \mathcal{R}^{\ell+1}$. Consequently, $p_i \in \mathcal{R}$ for all i. Since $p_i \in \mathcal{K}^\sigma$, $p_i \in \mathcal{R}^\sigma$. □

Proposition 6.11. *Let $H = \sigma\delta$-Gal$(\mathcal{K}/\mathcal{K}^\sigma(x))$. Then*

1. *$\mathcal{K}^\sigma(x)$ is the field of fractions of $\mathcal{R}^H$,*
2. *$\mathcal{K}^\sigma$ is the field of fractions of $\mathcal{R}^\sigma$.*

Proof. 1. Suppose $f \in \mathcal{K}^\sigma(x)$. Set $U = \{a \in \mathcal{R} \mid af \in \mathcal{R}\}$. We shall show that $U \cap \mathcal{R}^H \neq \{0\}$. Let $a \in U \setminus \{0\}$. Then $\{h(a) \mid h \in H\}$ generates a C-vector space of finite dimension. Suppose that $\{a_1, \ldots, a_m\}$ is a basis of this vector space. Since $\mathcal{R}$ is invariant under the action of H, this vector space is a subspace of $\mathcal{R}$. Moreover, as $\mathcal{R}$ is $\sigma\delta$-simple, by Proposition 3.4, there are $\theta_1, \ldots, \theta_m \in \Theta$ with $\theta_1 = 1$ such that $d = \det((\theta_i(a_j))_{1\le i,j\le m}) \neq 0$. Since $a_i \in U$ and $\theta_1 = 1$, $d \in U$. For each $h \in H$, $h(d) = \chi(h)d$ where χ is a character of H. We need to find an element $\tilde{a} \in \mathcal{R}$ such that $h(\tilde{a}) = \chi^{-1}(h)\tilde{a}$ for any $h \in H$. Once we have such $\tilde{a}$, $\tilde{a}d \in \mathcal{R}^H \cap U$ and thus $\tilde{a}df \in \mathcal{R}^H$. Corollary 3.8 implies that neither $\tilde{a}$ nor d is a zero divisor of $\breve{R}$ and thus none of them is a zero divisor of $\mathcal{R}$. This implies that $\tilde{a}d \in \mathcal{R}^H$ is not a zero divisor and so $f = r/\tilde{a}d$ for some $r \in \mathcal{R}^H$. We first find a nonzero $\beta \in R$ such that $h(\beta) = \chi^{-1}(h)\beta$. By Lemma 6.9, there is a nonzero $b \in C[G]$ such that $h(b) = \chi^{-1}(h)b$ for any $h \in H$. Define an action of G on $R \otimes_{k_0(x)} R$ and $R \otimes_C C[G]$ by $g(r_1 \otimes r_2) = r_1 \otimes g(r_2)$ and $g(r_1 \otimes r_2) = r_1 \otimes g(r_2)$ respectively. Then both $R \otimes_{k_0(x)} R$ and $R \otimes_C C[G]$ become G-modules and it is easy to see that the isomorphism $\varphi : R \otimes_{k_0(x)} R \to R \otimes_C C[G]$ given by $\varphi(r_1 \otimes r_2) = (r_1 \otimes 1)r_2(\mathcal{X} \otimes \mathcal{Z})$ is a G-module isomorphism. Write $\varphi^{-1}(b) = b(\mathcal{X}^{-1} \otimes \mathcal{X}) = \sum_{i=1}^s \alpha_i \otimes \beta_i$ where $\{\alpha_i\} \subset R$ is linearly independent over $k_0(x)$ and none of β_i is zero. Suppose $h \in H$. Then $h(\varphi^{-1}(b)) = \sum_{i=1}^s \alpha_i \otimes h(\beta_i)$. On the other hand, one has that $h(\varphi^{-1}(b)) = \varphi^{-1}(h(b)) = \varphi^{-1}(\chi^{-1}(h)b) = \sum_{i=1}^s \alpha_i \otimes \chi^{-1}(h)\beta_i$. Therefore for each $i = 1, \ldots, s$, $h(\beta_i) = \chi^{-1}(h)\beta_i$ for any $h \in H$. Thus we can take β to be any β_i. Let $q \in \mathcal{D}[x] \setminus \{0\}$ be such that $p\beta \in \mathcal{R}$. It is easy to verify that $h(p\beta) = ph(\beta) = \chi^{-1}(h)p\beta$. So $p\beta$ satisfies the requirement.

2. Suppose that $f \in \mathcal{K}^\sigma$. The previous result implies that $f = a/b$ with $a, b \in \mathcal{R}^H \subset \mathcal{K}^\sigma(x)$. Write $a = P_1/Q_1, b = P_2/Q_2$ where $P_i, Q_i \in \mathcal{K}^\sigma[x], \gcd(P_i, Q_i) = 1$ and Q_i is monic. Due to Lemma 6.10, there are nonzero $r_i \in \mathcal{K}^\sigma[x]$ such that all coefficients of r_iP_i, r_iQ_i are in $\mathcal{R}$. From $fP_2Q_1 = P_1Q_2$, one sees that $fr_2P_2r_1Q_1 = r_1P_1r_2Q_2$ and thus $\mathrm{lc}(r_2P_2r_1Q_1)f = \mathrm{lc}(r_1P_1r_2Q_2)$, where $\mathrm{lc}(\cdot)$ denotes the leading coefficient of a polynomial. As $\mathrm{lc}(r_iP_i), \mathrm{lc}(r_iQ_i) \in \mathcal{R} \cap \mathcal{K}^\sigma = \mathcal{R}^\sigma$, f is in the field of fractions of $\mathcal{R}^\sigma$. □

Proposition 6.12. *Let* $\mathbf{n}$ *be as in Theorem 6.6 and* $\mathcal{S}_{\delta,c_2} = (k_0 \otimes_D \mathcal{R})/\mathbf{n}$. *The* $\mathcal{D}$-δ-*homomorphism* $\phi : \mathcal{R}^\sigma \to \mathcal{S}_{\delta,c_2}$ *given by* $a(\mathcal{X}) \mapsto a^{c_2}(\bar{\mathcal{X}})$ *is injective, where* $\bar{\mathcal{X}} = 1 \otimes \mathcal{X} \mod \mathbf{n}$. *Furthermore,* $\phi(\mathcal{R}^\sigma)$ *is invariant under the action of* δ-$\mathrm{Gal}(\mathcal{S}_{\delta,c_2}/k_0)$.

Proof. Note that $\phi(a) = a^{c_2}(\bar{\mathcal{X}}) = \overline{1 \otimes a}$ for any $a \in \mathcal{R}^\sigma$. To prove the injectivity of ϕ, it suffices to show that $1 \otimes a \notin \mathbf{n}$ if $a \neq 0$. Suppose that $a \in \mathcal{R}^\sigma \setminus \{0\}$. Since $\sigma(a) = a$ and $\mathcal{R}$ is $\sigma\delta$-simple, there are $b_1, \ldots, b_s \in \mathcal{R}$ such that $1 = \sum_{i=1}^s b_i\delta^i(a)$. This implies that $1 \otimes 1 = \sum_{i=1}^s 1 \otimes b_i\delta^i(a) = \sum_{i=1}^s \delta^i(1 \otimes a)(1 \otimes b_i)$. In other words, the δ-ideal generated by $1 \otimes a$ equals $k_0 \otimes_D \mathcal{R}$. Hence $1 \otimes a \notin \mathbf{n}$.

Suppose that $\tau \in \delta$-$\mathrm{Gal}(\mathcal{S}_{\delta,c_2}/k_0)$ and $a \in \mathcal{R}^\sigma$. Using the identification given in Proposition 2.5 with $R = \mathcal{S}_{\delta,c_2}$, there is $\bar{\gamma} \in \mathrm{Hom}_C(C[G]/\Phi_{\mathbf{n},\mathbf{n}}, C)$ such that $\bar{\gamma}(\bar{\mathcal{Z}}) = \bar{\mathcal{X}}^{-1}\tau(\bar{\mathcal{X}})$, where $\bar{\mathcal{Z}} = \mathcal{Z} \mod \Phi_{\mathbf{n},\mathbf{n}}$. Let γ be the unique element in $\mathrm{Hom}_C(C[G], C)$ such that

$\gamma(\mathcal{Z}) = \bar{\gamma}(\bar{\mathcal{Z}})$. Then $\tau(\phi(a)) = a^{c_2}(\bar{\mathcal{X}}\bar{\gamma}(\bar{\mathcal{Z}})) = a^{c_2}(\bar{\mathcal{X}}\gamma(\mathcal{Z}))$. Since $\gamma \in \mathrm{Hom}_C(C[G], C)$ and $\mathcal{R}^\sigma$ is invariant under the action of $\sigma\delta$-$\mathrm{Gal}(R/k)$, $a(\mathcal{X}\gamma(\mathcal{Z})) \in \mathcal{R}^\sigma$. This implies that

$$\tau(\phi(a)) = a^{c_2}(\bar{\mathcal{X}}\gamma(\mathcal{Z})) = \phi(a(\mathcal{X}\gamma(\mathcal{Z}))) \in \phi(\mathcal{R}^\sigma).$$

Thus $\phi(\mathcal{R}^\sigma)$ is invariant under the action of δ-$\mathrm{Gal}(\mathcal{S}_{\delta,c_2}/k_0)$. $\square$

Let $\mathcal{F}_{\delta,c_2}$ be the field of fractions of $\mathcal{S}_{\delta,c_2}$. Then the δ-homomorphism ϕ given in Proposition 6.12 can be extended into an embedding of $\mathcal{K}^\sigma$ into $\mathcal{F}_{\delta,c_2}$ and $\phi(\mathcal{K}^\sigma)$ is invariant under the action of δ-$\mathrm{Gal}(F_{\delta,c_2}/k_0)$ that is an algebraic subgroup of G. By the Galois correspondence (see for example Proposition 1.34 on page 25 of [35]), $\phi(\mathcal{K}^\sigma)$ is a δ-Picard-Vessiot field for some δ-linear system over k_0 and the following canonical map

$$\begin{aligned} \pi : \delta\text{-}\mathrm{Gal}(\mathcal{F}_{\delta,c_2}/k_0) &\longrightarrow \delta\text{-}\mathrm{Gal}(\phi(\mathcal{K}^\sigma)/k_0) \\ g &\longmapsto g|_{\phi(\mathcal{K}^\sigma)}, \end{aligned}$$

is surjective and has kernel δ-$\mathrm{Gal}(\mathcal{F}_{\delta,c_2}/\phi(\mathcal{K}^\sigma))$.

Proposition 6.13. *Let $\sigma\delta$-$\mathrm{Gal}(\mathcal{K}/\mathcal{K}^\sigma(x))$ be identified with an algebraic subgroup H of G and $H' = \mathrm{stab}(\mathbf{n})$, where $\mathbf{n}$ is as in Theorem 6.6. Then $G = HH'$.*

Proof. Recall that for $g \in G = \mathrm{Hom}_C(C[G], C)$ the corresponding automorphism of R over k is $\tau_g \in \sigma\delta$-$\mathrm{Gal}(R/k)$ such that $g(\mathcal{Z}) = \mathcal{X}^{-1}\tau_g(\mathcal{X})$. Furthermore if $g \in \mathrm{stab}(\mathbf{n}) = \mathrm{Hom}_C(C[G]/\Phi_{\mathbf{n},\mathbf{n}}, C) \subset \mathrm{Hom}_C(C[G], C)$ then the corresponding automorphism of $\mathcal{S}_{\delta,c_2}$ over k_0 is $\gamma_g \in \delta$-$\mathrm{Gal}(\mathcal{S}_{\delta,c_2}/k_0)$ such that $\gamma_g(\bar{\mathcal{X}}) = \bar{\mathcal{X}}g(\mathcal{Z})$ where $\bar{\mathcal{X}} = 1 \otimes \mathcal{X} \mod \mathbf{n}$. Now suppose $g \in G$. Then $\tau_g \in \sigma\delta$-$\mathrm{Gal}(\mathcal{K}/k)$ and $\tau_g|_{\mathcal{K}^\sigma} \in \delta$-$\mathrm{Gal}(\mathcal{K}^\sigma/k_0)$. Let ϕ be given as in Proposition 6.12. Then ϕ can be viewed as a k_0-δ-isomorphism from $\mathcal{K}^\sigma$ to $\phi(\mathcal{K}^\sigma)$ and thus $\phi \circ \tau_g|_{\mathcal{K}^\sigma} \circ \phi^{-1} \in \delta$-$\mathrm{Gal}(\phi(\mathcal{K}^\sigma)/k_0)$. Let $h \in \mathrm{stab}(\mathbf{n})$ be such that $\gamma_h = \phi \circ \tau_g|_{\mathcal{K}^\sigma} \circ \phi^{-1}$, *i.e.* $\phi \circ \tau_g|_{\mathcal{K}^\sigma} \circ \phi^{-1} \circ \gamma_h^{-1} = \mathrm{id}$. We claim that $\tau_{gh^{-1}}(f) = f$ for all $f \in \mathcal{K}^\sigma(x)$. It suffices to show that $\tau_{gh^{-1}}(f) = f$ for all $f \in \mathcal{K}^\sigma$. Suppose $f \in \mathcal{K}^\sigma$. Due to Proposition 6.11, we may write $f = a/b$ with $a, b \in \mathcal{R}^\sigma$ and $b \neq 0$. We then have that

$$\begin{aligned} \phi\left(\tau_{gh^{-1}}\left(\frac{a}{b}\right)\right) &= \phi\left(\frac{a(\mathcal{X}g(\mathcal{Z})h(\mathcal{Z})^{-1})}{b(\mathcal{X}g(\mathcal{Z})h(\mathcal{Z})^{-1})}\right) = \frac{a^c(\bar{\mathcal{X}}g(\mathcal{Z})h(\mathcal{Z})^{-1})}{b^c(\bar{\mathcal{X}}g(\mathcal{Z})h(\mathcal{Z})^{-1})} \\ &= \phi \circ \tau_g|_{\mathcal{K}^\sigma} \circ \phi^{-1} \circ \gamma_h^{-1}\left(\frac{a^c(\bar{\mathcal{X}})}{b^c(\bar{\mathcal{X}})}\right) = \frac{a^c(\bar{\mathcal{X}})}{b^c(\bar{\mathcal{X}})} = \phi\left(\frac{a}{b}\right). \end{aligned}$$

Since ϕ is injective, $\tau_{gh^{-1}}(f) = f$. This proves our claim. Hence one has that $\tau_{gh^{-1}} \in \sigma\delta$-$\mathrm{Gal}(\mathcal{K}/\mathcal{K}^\sigma(x))$ and then $gh^{-1} \in H$ under the identification. Therefore $g \in HH'$. It is clear that $HH' \subset G$. So $G = HH'$. $\square$

Example 6.14. In Examples 5.7 and 6.8, we have already known that

$$\begin{aligned} G &= \{(g_{ij}) \in \mathrm{GL}_2(C) \mid g_{11}g_{12} = 0, g_{21}g_{22} = 0, g_{11}g_{22} + g_{12}g_{21} = 1\} \\ &= \left\{ \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix} \mid g_{11}g_{22} = 1 \right\} \cup \left\{ \begin{pmatrix} 0 & g_{12} \\ g_{21} & 0 \end{pmatrix} \mid g_{12}g_{21} = 1 \right\} \end{aligned}$$

and

$$H' = \begin{cases} G & c \notin \mathbb{Q} \\ \left\{ \begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix}, \begin{pmatrix} 0 & \xi \\ \xi^{-1} & 0 \end{pmatrix} \mid \xi^q = 1 \right\} & c = \frac{p}{q} \in \mathbb{Q}. \end{cases}$$

Now let us compute $H = \sigma\delta\text{-Gal}(\mathcal{K}/\mathcal{K}^\sigma(x))$. We first compute $\mathcal{K}^\sigma$. It is clear that $\mathcal{K}^\sigma \subseteq K = C(t, \sqrt{t^2-1})$. On the other hand, $(\mathcal{X}_{11}\mathcal{X}_{22})^2 - 2t\mathcal{X}_{11}\mathcal{X}_{22} + 1 = 0$ and $\mathcal{X}_{11}\mathcal{X}_{22} \in \mathcal{R}^\sigma$. As $[C(t, \mathcal{X}_{11}\mathcal{X}_{22}) : C(t)] = 2$, one has that $\mathcal{R}^\sigma = \mathcal{K}^\sigma = K = C(t)(\mathcal{X}_{11}\mathcal{X}_{22})$. Next, we compute $\sigma\delta\text{-Gal}(\mathcal{K}/K(m))$. Note that for any $g = (g_{ij}) \in \sigma\delta\text{-Gal}(\mathcal{K}/C(m,t))$, $g \in \sigma\delta\text{-Gal}(\mathcal{K}/K(m))$ if and only if $g(\mathcal{X}_{11}\mathcal{X}_{22}) = \mathcal{X}_{11}\mathcal{X}_{22}$. An easy calculation yields that if $g_{12}g_{21} = 1$ then $\rho_g(\mathcal{X}_{11}\mathcal{X}_{22}) = \mathcal{X}_{12}\mathcal{X}_{21}$. Since $(\mathcal{X}_{11}\mathcal{X}_{22})^2 \neq 1$, using the relations (9), $\mathcal{X}_{11}\mathcal{X}_{22} \neq \mathcal{X}_{12}\mathcal{X}_{21}$. Hence one has that $g \in \sigma\delta\text{-Gal}(\mathcal{K}/K(x))$ if and only if $g_{11}g_{22} = 1$ and $g_{12}g_{21} = 0$. Hence

$$H = \{(g_{ij}) \in G \mid g_{11}g_{22} = 1\} = \left\{ \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix} \mid g_{11}g_{22} = 1 \right\}.$$

It is easy to see that $G = HH'$.

In the remainder of this subsection, we shall prove that $H = \mathrm{stab}(\mathbf{m})$ for any maximal σ-ideal $\mathbf{m}$ of $\mathcal{S}_{\sigma,c_1}$ with suitable c_1. Since $\mathcal{K}^\sigma$ is a δ-Picard-Vessiot field for some δ-linear system over k_0, there is $\eta \in \mathrm{GL}_m(\mathcal{K}^\sigma)$ for some m such that $\mathcal{K}^\sigma = k_0(\eta)$ and moreover $k_0[\eta, 1/\det(\eta)]$ is a δ-Picard-Vessiot ring over k_0 for the corresponding δ-linear system. In particular, $k_0[\eta, 1/\det(\eta)]$ is δ-simple.

Notation 6.15. We assume that

- $\tilde{\mathcal{D}} = \mathcal{D}[\eta, 1/\det(\eta)]$;
- $\tilde{D} = \tilde{\mathcal{D}}[x][\{\frac{1}{\sigma^i(\mathfrak{h})} \mid \forall i \in \mathbb{Z}\}]$;
- $\tilde{\mathcal{R}} = \tilde{D}[\mathcal{X}, 1/\det(\mathcal{X})]$.

We see that $\tilde{\mathcal{D}}$ is finitely generated over C, because $\mathcal{D}$ is. Using an argument similar to the proof of Lemma 6.1, one sees that $\tilde{\mathcal{D}}$ is δ-simple and $\tilde{D}$ is $\sigma\delta$-simple. Let $\tilde{c}_1 : \tilde{\mathcal{D}} \to C$ be a C-homomorphism that uniquely lifts to a $C[x]$-homomorphism from $\tilde{D}$ to $C(x)$. As before, we consider $\tilde{T} = C(x) \otimes_{\tilde{D}} \tilde{\mathcal{R}}$. Then if $\tilde{T}$ is not the zero ring then it is a σ-ring.

Moreover, by Proposition 4.4 with $D = \tilde{D}, T = \tilde{T}$ and $G = H = \mathrm{Hom}_C((\breve{R} \otimes_{\mathcal{K}^\sigma(x)} \breve{R})^{\sigma\delta}, C)$, one has the following $\tilde{T}$-isomorphism:

$$\begin{aligned} \tilde{T} \otimes_{C(x)} \tilde{T} &\longrightarrow \tilde{T} \otimes_C C[H] \\ a \otimes b &\longmapsto (a \otimes 1)b(\tilde{\mathcal{X}} \otimes \mathcal{Z}) \end{aligned} \tag{10}$$

where $\tilde{\mathcal{X}} = 1 \otimes_{\tilde{D}} \mathcal{X}$.

Lemma 6.16. *Let H be as in Proposition 6.13. There is a Zariski dense subset U_1 of $\mathrm{Hom}_C(\tilde{\mathcal{D}}, C)$ such that for any $c_1 \in U_1$ if $C(x) \otimes_{\tilde{D}} \tilde{\mathcal{R}}$ is not the zero ring then it is σ-simple and H is the σ-Galois group of $C(x) \otimes_{\tilde{D}} \tilde{\mathcal{R}}$ over $C(x)$.*

Proof. Due to Proposition 2.4 of [8], $\bar{T} = \overline{\mathcal{K}^\sigma}(x) \otimes_{\mathcal{K}^\sigma(x)} \breve{R}$ is a σ-Picard-Vessiot ring over $\overline{\mathcal{K}^\sigma}(x)$ for $\sigma(Y) = AY$. By Lemma 5.12, $H(\mathcal{K}^\sigma)$ is the σ-Galois group of $\sigma(Y) = AY$ over $\mathcal{K}^\sigma(x)$. By Corollary 2.5 of [8] or using an argument similar to the proof of Proposition 5.11, one sees that $H(\overline{\mathcal{K}^\sigma})$ is the σ-Galois group of $\sigma(Y) = AY$ over $\overline{\mathcal{K}^\sigma}(x)$. Embedding H into $\mathrm{GL}_n(C)$, we consider H as an algebraic subgroup of $\mathrm{GL}_n(C)$ for the moment. Let $S \subset C[Z, 1/\det(Z)]$ be a finite set that generates the vanishing ideal of H. Then S also generates the vanishing ideal of $H(\overline{\mathcal{K}^\sigma})$. Due to Theorem 1.2 of [19] with $\mathbb{X} = \mathrm{Hom}_C(\tilde{\mathcal{D}}, C)$, there is a Zariski dense subset U_1 of $\mathrm{Hom}_C(\tilde{\mathcal{D}}, C)$ such that for any $c_1 \in U_1$, the variety in $\mathrm{GL}_n(C)$ defined by S^{c_1} is the σ-Galois group of $\sigma(Y) = A^{c_1}Y$ over $C(x)$. Since $S = S^{c_1}$, H is the σ-Galois group of $\sigma(Y) = A^{c_1}Y$ over $C(x)$ for any $c_1 \in U_1$.

Suppose $c_1 \in U_1$ and $\tilde{T} = C(x) \otimes_{\tilde{D}} \tilde{\mathcal{R}}$ is not the zero ring. We shall show that $\tilde{T}$ is σ-simple and then it is a σ-Picard-Vessiot ring over $C(x)$ for $\sigma(Y) = A^{c_1}Y$. Let $\tilde{\mathbf{m}}$ be a maximal σ-ideal of $\tilde{T}$. Due to Theorem 6.3 with $\mathbf{m} = \tilde{\mathbf{m}}$ and $G = H$, $\mathrm{stab}(\tilde{\mathbf{m}}) = \{h \in H \mid \rho_h(\tilde{\mathbf{m}}) \subset \tilde{\mathbf{m}}\}$ is the σ-Galois group of $\tilde{T}/\tilde{\mathbf{m}}$ over $C(x)$. In other words, $\mathrm{stab}(\tilde{\mathbf{m}})$ is also the σ-Galois group of $\sigma(Y) = A^{c_1}Y$ over $C(x)$. Hence there is a $g \in \mathrm{Hom}_C(C[\mathrm{GL}_n], C)$ such that $\mathrm{stab}(\tilde{\mathbf{m}}) = gHg^{-1}$. Since $\mathrm{stab}(\tilde{\mathbf{m}}) \subset H$, one has that $gHg^{-1} \subset H$ and thus $H \supset gHg^{-1} \supset g^2Hg^{-2} \supset \dots$. The Noetherian property of H then implies that $H = gHg^{-1} = \mathrm{stab}(\tilde{\mathbf{m}})$. This could happen only if $\tilde{\mathbf{m}} = (0)$ by (10). So $\tilde{T}$ is σ-simple. □

Proposition 6.17. *Let H be as in Proposition 6.13. Then there is a Zariski dense subset U_1 of $\mathrm{Hom}_C(\mathcal{D}, C)$ such that $H = \mathrm{stab}(\mathbf{m})$, where $\mathbf{m}$ is any maximal σ-ideal of $C(x) \otimes_D \mathcal{R}$.*

Proof. Note that $\mathcal{D} \subset \tilde{\mathcal{D}}, D \subset \tilde{D}$ and $\mathcal{R} \subset \tilde{\mathcal{R}}$. By Lemma 6.16, there is a Zariski dense subset $\tilde{U}_1$ of $\mathrm{Hom}_C(\tilde{\mathcal{D}}, C)$ such that for any $\tilde{c}_1 \in \tilde{U}_1$ if $\tilde{T} = C(x) \otimes_{\tilde{D}} \tilde{\mathcal{R}}$ is not the zero ring then $\tilde{T}$ is a σ-Picard-Vessiot ring over $C(x)$ for $\sigma(Y) = A^{\tilde{c}_1}Y$ and H is the σ-Galois group of $\tilde{T}$ over $C(x)$. By Lemma 4.5, there is a nonzero element in $\tilde{D}$, say a, such that for any $\tilde{c} \in \mathrm{Hom}_{C[x]}(\tilde{D}, C(x))$ if $\tilde{c}(a) \neq 0$ then $C(x) \otimes_{\tilde{D}} \tilde{\mathcal{R}}$ is not the zero ring. Write $a = a_1/a_2$ with $a_i \in \tilde{\mathcal{D}}[x]$. Let

$$U_1 = \{\tilde{c}_1|_{\mathcal{D}} \mid \tilde{c}_1 \in \tilde{U}_1 \text{ and } \tilde{c}_1(\mathrm{lc}(a_1a_2\mathfrak{h})) \neq 0\}$$

where $\mathrm{lc}(\cdot)$ denotes the leading coefficient of a polynomial in x. We shall prove that U_1 has the desired property. It is clear that U_1 is a Zariski dense subset of $\mathrm{Hom}_C(\mathcal{D}, C)$. Suppose $c_1 \in U_1$ and $\tilde{c}_1 \in \tilde{U}_1$ such that $c_1 = \tilde{c}_1|_D$. Then $\tilde{T} = C(x) \otimes_{\tilde{D}} \tilde{\mathcal{R}}$ is not the zero ring. We have the natural homomorphisms $\psi : T \to \tilde{T}, 1 \otimes_D \mathcal{X} \mapsto 1 \otimes_{\tilde{D}} \mathcal{X}$. Thus T is not the zero ring too. Set $\mathbf{m} = \ker(\psi)$. Then $\mathbf{m}$ is a maximal σ-ideal. Suppose $h \in H \subset \mathrm{Hom}_C(C[G], C)$. Then h induces a σ-$C(x)$-automorphism of T defined as $a \otimes_D b(\mathcal{X}) \mapsto a \otimes_D b(\mathcal{X}h(\mathcal{Z}))$ and also induces a σ-$C(x)$-automorphism of $\tilde{T}$ defined as $a \otimes_{\tilde{D}} b(\mathcal{X}) \mapsto a \otimes_{\tilde{D}} b(\mathcal{X}h(\mathcal{Z}))$. Suppose $\sum a_i \otimes_D b_i \in \mathbf{m}$, *i.e.* $\sum a_i \otimes_{\tilde{D}} b_i = 0$. Then $\sum a_i \otimes_{\tilde{D}} b_i(\mathcal{X}h(\mathcal{Z})) = 0$ by Lemma 6.16. This implies that $\sum a_i \otimes_D b_i(\mathcal{X}h(\mathcal{Z})) \in \mathbf{m}$. Hence $h \in \mathrm{stab}(\mathbf{m})$ and thus $H \subset \mathrm{stab}(\mathbf{m})$. As both H and $\mathrm{stab}(\mathbf{m})$ are the σ-Galois group of $\sigma(Y) = A^{c_1}Y$ over $C(x)$, $H = \mathrm{stab}(\mathbf{m})$ by an argument similar to that in Lemma 6.16. Finally, suppose that $\mathbf{m}'$ is another maximal σ-ideal of T. Then there is a $g \in G$ such that $\mathrm{stab}(\mathbf{m}') = g\mathrm{stab}(\mathbf{m})g^{-1}$ by Corollary 4.8. Since H is a normal subgroup of G, $\mathrm{stab}(\mathbf{m}') = gHg^{-1} = H$. □

Theorem 6.18. *There is a Zariski dense subset U_1 of $\mathrm{Hom}_C(\mathcal{D}, C)$ and a Zariski dense subset U_2 of $\mathrm{Hom}_C(C[x], C)$ such that for any $c_1 \in U_1$ and any $c_2 \in U_2$, $G = \mathrm{stab}(\mathbf{m})\mathrm{stab}(\mathbf{n})$, where $\mathbf{m}, \mathbf{n}$ are given as in Theorem 6.3 and Theorem 6.6 respectively.*

Proof. Let b be a nonzero element in D such that for any $c \in \mathrm{Hom}_C(D, k_0)$ with $c(b) \neq 0$, $k_0 \otimes_D \mathcal{R}$ is not the zero ring. Write $b = b_1/b_2$ where $b_1, b_2 \in \mathcal{D}[x]$ and $b_2 \neq 0$ and set

$$U_2 = \{c_2 \in \mathrm{Hom}_C(C[x], C) \mid b_1(c_2(x))b_2(c_2(x))\mathfrak{h}(c_2(x) + i) \neq 0 \ \forall i \in \mathbb{Z}\}.$$

We claim that U_2 is Zariski dense. Otherwise assume that $U_2 = \{u_1, \ldots, u_\ell\}$. Let $u_{\ell+1}, \ldots, u_\nu$ be all zeroes of b_1b_2 in C and $v_1, \ldots, v_s$ are all zeroes of $\mathfrak{h}$ in C. Then

$$C = \{u_i \mid 1 \leq i \leq \nu\} \cup \cup_{i=1}^{s}\{v_i + \mathbb{Z}\} \subseteq \mathbb{Z} + \sum_{i=1}^{\nu} \mathbb{Z}u_i + \sum_{i=1}^{s} \mathbb{Z}v_i \subseteq C.$$

This implies that C is a finitely generated $\mathbb{Z}$-module. However this is impossible, since $\mathbb{Q} \subset C$ is not finitely generated as a $\mathbb{Z}$-module. Due to Proposition 6.17, let U_1 be a Zariski dense subset of $\mathrm{Hom}_C(\mathcal{D}, C)$ such that for any $c_1 \in U_1$, $H = \mathrm{stab}(\mathbf{m})$. The theorem then follows from Proposition 6.13. □

Example 6.19. Let $U_1 = \{c \in C \mid c + \sqrt{c^2 - 1}$ is not a root of unit$\}$ and $U_2 = C$. From Examples 6.7, 6.8 and 6.14, one has that $G = \mathrm{stab}(\mathbf{m})\mathrm{stab}(\mathbf{n})$ for any $c_1 \in U_1$ and any $c_2 \in U_2$.

Data availability

No data was used for the research described in the article.

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