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RESEARCH

Rational First Integrals of Separable Differential Equations

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Abstract

In this paper, we present a necessary and sufficient condition for the existence of rational first integrals of the following separable differential equation:

$$\frac{dy}{dx} = f(x)g(y)$$

where $f(x)$, $g(y)$ are two univariate rational functions. We also present an algorithm to verify the condition and to compute a rational first integral when the condition is satisfied.

Keywords: Separable differential equation, Rational first integral, Hermite reduction

Mathematics Subject Classification: 16S32, 68W30

1 Introduction

Let $\mathbb{K}$ denote a field of characteristic zero, and $\mathbb{K}(x)$ be the differential field with usual derivation $\delta_x = \frac{d}{dx}$. Consider the following first order differential system

$$\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)} \tag{1}$$

where $P, Q \in \mathbb{K}[x,y]$ and $PQ \neq 0$. A rational first integral of the system (1) is a nonconstant rational function $R(x,y)$ such that $R(x,\eta(x))$ is a constant, i.e., $\delta_x(R(x,\eta(x))) = 0$, for any solution $\eta(x)$ of (1) that does not make the denominator of $R(x,y)$ vanish. A polynomial $S \in \mathbb{K}[x,y] \setminus \mathbb{K}$ is called a Darboux polynomial or special polynomial if all $\xi \in \overline{\mathbb{K}(x)}$ with $S(x,\xi) = 0$ are solutions of (1). Here and henceforth, the overline of a field denotes its algebraic closure. Rational first integrals are closely related to Darboux polynomials: the numerator and denominator of a rational first integral are Darboux polynomials, and conversely, the quotient of two different Darboux polynomials with the same cofactor is a rational first integral.

The problem of computing rational first integrals was already studied by Darboux in 1878. Darboux [1] showed that if there are enough Darboux polynomials, then the system will admit a rational first integral. Due to the lack of degree bounds, the computation of Darboux polynomials is quite difficult, and there is no complete algorithm for this task so far. In the celebrated work by Prelle and Singer [2], a procedure was presented to compute Darboux polynomials when the degree bound of the Darboux polynomials is given. Since then, there has been various literature regarding the computation of rational or elementary first integrals [3–12] and Darboux polynomials [5,13–15].

In this paper, we do not intend to develop algorithms for computing rational first integrals or Darboux polynomials for general first-order differential equations. Instead, we will focus on the necessary and sufficient conditions for the existence of rational first integrals for a first-order differential system. The first such condition was given by Darboux in [1]. Modern proofs were later presented in [7,16], and a generalization was also provided in [8,16].

Theorem 1 *Let $d = \max\{\deg(P), \deg(Q)\}$. Then the system (1) has a rational first integral if and only if (1) has at least $\frac{d(d+1)}{2} + 2$ irreducible Darboux polynomials.*

Due to the difficulty in computing Darboux polynomials, the condition given in Theorem 1 is hard to verify. In this paper, we will consider the following separable differential system:

$$\frac{dy}{dx} = f(x)g(y) \tag{2}$$

where $f(x) \in \mathbb{K}(x), g(y) \in \mathbb{K}(y)$ and $f(x)g(y) \neq 0$. Note that the system (2) always has an elementary first integral, see Remark 1. Here, we provide an effective necessary and sufficient condition for the existence of a rational first integral. To describe our result, we need to introduce Hermite reduction of rational functions. Suppose $h(x) \in \mathbb{K}(x)$. A Hermite reduction of $h(x)$ is

$$h(x) = \delta_x(h_1(x)) + h_2(x)$$

where $h_1(x), h_2(x) \in \mathbb{K}(x)$ satisfy

(a) $h_2(x)$ is proper, i.e., the degree of the numerator of $h_2(x)$ is less than the degree of its denominator;
(b) the denominator of $h_2(x)$ is squarefree.

The readers are referred to Chapter 2 of [17] for the details of Hermite reduction and referred to [18–21] for its various generalizations. Since the denominator of $\delta_x(h_1(x))$ can not be square-free, the Hermite reduction of $h(x)$ is unique, i.e., if $h(x) = \delta_x(\bar{h}_1(x)) + \bar{h}_2(x)$ is another Hermite reduction of $h(x)$ then $\delta_x(h_1(x)) = \delta_x(\bar{h}_1(x))$ and $h_2(x) = \bar{h}_2(x)$. The specific proof can be found in Lemma 2.1 of [22], where the uniqueness of a more general reduction is proven. Let

$$f(x) = \delta_x(f_1(x)) + f_2(x) \text{ and } \frac{1}{g(y)} = \delta_y(g_1(y)) + g_2(y) \tag{3}$$

be the Hermite reductions of $f(x)$ and $\frac{1}{g(y)}$ respectively. The main result of this paper is the following theorem.

Theorem 2 *The system (2) admits a rational first integral if and only if one of the following conditions holds:*

(a) $f_2(x) = 0 = g_2(y)$; *or*
(b) $\delta_x(f_1(x)) = \delta_y(g_1(y)) = 0$, *and there exists a nonzero* $\lambda \in \overline{\mathbb{K}}$ *such that*

$$f_2(x) = \lambda \frac{\delta_x(\tilde{f}(x))}{\tilde{f}(x)} \text{ and } g_2(y) = \lambda \frac{\delta_y(\tilde{g}(y))}{\tilde{g}(y)}$$

for some $\tilde{f}(x) \in \overline{\mathbb{K}}(x) \setminus \{0\}, \tilde{g}(y) \in \overline{\mathbb{K}}(y) \setminus \{0\}$.

The condition (b) of Theorem 2 implies that all residues of $f_2(x)$ and $g_2(y)$ are integer multiples of λ. In Sect. 2, we shall prove that these residues (and consequently λ) belong to an extension of $\mathbb{K}$ with a degree of at most two. Furthermore, we will demonstrate that the conditions of Theorem 2 can be verified efficiently, and rational first integrals, if they exist, can be computed in $O(\ell^3 \log(\ell))$ operations in $\mathbb{K}$, plus the cost of factoring a univariate polynomial over $\mathbb{K}$ of degree not greater than ℓ, where ℓ is the maximum of the degrees of the numerators and denominators of $f(x)$ and $g(y)$. See Algorithm 1 and the proof of its correctness.

The paper is organized as follows: In Sect. 2, we shall prove Theorem 2 and give a method to verify its conditions and to compute a rational first integral when one of these conditions is satisfied. In Sect. 3, we summarize the results from Sect. 2 into an algorithm and present some examples.

2 Proof of Theorem 2

Before presenting the proof of Theorem 2, let us define rational first integrals and Darboux polynomials in the language of differential algebra. As before, denote by δ_x and δ_y the usual derivations with respect to x and y. Equipped with δ_x and δ_y, $\overline{\mathbb{K}}(x, y)$ becomes a differential field and $\overline{\mathbb{K}}$ is the field of constants of $\overline{\mathbb{K}}(x, y)$. Set

$$D = P(x, y)\delta_x + Q(x, y)\delta_y,$$

where P, Q are given in (1). Then D is a new derivation on $\overline{\mathbb{K}}(x, y)$. It is clear that the field $\{a \in \overline{\mathbb{K}}(x, y) \mid D(a) = 0\}$ contains $\overline{\mathbb{K}}$. In the following, we shall use C_F to denote the field of constants of a differential field F.

Definition 1 A nonzero polynomial $h(x, y) \in \overline{\mathbb{K}}[x, y]$ such that h divides $D(h)$ is called a Darboux polynomial of D or the system (1). An element $h \in \overline{\mathbb{K}}(x, y) \setminus \overline{\mathbb{K}}$ such that $D(h) = 0$ is called a rational first integral of D or the system (1).

It is easy to see that if the system (1) has a rational first integral in $\overline{\mathbb{K}}(x, y)$ then it has a rational first integral in $\mathbb{K}(x, y)$, because $P, Q \in \mathbb{K}[x, y]$. Suppose that $\eta \in \overline{\mathbb{K}(x)}$ is an algebraic solution of the system (1) then the defining polynomial $h(x, y) \in \mathbb{K}[x, y]$ of η is an irreducible Darboux polynomial of D. Therefore Theorem 1 implies that if (1) has enough algebraic solutions then it admits a rational first integral. Conversely, we have the following well-known result.

Lemma 3 *Suppose that the system (1) admits a rational first integral and F is a differential field extension of $\mathbb{K}(x)$. Then any solution $\eta \in F$ of (1) is algebraic over $C_F(x)$.*

Proof Assume that $h(x, y) \in \mathbb{K}(x, y)$ is a rational first integral of (1) and that η is a solution of (1) in F. If η is a zero of the denominator of $h(x, y)$ then η is algebraic over $\mathbb{K}(x)$. Suppose that η is not a zero of the denominator of $h(x, y)$. Then $h(x, \eta) \in C_F$ and thus η is algebraic over $C_F(x)$. □

Let $f_2(x), g_2(y)$ be as in (3). We further write

$$f_2(x) = \sum_{i=1}^{m} \frac{\alpha_i}{x - a_i} \text{ and } g_2(y) = \sum_{j=1}^{n} \frac{\beta_j}{y - b_j}$$

where $a_i, b_j, \alpha_i, \beta_j \in \overline{\mathbb{K}}$ and $a_1, \ldots, a_m$ are distinct, $b_1, \ldots, b_n$ are distinct. Set

$$V = \mathrm{Span}_{\mathbb{Q}}\{\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n\} \tag{4}$$

the vector space over $\mathbb{Q}$ spanned by $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$. Let $\gamma_1, \ldots, \gamma_\ell$ be a basis of V. Write $\alpha_i = \sum_{s=1}^{\ell} d_{i,s}\gamma_s$, $\beta_j = \sum_{s=1}^{\ell} e_{j,s}\gamma_s$, where $d_{i,s}, e_{j,s} \in \mathbb{Q}$. Let p be the least common multiple of the denominators of the $d_{i,s}$ and the $e_{j,s}$, and set

$$R_s(x, y) = \left(\prod_{i=1}^{m}(x - a_i)^{pd_{i,s}}\right)\left(\prod_{j=1}^{n}(y - b_j)^{-pe_{j,s}}\right) \in \overline{\mathbb{K}}(x, y).$$

Remark 1 Set

$$I(x, y) = p(f_1(x) - g_1(y)) + \sum_{s=1}^{\ell} \gamma_s \ln(R_s(x, y)).$$

Applying δ_x to $I(x, y)$ yields that

$$\begin{aligned}\delta_x(I(x, y)) &= \delta_x(pf_1(x)) + \sum_{s=1}^{\ell} \gamma_s \frac{\delta_x(R_s(x, y))}{R_s(x, y)} \\ &= p\delta_x(f_1(x)) + \sum_{s=1}^{\ell} \gamma_s \left(\sum_{i=1}^{m} \frac{pd_{i,s}}{x - a_i}\right) = pf(x).\end{aligned}$$

Applying $f(x)g(y)\delta_y$ to $I(x, y)$ yields that

$$f(x)g(y)\delta_y(I(x, y)) = f(x)g(y)\left(-p\frac{1}{g(y)}\right) = -pf(x).$$

Since $f(x)g(y) \neq 0$, $\delta_x(I(x, y)) \neq 0$ and thus $I(x, y) \notin \overline{\mathbb{K}}$. On the other hand,

$$(\delta_x + f(x)g(y)\delta_y)(I(x, y)) = 0$$

and so $I(x, y)$ is an elementary first integral of the system (2).

The following lemma is essential. It demonstrates that if the system (2) has a rational first integral, then we can find a rational first integral in the set $\{f_1(x) - g_1(y), R_1(x, y), \ldots, R_\ell(x, y)\}$.

Lemma 4 *Suppose that the system (2) admits a rational first integral. Then for any* $h \in \{f_1(x) - g_1(y), R_1(x, y), \ldots, R_\ell(x, y)\}$, *if* $h \notin \overline{\mathbb{K}}$ *then h is a rational first integral of (2).*

Proof We have that

$$\begin{aligned}0 = f(x) - \frac{1}{g(y)}\frac{dy}{dx} &= (\delta_x(f_1(x)) + f_2(x)) - (\delta_y(g_1(y)) + g_2(y))\frac{dy}{dx} \\ &= \delta_x(f_1(x)) - \delta_y(g_1(y))\frac{dy}{dx} + \sum_{s=1}^{\ell} \gamma_s \left(\sum_{i=1}^{m} \frac{d_{i,s}}{x - a_i} - \sum_{j=1}^{n} \frac{e_{j,s}}{y - b_j}\frac{dy}{dx}\right).\end{aligned}$$

Suppose that η is a solution of (2) in F, a differential field extension of $\mathbb{K}(x)$. Replacing y with η in the above equalities yields that

$$\delta_x(f_1(x) - g_1(\eta)) + \sum_{s=1}^{\ell} \frac{\gamma_s}{p}\frac{\delta_x(R_s(x, \eta))}{R_s(x, \eta)} = 0.$$

Due to Lemma 3, η is algebraic over $C_F(x)$ and thus so are $f_1(x) - g_1(\eta)$ and the $R_s(x, \eta)$. Set $K = C_F(x)(f_1(x) - g_1(\eta), R_1(x, \eta), \ldots, R_\ell(x, \eta))$. Then the transcendence degree of K

over $C_F(x)$ is zero. By Theorem 1 of [23] (with $i=1$, $u_j = R_j(x,\eta), j=1,\dots,\ell$ and $v_1 = f_1(x) - g_1(\eta)$),

$$d(f_1(x) - g_1(\eta)) + \sum_{s=1}^{\ell} \frac{\gamma_s}{p} \frac{d(R_s(x,\eta))}{R_s(x,\eta)} = 0$$

where $d : K \longrightarrow \Omega_{K/C_F}$ is the universal derivation from K to the module of Kähler differentials. Proposition 4 of [23] implies that $f_1(x) - g_1(\eta)$ and the $R_s(x,\eta)$ are algebraic over C_F. Hence, $\delta_x(f_1(x) - g_1(\eta)) = 0$ and $\delta_x(R_s(x,\eta)) = 0$ for all $1 \le s \le \ell$. This concludes the lemma. □

Proposition 5 *Suppose that $\delta_x(f_1(x)) = \delta_y(g_1(y)) = 0$. If the system (2) admits a rational first integral then $\ell = \dim_{\mathbb{Q}}(V) = 1$, where V is given in (4).*

Proof We have that $f(x) = f_2(x)$ and $\frac{1}{g(y)} = g_2(y)$. Since $f(x)g(y) \neq 0$ and both $f_2(x)$ and $g_2(y)$ are proper, $f_2(x), g_2(y) \notin \mathbb{K}$. Without loss of generality, we may assume that $\prod_{i=1}^m \alpha_i \prod_{j=1}^n \beta_j \neq 0$, where α_i, β_j are as in (4). Then for each $1 \le i \le m$ and each $1 \le j \le n$, there are at least one of the $d_{i,s}$ that is not zero and at least one of the $e_{j,s}$ that is not zero. Hence $R_s \notin \overline{\mathbb{K}}$ for all $1 \le s \le \ell$. Lemma 4 implies that R_s is a rational first integral for all $1 \le s \le \ell$.

Applying $\frac{1}{g(y)}\delta_x + f(x)\delta_y$ $(= g_2(y)\delta_x + f_2(x)\delta_y)$ to R_s yields that

$$\begin{aligned} 0 &= \frac{g_2(y)\delta_x(R_s) + f_2(x)\delta_y(R_s)}{R_s} = g_2(y)\frac{\delta_x(R_s)}{R_s} + f_2(x)\frac{\delta_y(R_s)}{R_s} \\ &= \sum_{j=1}^n \frac{\beta_j}{y - b_j} \sum_{i=1}^m \frac{p d_{i,s}}{x - a_i} - \sum_{i=1}^m \frac{\alpha_i}{x - a_i} \sum_{j=1}^n \frac{p e_{j,s}}{y - b_j} \\ &= \sum_{i=1}^m \sum_{j=1}^n \frac{p(\beta_j d_{i,s} - \alpha_i e_{j,s})}{(x - a_i)(y - b_j)}. \end{aligned}$$

Since $a_1,\dots,a_m$ are distinct and $b_1,\dots,b_n$ are distinct, $\beta_j d_{i,s} - \alpha_i e_{j,s} = 0$ for all i,j,s. As not all $d_{1,1},\dots,d_{1,\ell}$ are zero, assume that $d_{1,s_1} \neq 0$. Then $\beta_j = \frac{e_{j,s_1}}{d_{1,s_1}}\alpha_1$ for all j, i.e., α_1, β_j are linearly dependent over $\mathbb{Q}$ for all j. Similarly, we have that β_1, α_i are linearly dependent over $\mathbb{Q}$ for all i. Then α_i, α_1 are also linearly dependent over $\mathbb{Q}$ and thus $\dim_{\mathbb{Q}}(V) = 1$. □

Proof of Theorem 2 Write $f(x) = \frac{p_1(x)}{q_1(x)}, g(y) = \frac{p_2(y)}{q_2(y)}$, where p_i and q_i are coprime polynomials. Set

$$D = q_1(x)q_2(y)\delta_x + p_1(x)p_2(y)\delta_y.$$

Suppose that (a) holds. Then $\frac{p_1(x)}{q_1(x)} = \delta_x(f_1(x))$ and $\frac{p_2(y)}{q_2(y)} = \frac{1}{\delta_y(g_1(y))}$. This implies that

$$\begin{aligned} D(f_1(x) - g_1(y)) &= q_1(x)q_2(y)\delta_x(f_1(x) - g_1(y)) + p_1(x)p_2(y)\delta_y(f_1(x) - g_1(y)) \\ &= p_1(x)q_2(y) - p_1(x)q_2(y) = 0. \end{aligned}$$

If $f_1(x) - g_1(y) \in \overline{\mathbb{K}}$ then both $f_1(x)$ and $g_1(y)$ must be in $\overline{\mathbb{K}}$. This implies that $f(x) = \delta_x(f_1(x)) = 0$ and $1/g(y) = \delta_y(g_1(y)) = 0$, which leads to a contradiction. Therefore, $f_1(x) - g_1(y) \notin \overline{\mathbb{K}}$, and thus $f_1(x) - g_1(y)$ is a rational first integral.

Now assume that the item (b) holds. Then

$$f(x) = \frac{p_1(x)}{q_1(x)} = \lambda \frac{\delta_x(\tilde{f}(x))}{\tilde{f}(x)} \text{ and } g(y) = \frac{p_2(y)}{q_2(y)} = \frac{\tilde{g}(y)}{\lambda \delta_y(\tilde{g}(y))}.$$

We have that

$$\begin{aligned} D(\tilde{f}(x)\tilde{g}(y)^{-1}) &= q_1(x)q_2(y)\delta_x(\tilde{f}(x))\tilde{g}(y)^{-1} - p_1(x)p_2(y)\delta_y(\tilde{g}(y))\tilde{g}(y)^{-2}\tilde{f}(x) \\ &= \left(\frac{1}{\lambda}p_1(x)q_2(y) - \frac{1}{\lambda}p_1(x)q_2(y)\right)\tilde{g}(y)^{-1}\tilde{f}(x) = 0. \end{aligned}$$

If $\tilde{f}(x) \in \overline{\mathbb{K}}$, then since $f(x) = \lambda \frac{\delta_x(\tilde{f}(x))}{\tilde{f}(x)}$, it follows that $f(x) = 0$, which contradicts the assumption that $f(x) \neq 0$. Therefore, $\tilde{f}(x) \notin \overline{\mathbb{K}}$. Similarly, $\tilde{g}(y) \notin \overline{\mathbb{K}}$. Thus $\tilde{f}(x)\tilde{g}(y)^{-1} \notin \overline{\mathbb{K}}$, and so $\tilde{f}(x)\tilde{g}(y)^{-1}$ is a rational first integral.

It remains to show that the item (a) or (b) is necessary. Suppose that the system (2) admits a rational first integral. By Lemma 4, for any $h \in \{f_1(x) - g_1(y), R_1(x,y), \ldots, R_\ell(x,y)\}$, if $h \notin \overline{\mathbb{K}}$ then h is a rational first integral. We first show that if $f_1(x) - g_1(y) \notin \overline{\mathbb{K}}$ then $f_2(x) = 0 = g_2(y)$. Assume that $f_1(x) - g_1(y) \notin \overline{\mathbb{K}}$. Then $f_1(x) - g_1(y)$ is a rational first integral, and we have that

$$\begin{aligned} 0 &= \frac{D(f_1(x) - g_1(y))}{q_1(x)q_2(y)} = \delta_x(f_1(x)) - f(x)g(y)\delta_y(g_1(y)) \\ &= \delta_x(f_1(x)) - f(x)g(y)\left(\frac{1}{g(y)} - g_2(y)\right) = \delta_x(f_1(x)) - f(x) + f(x)g(y)g_2(y) \\ &= -f_2(x) + f(x)g(y)g_2(y). \end{aligned}$$

Hence $\frac{f_2(x)}{f(x)} = g(y)g_2(y)$ and so $c = \frac{f_2(x)}{f(x)} = g(y)g_2(y) \in \mathbb{K}$. This implies that

$$0 = cf(x) - f_2(x) = \delta_x(cf_1(x)) + (c-1)f_2(x).$$

By the uniqueness of Hermite reduction, $(c-1)f_2(x) = 0 = \delta_x(cf_1(x))$. If $f_2(x) \neq 0$ then $c = 1$ and $\delta_x(f_1(x)) = 0$. So $b = f_1(x) \in \mathbb{K}$ and $f_1(x) - g_1(y) = b - g_1(y)$ is a rational first integral. This is impossible because it implies that $\delta_y(g_1(y)) = 0$, but $g_1(y) \notin \mathbb{K}$. Therefore, $f_2(x) = 0$. Similarly, $g_2(y) = 0$.

Now assume that $f_1(x) - g_1(y) \in \overline{\mathbb{K}}$. Then $f_1(x) \in \mathbb{K}$ and $g_1(y) \in \mathbb{K}$, i.e., $\delta_x(f_1(x)) = 0 = \delta_y(g_1(y))$. Proposition 5 implies that $\ell = \dim_{\mathbb{Q}}(V) = 1$. Using the notation introduced after Lemma 3, we set $w(x) = \prod_{i=1}^m (x - a_i)^{pd_{i,1}}$. Then

$$f_2(x) = \sum_{i=1}^m \frac{\alpha_i}{x - a_i} = \gamma_1 \sum_{i=1}^m \frac{d_{i,1}}{x - a_i} = \frac{\gamma_1}{p}\frac{\delta_x(w(x))}{w(x)}.$$

Similarly, set $v(y) = \prod_{j=1}^n (y - b_j)^{pe_{j,1}}$. Then we have that $g_2(y) = \frac{\gamma_1}{p}\frac{\delta_y(v(y))}{v(y)}$. This concludes the theorem. □

Suppose that $\delta_x(f_1(x)) = \delta_y(g_1(y)) = 0$. In the following, we shall explain how to verify the item (b) of Theorem 2. To this end, we first demonstrate that if the system (2) admits a rational first integral then the residues of $f_2(x)$ and $g_2(y)$ lie in a finite extension of $\mathbb{K}$ of degree at most two. This result has already been established in [24] for the case $\mathbb{K} = \mathbb{Q}$. For completeness, we provide an alternative proof here. For $A, B \in \mathbb{K}[x]$, $\mathrm{res}_x(A, B)$ denotes the Sylvester resultant of A and B with respect to x.

Definition 2 Let t be an indeterminate over $\mathbb{K}(x)$ and A, B be in $\mathbb{K}[x]$ with $\deg(B) > 0$, B squarefree and $\gcd(A, B) = 1$. We call $\mathrm{res}_x(B, A - t\delta_x(B)) \in \mathbb{K}[t]$ the residue polynomial of $\frac{A}{B}$, denoted by $\mathrm{respoly}(\frac{A}{B})$.

Concerning the residue polynomial, we have the following theorem, originally proved by Trager and Rothstein (see Theorem 22.8 on page 601 of [25]).

Theorem 6 *Let $A, B \in \mathbb{K}[x]$ be coprime with $\deg(A) < \deg(B)$, and B monic and squarefree. If $\mathbb{E}$ is an algebraic extension of $\mathbb{K}$, $c_1, \ldots, c_l \in \mathbb{E} \setminus \{0\}$ are pairwise distinct, and $v_1, \ldots, v_l \in \mathbb{E}[x] \setminus \mathbb{E}$ are monic, squarefree, and pairwise coprime, then the following are equivalent:*

(1) $\frac{A}{B} = \sum_{i=1}^{l} c_i \frac{\delta_x(v_i)}{v_i}$.
(2) The polynomial $\mathrm{respoly}(\frac{A}{B})$ *splits over $\mathbb{E}$ in linear factors, $c_1, \ldots, c_l$ are precisely the distinct roots of* $\mathrm{respoly}(\frac{A}{B})$*, and $v_i = \gcd(B, A - c_i\delta_x(B))$ for all $1 \le i \le l$.*

Proposition 7 *Assume that the system (2) admits a rational first integral and $\delta_x(f_1(x)) = 0 = \delta_y(g_1(y))$. Let $S = \mathrm{respoly}(f_2(x))\mathrm{respoly}(g_2(y))$. Then S takes one of the following forms:*

$$S = a\prod_{i=1}^{\ell}(t - r_i b)^{d_i}, \tag{5}$$

or

$$S = a\prod_{i=1}^{\ell}(t^2 - r_i^2 b)^{d_i}, \tag{6}$$

where $a, b \in \mathbb{K} \setminus \{0\}$, $r_i \in \mathbb{Q} \setminus \{0\}$, $d_i > 0$, and $r_i \neq r_j$ if $i \neq j$ in case (5), while $r_i^2 \neq r_j^2$ if $i \neq j$ and $t^2 - b$ is irreducible over $\mathbb{K}$ in case (6).

Proof Note that the denominators of $f_2(x)$ and $g_2(y)$ are squarefree. By Theorem 6, all residues of $f_2(x)$ and $g_2(y)$ are exactly the roots of S. Assume that $\gamma \in \overline{\mathbb{K}}$ is a zero of S, i.e., γ is a residue of $f_2(x)$ or $g_2(y)$. Due to Proposition 5, every root of S in $\overline{\mathbb{K}}$ is of the form $r\gamma$ for some nonzero $r \in \mathbb{Q}$. If $\gamma \in \mathbb{K}$ then it is easy to see that S takes the form (5). Now assume that $\gamma \notin \mathbb{K}$. Then $d = [\mathbb{K}(\gamma) : \mathbb{K}] > 1$. We shall show that S takes the form (6). Since $S \in \mathbb{K}[t]$, each conjugate of γ over $\mathbb{K}$ is still a root of S and thus it is of the form $r\gamma$ for some nonzero $r \in \mathbb{Q}$. Let $r_i\gamma$ be all conjugates of γ over $\mathbb{K}$, where $i = 1, \ldots, d$ and $r_1 = 1$. Denote by $\sigma_i(r_1, \ldots, r_d)$ the i-th elementary symmetric polynomial on $r_1, \ldots, r_d$. Suppose that $\sigma_{i'}(r_1, \ldots, r_d) \neq 0$ for some $1 \le i' \le d - 1$. Then $\sigma_{i'}(r_1\gamma, \ldots, r_d\gamma) = \sigma_{i'}(r_1, \ldots, r_d)\gamma^{i'} \in \mathbb{K}$ and thus $\gamma^{i'} \in \mathbb{K}$, which contradicts with $[\mathbb{K}(\gamma) : \mathbb{K}] = d > i'$. Hence $\sigma_i(r_1, \ldots, r_d) = 0$ for all $i = 1, \ldots, d - 1$. Therefore, the minimal polynomial of γ over $\mathbb{K}$ must be of the form $t^d - c$ for some $c \in \mathbb{K}$. This implies that every conjugate of γ over $\mathbb{K}$ is of the form $\xi\gamma$, where ξ is a d-th root of unity, and conversely. Hence, all d-th roots of unity must be rational numbers, so d must equal two. Consequently, the minimal polynomial of γ over $\mathbb{K}$ is of the form $t^2 - c$, and S takes the form (6). □

Remark 2 Assume that $\mathbb{K}$ is a field of characteristic zero, equipped with a polynomial factorization algorithm. Then we can decide whether $S \in \mathbb{K}[t]$ takes the form (5) or (6) in Proposition 7 as follows. Let $S = aS_1^{d_1} \ldots S_v^{d_v}$ be the irreducible factorization of S over $\mathbb{K}$.

(1) If $S_i = t - b_i$ for all $i = 1, \ldots, \nu$ and for each pair (i, j), b_i and b_j are linearly dependent over $\mathbb{Q}$ then S takes the form (5).
(2) If $S_i = t^2 - b_i$ for all $i = 1, \ldots, \nu$ and for each pair (i, j), b_i/b_j is a square of some rational number then S takes the form (6).
(3) In other cases, S does not take the form (5) or (6).

3 Algorithm and Examples

In this section, we summarize the previous results into an algorithm (see Algorithm 1). The correctness of the algorithm is guaranteed by Theorem 2 and Proposition 7. Additionally, we will illustrate our results with several examples. In the following, we assume that $\mathbb{K}$ is a field of characteristic zero, equipped with a polynomial factorization algorithm. We shall use $\mathrm{num}(\cdot)$ and $\mathrm{den}(\cdot)$ to denote the numerator and denominator of a rational function respectively. Furthermore, $\mathrm{den}(\cdot)$ is assumed to be monic.

Algorithm 1 FindRationalFirstIntegral
Input: $f(x) \in \mathbb{K}(x) \setminus \{0\}, g(y) \in \mathbb{K}(y) \setminus \{0\}$.
Output: *a rational first integral of (2) if it exists, or "No" otherwise.*

1. *Compute the Hermite reduction:*

$$f(x) = \delta_x(f_1(x)) + f_2(x) \text{ and } \frac{1}{g(y)} = \delta_y(g_1(y)) + g_2(y).$$

2. *If* $f_2(x) = g_2(y) = 0$ *then return* $f_1(x) - g_1(y)$.
3. *If* $\delta_x(f_1(x)) = \delta_y(g_1(y)) = 0$ *then*

   3.1. *Compute* $S = \mathrm{respoly}(f_2(x))$ *and* $T = \mathrm{respoly}(g_2(y))$.
   3.2. *Compute* $S = aS_1^{d_1} \ldots S_m^{d_m}$ *and* $T = bT_1^{e_1} \ldots T_n^{e_n}$, *irreducible factorization.*
   3.3. *If for all* i, j, $S_i = t - r_i\alpha$ *and* $T_j = t - s_j\alpha$ *with* $r_i, s_j \in \mathbb{Q}\setminus\{0\}, r_1 = 1$ *and* $\alpha \in \mathbb{K}\setminus\{0\}$ *then compute*

$$G_i = \gcd(\mathrm{den}(f_2), \mathrm{num}(f_2) - r_i\alpha\delta_x(\mathrm{den}(f_2))),$$
$$H_j = \gcd(\mathrm{den}(g_2), \mathrm{num}(g_2) - s_j\alpha\delta_y(\mathrm{den}(g_2)))$$

   and return

$$\left(\prod_{i=1}^{m} G_i^{r_i p}\right)\left(\prod_{j=1}^{n} H_j^{s_j p}\right)^{-1},$$

   where p is the least common multiple of the denominators of the r_i *and the* s_j.
   3.4. *If for all* i, j, $S_i = t^2 - r_i^2\alpha$ *and* $T_j = t^2 - s_j^2\alpha$ *with* $r_1 = 1$ *then compute*

$$G_{\pm i} = \gcd(\mathrm{den}(f_2), \mathrm{num}(f_2) \pm r_i\sqrt{\alpha}\delta_x(\mathrm{den}(f_2))),$$
$$H_{\pm j} = \gcd(\mathrm{den}(g_2), \mathrm{num}(g_2) \pm s_j\sqrt{\alpha}\delta_y(\mathrm{den}(g_2)))$$

   and return

$$\left(\prod_{i=1}^{m}\left(\frac{G_i}{G_{-i}}\right)^{r_i p}\right)\left(\prod_{j=1}^{n}\left(\frac{H_j}{H_{-j}}\right)^{s_j p}\right)^{-1}.$$

4. *Otherwise, return "No".*

As defined in Definition 8.26 on page 232 of [25], $\boldsymbol{M}(\ell)$ denotes the multiplication time for polynomials of degree less than ℓ, meaning that polynomials in $\mathbb{K}[x]$ of degree less than ℓ can be multiplied using at most $\boldsymbol{M}(\ell)$ operations in $\mathbb{K}$. For instance, if the classical algorithm is used, $\boldsymbol{M}(\ell) = 2\ell^2$.

Proposition 8 *Algorithm 1 works correctly and its runtime is $O(\ell \boldsymbol{M}(\ell)\log(\ell))$ operations in $\mathbb{K}$, plus the cost for factoring S and T over $\mathbb{K}$ in Step 3.2, where* $\ell = \max\{\deg(\mathrm{num}(f)), \deg(\mathrm{num}(g)), \deg(\mathrm{den}(f)), \deg(\mathrm{den}(g))\}$.

Proof From the proof of Theorem 2, if $f_2(x) = 0 = g_2(y)$ then $f_1(x) - g_1(y)$ is a rational first integral of the system (2). It remains to show that Step 3 works correctly. By Proposition 7, if the system (2) admits a rational first integral and $\delta_x(f_1(x)) = 0 = \delta_y(g_1(y))$ then ST takes the form (5) or (6). If ST takes the form (5) then Step 3.3 is performed. In this case, set $\tilde{f}(x) = \prod_{i=1}^m G_i^{r_i p}$ and $\tilde{g}(y) = \prod_{j=1}^n H_j^{s_j p}$, where G_i and H_j are given as in Step 3.3. Due to Theorem 6, one has that

$$f_2(x) = \sum_{i=1}^m r_i \alpha \frac{\delta_x(G_i)}{G_i} = \sum_{i=1}^m \frac{\alpha}{p} \frac{\delta_x(G_i^{r_i p})}{G_i^{r_i p}} = \frac{\alpha}{p} \frac{\delta_x\left(\prod_{i=1}^m G_i^{r_i p}\right)}{\prod_{i=1}^m G_i^{r_i p}} = \frac{\alpha}{p} \frac{\delta_x(\tilde{f}(x))}{\tilde{f}(x)},$$

$$g_2(y) = \sum_{j=1}^n s_j \alpha \frac{\delta_y(H_j)}{H_j} = \sum_{j=1}^n \frac{\alpha}{p} \frac{\delta_y(H_j^{s_j p})}{H_j^{s_j p}} = \frac{\alpha}{p} \frac{\delta_y\left(\prod_{j=1}^n H_j^{s_j p}\right)}{\prod_{j=1}^n H_j^{s_j p}} = \frac{\alpha}{p} \frac{\delta_y(\tilde{g}(y))}{\tilde{g}(y)}.$$

From the proof of Theorem 2, the output in Step 3.3, which is $\tilde{f}(x)\tilde{g}(y)^{-1}$, is a rational first integral. If ST takes the form (6) then Step 3.4 is performed. In this case, set

$$\tilde{f}(x) = \prod_{i=1}^m \left(\frac{G_i}{G_{-i}}\right)^{r_i p} \text{ and } \tilde{g}(y) = \prod_{j=1}^n \left(\frac{H_j}{H_{-j}}\right)^{s_j p}.$$

Again, by Theorem 6, one has that

$$f_2(x) = \sum_{i=1}^m r_i \sqrt{\alpha}\left(\frac{\delta_x(G_i)}{G_i} - \frac{\delta_x(G_{-i})}{G_{-i}}\right) = \frac{\sqrt{\alpha}}{p} \frac{\delta_x\left(\prod_{i=1}^m (G_i/G_{-i})^{r_i p}\right)}{\prod_{i=1}^m (G_i/G_{-i})^{r_i p}} = \frac{\sqrt{\alpha}}{p} \frac{\delta_x(\tilde{f}(x))}{\tilde{f}(x)},$$

$$g_2(y) = \sum_{j=1}^n s_j \sqrt{\alpha}\left(\frac{\delta_y(H_j)}{H_j} - \frac{\delta_y(H_{-j})}{H_{-j}}\right) = \frac{\sqrt{\alpha}}{p} \frac{\delta_y\left(\prod_{j=1}^n (H_j/H_{-j})^{s_j p}\right)}{\prod_{j=1}^n (H_j/H_{-j})^{s_j p}} = \frac{\sqrt{\alpha}}{p} \frac{\delta_y(\tilde{g}(y))}{\tilde{g}(y)}.$$

From the proof of Theorem 2 again, one sees that the output in Step 3.4, which is $\tilde{f}(x)\tilde{g}(y)^{-1}$, is a rational first integral. This concludes the correctness of the algorithm.

By Theorem 22.7 on page 600 of [25], Step 1 requires $O(\boldsymbol{M}(\ell)\log(\ell))$ operations in $\mathbb{K}$ and one sees that $\deg(\mathrm{den}(g_2)), \deg(\mathrm{den}(f_2)) \leq \ell$. Due to Corollary 11.18 on page 310 of [25], Step 3.1 requires $O(\ell \boldsymbol{M}(\ell)\log(\ell))$ operations in $\mathbb{K}$. It is straightforward to verify that $\deg(S), \deg(T) \leq \ell$. Note that both S_i and T_j are of degree not greater than two. By Corollary 11.6 on page 304 and Corollary 11.8 on page 305 of [25], computing $G_{\pm i}$ or $H_{\pm j}$ requires $O(\boldsymbol{M}(\ell)\log(\ell))$ operations in $\mathbb{K}$. So the total cost for Steps 3.3 and 3.4 is $O(\ell \boldsymbol{M}(\ell)\log(\ell))$. Therefore, the overall runtime for Algorithm 1 is $O(\ell \boldsymbol{M}(\ell)\log(\ell))$ operations in $\mathbb{K}$, plus the cost for factoring S and T in Step 3.2. ◻

Set

$$C=\left\{h\in\overline{\mathbb{K}}(x,y)\mid \frac{1}{g(y)}\delta_x(h)+f(x)\delta_y(h)=0\right\}. \tag{7}$$

Then C is a subfield of $\overline{\mathbb{K}}(x,y)$ containing $\overline{\mathbb{K}}$. Suppose that the system (2) admits a rational first integral. Then $C\setminus\overline{\mathbb{K}}\neq\emptyset$. We claim that tr.deg$(C/\overline{\mathbb{K}})$, the transcendence degree of C over $\overline{\mathbb{K}}$, is equal to one. Assume, for contradiction, that tr.deg$(C/\overline{\mathbb{K}})=2$. Then y is algebraic over C and thus $y\in C$. This implies

$$0=\frac{1}{g(y)}\delta_x(y)+f(x)\delta_y(y)=f(x),$$

which contradicts the assumption that $f(x)\neq 0$. This proves the claim. Due to a theorem of Gordan and Igusa (see Theorem 3 on page 15 of [26]), we have $C=\overline{\mathbb{K}}(\tilde h)$ for some $\tilde h\in C$. We refer to such $\tilde h$ as a generator of C over $\overline{\mathbb{K}}$.

Lemma 9 *Assume that the system (2) admits a rational first integral and C is as in (7). Let $\tilde h$ be the output of Algorithm 1, and let h be a generator of C over $\overline{\mathbb{K}}$. Then h can be chosen such that* den(h) *divides* den$(\tilde h)$. *Furthermore, if* $\deg(\text{den}(\tilde h),y)>0$ *then* $\deg(\text{den}(h),y)>0$, *and if $\tilde h=u(x)v(y)$ for some $u(x)\in\overline{\mathbb{K}}(x)$, $v(y)\in\overline{\mathbb{K}}(y)$ then h can be chosen to have the same form.*

Proof Since $\tilde h\in C=\overline{\mathbb{K}}(h)$, there exist two coprime polynomials $P,Q\in\overline{\mathbb{K}}[z]$ such that $\tilde h=P(h)/Q(h)$. Write $h=h_1/h_2$, where $h_1,h_2\in\overline{\mathbb{K}}[x,y]$ and they are coprime. Set

$$\bar P=Y^dP\left(\frac{X}{Y}\right),\ \bar Q=Y^dQ\left(\frac{X}{Y}\right),$$

where $d=\max\{\deg(P),\deg(Q)\}$. Then

$$\tilde h=\frac{\bar P(h_1,h_2)}{\bar Q(h_1,h_2)}. \tag{8}$$

By Lemma on page 16 of [26], $\bar P(h_1,h_2)$ and $\bar Q(h_1,h_2)$ are coprime. Hence $\bar Q(h_1,h_2)=\alpha\,\text{den}(\tilde h)$ for some nonzero $\alpha\in\overline{\mathbb{K}}$. Let $a_1h_1-b_1h_2$ be a factor of $\bar P(h_1,h_2)$ and $a_2h_1-b_2h_2$ be a factor of $\bar Q(h_1,h_2)$, where $a_i,b_i\in\overline{\mathbb{K}}$ and $a_1b_2-a_2b_1\neq 0$. Then $a_2h_1-b_2h_2$ divides den$(\tilde h)$ and $(a_1h_1-b_1h_2)/(a_2h_1-b_2h_2)$ is also a generator of C.

Suppose that $\deg(\text{den}(\tilde h),y)>0$. We take $a_2h_1-b_2h_2$ to be a factor of $\bar Q(h_1,h_2)$ such that $\deg(a_2h_1-b_2h_2,y)>0$. In this case, we take $h=(a_1h_1-b_1h_2)/(a_2h_1-b_2h_2)$ and then $\deg(\text{den}(h),y)>0$. Now assume that $\tilde h=u(x)v(y)$. Then both $a_1h_1-b_1h_2$ and $a_2h_1-b_2h_2$ are of the form $p(x)q(y)$, where $p(x)\in\overline{\mathbb{K}}[x]$, $q(y)\in\overline{\mathbb{K}}[y]$. Hence h can be taken to have the same form as $\tilde h$. □

Proposition 10 *Suppose that C is as in (7) and $f_1(x)-g_1(y)$ is the output of Algorithm 1. Then $f_1(x)-g_1(y)$ is a generator of C.*

Proof Note that in this case, $f(x)=\delta_x(f_1(x))$ and $\frac{1}{g(y)}=\delta_y(g_1(y))$. Since $fg\neq 0$, we have $\delta_x(f_1(x))\neq 0$ and $\delta_y(g_1(y))\neq 0$. Let $h(x,y)$ be a generator of C. We first show that $h(x,y)$ can be chosen to have the same form as $f_1(x)-g_1(y)$. By Lemma 9, we may write $h(x,y)=\frac{u(x,y)}{p(x)q(y)}$, where $p(x)$ divides den$(f_1(x))$, $q(y)$ divides den$(g_1(y))$, $u(x,y)\in\overline{\mathbb{K}}[x,y]$, and $u(x,y)$ and $p(x)q(y)$ are coprime. Moreover, if $\deg(\text{den}(g_1))>0$ then $\deg(q(y))>0$. We regard h as a rational function in $\mathcal{K}(y)$, where $\mathcal{K}=\overline{\mathbb{K}}(x)$. For a rational function $w(y)\in\mathcal{K}(y)$, denote by $\nabla_w=w_n(Y)w_d(Z)-w_n(Z)w_d(Y)$, where w_n and w_d are the numerator and denominator of w, respectively. We have $\nabla_{f_1(x)-g_1(y)} = -\nabla_{g_1(y)}$. Since $f_1(x) - g_1(y) \in C = \overline{\mathbb{K}}(h)$, by Proposition 3.1 of [27], ∇_h divides $\nabla_{f_1(x)-g_1(y)}$. Since $\nabla_{f_1(x)-g_1(y)} = -\nabla_{g_1(y)} \in \overline{\mathbb{K}}[Y,Z]$, we have

$$\nabla_h = p(x)(u(x,Y)q(Z) - u(x,Z)q(Y)) = a(x)M(Y,Z)$$

where $M(Y,Z) \in \overline{\mathbb{K}}[Y,Z]$ and $a(x) \in \mathcal{K}$. Let $c_1 \in \overline{\mathbb{K}}$ be such that $p(c_1), a(c_1)$ are well-defined, $a(c_1) \neq 0$, and $u(c_1,y)$ and $q(y)$ are coprime. We then have that

$$M(Y,Z) = \frac{1}{a(c_1)} p(c_1)(u(c_1,Y)q(Z) - u(c_1,Z)q(Y)) = \frac{1}{a(c_1)} \nabla_{h(c_1,y)},$$

and thus

$$\nabla_h = a(x)M(Y,Z) = \frac{a(x)}{a(c_1)} \nabla_{h(c_1,y)}.$$

By Proposition 3.1 of [27] again, we further have $h(x,y) = w_1(h(c_1,y))$ and $h(c_1,y) = w_2(h(x,y))$ for some $w_1, w_2 \in \mathcal{K}(z)$. This implies that $w_1(z) = (b_{11}z + b_{12})/(b_{21}z + b_{22})$, where $b_{ij} \in \overline{\mathbb{K}}[x]$ and $b_{11}b_{22} - b_{12}b_{21} \neq 0$. In other words,

$$h(x,y) = \frac{u(x,y)}{p(x)q(y)} = \frac{b_{11}u(c_1,y) + b_{12}p(c_1)q(y)}{b_{21}u(c_1,y) + b_{22}p(c_1)q(y)}.$$

Since $u(x,y)$ and $p(x)q(y)$ are coprime, $q(y)$ divides $b_{21}u(c_1,y) + b_{22}p(c_1)q(y)$, viewed as polynomials in y. Furthermore, because $u(c_1,y)$ and $q(y)$ are coprime, $b_{21} = 0$. Therefore $h(x,y) = \tilde{a}(x)h(c_1,y) + \tilde{b}(x)$ for some $\tilde{a}, \tilde{b} \in \overline{\mathbb{K}}(x)$. Note that if $h(c_1,y) \in \overline{\mathbb{K}}$ or $\tilde{a}(x) = 0$ then $\delta_y(h(x,y)) = 0$. In this case, the equality $\delta_x(f_1)\delta_y(h(x,y)) + \delta_y(g_1)\delta_x(h(x,y)) = 0$ with $\delta_y(g_1) \neq 0$ implies that $\delta_x(h(x,y)) = 0$. This leads to the conclusion that $h(x,y) \in \overline{\mathbb{K}}$, which is a contradiction since $\mathrm{tr.deg}(C/\overline{\mathbb{K}}) = 1$ and $C = \overline{\mathbb{K}}(h(x,y))$. Hence $h(c_1,y) \notin \overline{\mathbb{K}}$ and $\tilde{a}(x) \neq 0$.

Now, substituting $h(x,y) = \tilde{a}(x)h(c_1,y) + \tilde{b}(x)$ into $\delta_y(g_1)\delta_x + \delta_x(f_1)\delta_y$ gives

$$\delta_y(g_1)\delta_x(\tilde{a})h(c_1,y) + \delta_y(g_1)\delta_x(\tilde{b}) + \delta_x(f_1)\delta_y(h(c_1,y))\tilde{a} = 0. \tag{9}$$

Assume that $h(c_1,y)$ is a polynomial in y, i.e., $\deg(q(y)) = 0$. Then $h(x,y)$ is also a polynomial in y. By the choice of $h(x,y)$, g_1 is a polynomial in y as well. Since $\delta_y(g_1) \neq 0$, $\deg(\delta_y(g_1)) \geq 0$. This together with $\deg(h(c_1,y)) > 0$ implies

$$\deg(\delta_y(g_1)h(c_1,y)) > \max\{\deg(\delta_y(g_1)), \deg(\delta_y(h(c_1,y)))\}.$$

Equality (9) then implies that $\delta_x(\tilde{a}(x)) = 0$. Suppose instead that $h(c_1,y)$ is not a polynomial, i.e., $\deg(q(y)) > 0$. Let c_2 be a zero of $q(y)$ in $\overline{\mathbb{K}}$. Then $\mathrm{ord}_{c_2}(h(c_1,y)) < 0$, where $\mathrm{ord}_{c_2}(\cdot)$ denotes the order of a rational function at $y = c_2$. Because $q(y)$ divides $\mathrm{den}(g_1)$, c_2 is a pole of g_1 and so $\mathrm{ord}_{c_2}(\delta_y(g_1)) \leq -2$. Consequently,

$$\mathrm{ord}_{c_2}(\delta_y(g_1)h(c_1,y)) < \min\{\mathrm{ord}_{c_2}(\delta_y(g_1)), \mathrm{ord}_{c_2}(\delta_y(h(c_1,y)))\}.$$

Similarly, equality (9) implies that $\delta_x(\tilde{a}(x)) = 0$. In summary, $\tilde{a}(x) \in \overline{\mathbb{K}} \setminus \{0\}$ and $h(x,y) = \tilde{a}(x)h(c_1,y) + \tilde{b}(x)$, which is of the form as $f_1(x) - g_1(y)$. Finally, since $\delta_x(f_1)\delta_y(g_1) \neq 0$ and $\delta_x(\tilde{a}) = 0$, by (9), we have

$$\frac{\delta_x(\tilde{b})}{\delta_x(f_1)} = -\frac{\delta_y(\tilde{a}h(c_1,y))}{\delta_y(g_1)}.$$

Because the left-hand side of the above equality is independent of y and the right-hand side is independent of x, it follows that $\alpha = -\delta_y(\tilde{a}h(c_1,y))/\delta_y(g_1) \in \overline{\mathbb{K}}$. Furthermore, since

$\tilde{a} \in \overline{\mathbb{K}}\backslash\{0\}$ and $h(c_1, y) \in \overline{\mathbb{K}}(y)\backslash\overline{\mathbb{K}}$, we conclude that $\alpha \neq 0$, $\tilde{a}h(c_1, y) = -\alpha g_1(y) + \beta_1$ and $\tilde{b}(x) = \alpha f_1(x) + \beta_2$ for some $\beta_1, \beta_2 \in \overline{\mathbb{K}}$. Consequently,

$$h(x, y) = \alpha(f_1(x) - g_1(y)) + \beta_1 + \beta_2$$

and thus $f_1(x) - g_1(y) = \frac{1}{\alpha}(h(x, y) - \beta_1 - \beta_2)$ is a generator of C. □

Proposition 11 *Suppose that C is as in (7) and $\tilde{f}(x)\tilde{g}(y)^{-1}$ is the output of Algorithm 1. Then $\tilde{f}(x)\tilde{g}(y)^{-1}$ is a generator of C.*

Proof Note that in this case $f(x) = f_2(x)$ and $\frac{1}{g(y)} = g_2(y)$. Suppose that $h(x, y)$ is a generator of C. By Lemma 9, $h(x, y)$ can be chosen to have the form $u(x)v(y)$, where $u(x) \in \overline{\mathbb{K}}(x)$ and $v(y) \in \overline{\mathbb{K}}(y)$. Substituting $h(x, y)$ into $f_2(x)\delta_y + g_2(y)\delta_x$ yields

$$f_2(x)\frac{\delta_y(v(y))}{v(y)} + g_2(y)\frac{\delta_x(u(x))}{u(x)} = 0.$$

This implies that $\frac{\delta_x(u(x))}{u(x)} = \beta f_2(x)$ and $\frac{\delta_y(v(y))}{v(y)} = -\beta g_2(y)$ for some nonzero $\beta \in \overline{\mathbb{K}}$. From the proof of Proposition 8, we have

$$\frac{\delta_x(u(x))}{u(x)} = \beta f_2(x) = \beta\tilde{\alpha}\frac{\delta_x(\tilde{f}(x))}{\tilde{f}(x)} \text{ and } \frac{\delta_y(v(y))}{v(y)} = -\beta g_2(y) = -\beta\tilde{\alpha}\frac{\delta_y(\tilde{g}(y))}{\tilde{g}(y)}, \tag{10}$$

where $\tilde{\alpha} = \frac{\alpha}{p}$ or $\tilde{\alpha} = \frac{\sqrt{\alpha}}{p}$, and the residues of $\frac{\delta_x(\tilde{f}(x))}{\tilde{f}(x)}$ and $\frac{\delta_y(\tilde{g}(y))}{\tilde{g}(y)}$ are the r_ip and s_jp, respectively. On the other hand, the residues of $\frac{\delta_x(u(x))}{u(x)}$ and $\frac{\delta_y(v(y))}{v(y)}$ are integers. Therefore,

$$\beta\tilde{\alpha}r_ip = d_i \in \mathbb{Z} \text{ and } \beta\tilde{\alpha}s_jp = e_j \in \mathbb{Z}$$

for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Since $r_1 = 1$, $\beta\tilde{\alpha}p = d_1$, which implies $r_i = \frac{d_i}{d_1}$ and $s_j = \frac{e_j}{d_1}$ for all i, j. Note that p is the least common multiple of the denominators of the r_i and s_j. Thus, $p \mid d_1$, and we conclude that $\nu = \beta\tilde{\alpha} \in \mathbb{Z}$. From (10), $u(x) = c_1\tilde{f}(x)^{\nu}$ and $v(y) = c_2(\tilde{g}(y))^{-\nu}$ for some nonzero $c_1, c_2 \in \overline{\mathbb{K}}$. This implies

$$h(x, y) = u(x)v(y) = c_1c_2\left(\tilde{f}(x)\tilde{g}(y)^{-1}\right)^{\nu} \in \overline{\mathbb{K}}(\tilde{f}(x)\tilde{g}(y)^{-1}),$$

and so $C = \overline{\mathbb{K}}(h(x, y)) = \overline{\mathbb{K}}(\tilde{f}(x)\tilde{g}(y)^{-1})$, i.e., $\tilde{f}(x)\tilde{g}(y)^{-1}$ is a generator of C. □

Example 1 Consider

$$\frac{dy}{dx} = \frac{y^2 - 2}{x^2 - 2}.$$

We have that

$$S = \text{respoly}\left(\frac{1}{x^2 - 2}\right) = 1 - 8t^2 \text{ and } T = \text{respoly}\left(\frac{1}{y^2 - 2}\right) = 1 - 8t^2.$$

So ST takes the form (6). In Step 3.4, we set $\alpha = 1/8$, $r_1 = s_1 = 1$. Then

$$\begin{aligned} G_1 &= \gcd(x^2 - 2, 1 - \sqrt{2}x/2) = x - \sqrt{2}, \\ G_{-1} &= \gcd(x^2 - 2, 1 + \sqrt{2}x/2) = x + \sqrt{2}, \\ H_1 &= \gcd(y^2 - 2, 1 - \sqrt{2}y/2) = y - \sqrt{2}, \\ H_{-1} &= \gcd(y^2 - 2, 1 + \sqrt{2}y/2) = y + \sqrt{2}, \end{aligned}$$

and a rational first integral of the system is

$$\frac{(x-\sqrt{2})(y+\sqrt{2})}{(x+\sqrt{2})(y-\sqrt{2})}.$$

Example 2 Consider

$$\frac{dy}{dx}=\frac{y(ax-b)}{x(cy-d)}$$

where $a, b, c, d \in \mathbb{K}$, which corresponds to the Lotka-Volterra equation

$$\frac{dx}{dt}=x(cy-d);$$

$$\frac{dy}{dt}=y(ax-b).$$

To apply our result, we assume that $(ax-b)(cy-d)\neq 0$. We have that

$$f(x)=\frac{ax-b}{x}=\delta_x(ax)-\frac{b}{x} \text{ and } \frac{1}{g(y)}=\frac{cy-d}{y}=\delta_y(cy)-\frac{d}{y}.$$

By Theorem 2, the above system admits a rational first integral if and only if one of the following conditions holds:

(a) $b=d=0$;
(b) $a=c=0$ and there exists a $\lambda\in\overline{\mathbb{K}}\setminus\{0\}$ such that $-\frac{b}{x}=\lambda\frac{\delta_x(\tilde{f}(x))}{\tilde{f}(x)}$ and $-\frac{d}{y}=\lambda\frac{\delta_y(\tilde{g}(y))}{\tilde{g}(y)}$ for some nonzero $\tilde{f}(x)\in\overline{\mathbb{K}}(x)$ and nonzero $\tilde{g}(y)\in\overline{\mathbb{K}}(y)$.

In case (a), $ax-cy$ is a rational first integral. In case (b), $\tilde{f}(x)$ must be of the form $c_1x^{e_1}$ and $\tilde{g}(y)$ must be of the form $c_2y^{e_2}$, where e_1, e_2 are nonzero integers and $c_1, c_2\in\overline{\mathbb{K}}$ are nonzero constants. This implies that $\frac{b}{d}=\frac{e_1}{e_2}$ and $x^{e_1}y^{-e_2}$ is a rational first integral.

Example 3 Consider

$$\frac{dy}{dx}=\frac{y^3+1}{x^3+1}. \tag{11}$$

We have that

$$S=\text{respoly}\left(\frac{1}{x^3+1}\right)=1-27t^3=-\frac{1}{27}(t-1)(t^2+t+1).$$

Since S does not take the form (5) or (6), the system (11) has no rational first integral. Note that if $\eta\in\overline{\mathbb{K}(x)}$ is a solution of the system (11) then the irreducible polynomial $h(x,y)\in\mathbb{K}[x,y]$ with $h(x,\eta)=0$ is an irreducible Darboux polynomial of D, where

$$D=(x^3+1)\delta_x+(y^3+1)\delta_y.$$

On the other hand, Theorem 1 implies that the system (11) has at most seven irreducible Darboux polynomials. Thus, it has at most seven solutions in $\overline{\mathbb{K}(x)}$. In fact, we shall show that, except for $y=x,-1,\frac{1\pm\sqrt{-3}}{2}$, the system has no other solutions in $\overline{\mathbb{K}(x)}$. A straightforward calculation implies that

$$y-x, x+1, x-\frac{1\pm\sqrt{-3}}{2}, y+1 \text{ and } y-\frac{1\pm\sqrt{-3}}{2}$$

are irreducible Darboux polynomials of D. If the system (11) had a solution in $\overline{\mathbb{K}(x)}$ other than x, -1 or $\frac{1\pm\sqrt{-3}}{2}$ then it would have one more irreducible Darboux polynomial beyond the seven polynomials listed above, implying that it admits a rational first integral, which leads to a contradiction. We conclude that $y = x, -1, \frac{1\pm\sqrt{-3}}{2}$ are the only solutions of the system in $\overline{\mathbb{K}(x)}$.

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Data Availability
No datasets were generated or analysed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

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