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Mechanical theorem proving in the surfaces using the characteristic set method and Wronskian determinant

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Abstract In this paper, we generalize the method of mechanical theorem proving in curves to prove theorems about surfaces in differential geometry with a mechanical procedure. We improve the classical result on Wronskian determinant, which can be used to decide whether the elements in a partial differential field are linearly dependent over its constant field. Based on Wronskian determinant, we can describe the geometry statements in the surfaces by an algebraic language and then prove them by the characteristic set method.

Keywords: mechanical theorem proving, Wu-Ritt's characteristic set method, local theory of surface, Wronskian determinant

MSC(2000): 12H99, 53A05

1 Introduction

Mechanical theorem proving in geometry on computers began in the 1950s with the work of Gelernter[1] and his collaborators. At that time, their method was limited to the traditional proofs and therefore was not able to prove nontrivial geometry theorems effectively. Since late 1970s, the appearance of several successful methods, such as Wu's method[2–4] and Gröbner Bases method[5], has turned mechanical theorem proving in geometry into a new era. Based on the Wu-Ritt's characteristic set method, Chou[6] has mechanically proved and discovered hundreds of nontrivial theorems in Euclidean geometry and non-Euclidean geometry. In 1993, Chou and Gao presented an improved version of Wu-Ritt's decomposition algorithm on differential polynomials and proved many nontrivial theorems in differential geometry and elementary mechanics such as Bertrand's Theorem and Newton's gravitational laws (cf. [7–10]). Cao and Li[11] proposed a mechanical theorem proving algorithm on local properties of curves on space surfaces through the exterior differential calculation and vector formulation. With the help of the characteristic set method in algebraic differential polynomials, Li[12] discovered a new relation between the first and second fundamental forms of a surface without umbilici and proved

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some nontrivial theorems. Li[13] proved a famous theorem of Chern in differential geometry using the characteristic set method and the calculus of differential forms. Ferro[14] proposed a method to prove theorems in differential geometry based on the Gröbner basis method.

In this paper, we generalize the results given by Chou and Gao[8] to surface theory. The method is based on Wu-Ritt's characteristic set method and a simplified version of Wronskian determinants. An important step of mechanical theorem proving in differential geometry is to convert the geometry statements into the algebraic statements. The classical Wronskian determinant provides an algebraic language to describe the geometry statements such as "The tangent lines pass through a fixed point" in the case of curves. In the case of surfaces, Wronskian determinant needs to be generalized to describe similar statements as above. This has been done by Kolchin (cf. [15, p. 86, Theorem 1]). In Kolchin's result, in order to decide whether n elements in a differential field with two derivation operators are linearly dependent, the number of the determinants to be calculated in the worst case is almost $n!(n+1)!/2^n$. In this paper, we improve this result. We only need to compute n^2-2n+2 determinants in the worst case. This greatly simplifies the computation.

We implemented our method in Maple and used the program to prove many theorems of surfaces. In the process of computing characteristic sets, the main difference between ordinary differential polynomials and partial differential polynomials is that the integrability conditions are introduced in the case of partial differential polynomials. The integrability conditions make the computation costly. In order to prove more difficult theorems about surfaces, we need more efficient methods to compute the characteristic sets for partial differential polynomials.

The rest of this paper is organized as follows. First, we introduce Wu-Ritt's characteristic set method which we will use and the formulation of mechanical theorem proving in differential geometry. In the third part of this paper we improve the results in [15, p. 86, Theorem 1] and give some basic languages to describe the geometry statements. Finally some examples are given.

2 Wu-Ritt's characteristic set method

In this section, we will introduce some basic concepts and results on differential algebra and characteristic set method. More details can be found in [15, Chapter 1]; [4, Chapter 1]; [16]. A differential field is an algebraic field $\mathscr{F}$ with a finite set $\Delta = \{\delta_i, i = 1, \ldots, m\}$ of derivation operators such that for every $a, b \in \mathscr{F}$,

(1) $\delta_i(a+b) = \delta_i a + \delta_i b$;

(2) $\delta_i(ab) = b\delta_i a + a\delta_i b$;

(3) $\delta_j(\delta_i a) = \delta_i(\delta_j a)$, for every i and j.

Let Θ be the free commutative monoid generated by the elements of Δ. Every element θ in Θ has the form $\prod \delta_i^{k_i}$. We call the integer $\sum k_i$ the order of θ, denoted by $\mathrm{ord}(\theta)$. Let $\mathscr{E}, \mathscr{F}$ be two differential fields. If $\mathscr{F} \subseteq \mathscr{E}$ and when the derivation operators in $\mathscr{E}$ are restricted in $\mathscr{F}$, they are compatible with those in $\mathscr{F}$, then $\mathscr{F}$ is called a subfield of $\mathscr{E}$ and $\mathscr{E}$ is called an extension field of $\mathscr{F}$. In this paper, we always let $\mathscr{F}$ be the rational function field $\mathbb{R}(u,v)$ in variables u and v with derivation operators $\partial/\partial u$ and $\partial/\partial v$ where $\mathbb{R}$ is the real number field. For convenience, we use $\partial_{i,j}$ to denote $\frac{\partial^{i+j}}{\partial u^i \partial v^j}$ for any non-negative integers i and j. Let $x_1, \ldots, x_n$ be indeterminates. An ordinary polynomial P in variables $\frac{\partial^{i+j} x_k}{\partial u^i \partial v^j}$, where $k = 1, 2, \ldots, n$ and i, j are nonnegative integers with coefficients in $\mathscr{F}$ is called a differential polynomial in $x_1, x_2, \dots, x_n$. We denote the set of all differential polynomials in $x_1, x_2, \dots, x_n$ by $\mathscr{F}\{x_1, x_2, \dots, x_n\} = \mathscr{F}\{X\}$. A non-empty subset I of $\mathscr{F}\{X\}$ is called an ideal of $\mathscr{F}\{X\}$ if for any $f, g \in I, h \in \mathscr{F}\{X\}$, we have $f+g \in I$, $fg \in I$, $hf \in I$, $\partial g/\partial u$ and $\partial g/\partial v \in I$. An ideal I is called a radical ideal of $\mathscr{F}\{X\}$ if for any $f \in \mathscr{F}\{X\}$ and a positive integer n, $f^n \in I$ implies $f \in I$. Let $\mathbb{S}$ be a non-empty set in $\mathscr{F}\{X\}$, the minimal ideal I containing the set $\mathbb{S}$ is called the ideal generated by $\mathbb{S}$ and denoted by Ideal($\mathbb{S}$). In fact, Ideal($\mathbb{S}$) is the set of all linear combinations of the partial differential polynomials in $\mathbb{S}$ and their partial derivatives. We will use $\{\mathbb{S}\}$ to denote the radical ideal generated by $\mathbb{S}$.

Definition 2.1. *A ranking of $(x_1, x_2, \dots, x_n)$ is a total order on $\Theta X = \{\theta x_i, i = 1, 2, \dots, n, \theta \in \Theta\}$ such that for any derivative $u \in \Theta X$ and $\partial \in \Delta$, we have $\partial u \geqslant u$, and for any pair of derivatives $u, v \in \Theta X$, and $\partial \in \Delta$ with $u \geqslant v$ we have that $\partial u \geqslant \partial v$.*

By Dickson's lemma, we have that any decreasing sequence of derivatives is finite. For a given ranking of $(x_1, x_2, \dots, x_n)$, we use $u < v$ to denote the fact that the rank of u is lower than that of v or the rank of v is higher than that of u where $u, v \in \Theta X$.

Remark 2.2. In this paper, we will always use the following ranking on $\Theta X = \{\partial_{i,j} x_k, k = 1, 2, \dots, n; i, j$ are nonnegative integers$\}$, where $\partial_{i,j} = \frac{\partial^{i+j}}{\partial u^i \partial v^j}$. Then $\partial_{i_1,j_1} x_{k_1} > \partial_{i_2,j_2} x_{k_2}$ if it satisfies one of the following conditions:

(a) $x_{k_1} > x_{k_2}$;

(b) $x_{k_1} = x_{k_2}$ and $i_1 + j_1 > i_2 + j_2$;

(c) $x_{k_1} = x_{k_2}$, $i_1 + j_1 = i_2 + j_2$ and $i_1 > i_2$.

Definition 2.3. *Let P be a differential polynomial in $\mathscr{F}\{X\}$. The leader of P is the highest ranking derivative appearing in P, denoted by L_P. View P as a univariate polynomial in its leader, then the leading coefficient of P is called the initial of P and denoted by I_P. The separant of P is $\partial P/\partial L_P$ denoted by S_P.*

By the above definition, for a differential polynomial $P \in \mathscr{F}\{X\}$ with initial I_p, P can be written as $P = I_P L_P^d + I_1 L_P^{d-1} + \cdots + I_d$ and the separant of P is $\partial P/\partial L_P = d I_P L_P^{d-1} + (d-1) I_1 L_P^{d-2} + \cdots + I_{d-1}$.

Definition 2.4. *For differential polynomials P and Q in $\mathscr{F}\{X\}$, P is said to have higher rank than Q, denoted by $P \succ Q$, if $L_P > L_Q$ or $L_P = L_Q$ but $\deg(P, L_P) > \deg(Q, L_P)$. If neither $P \succ Q$ nor $P \prec Q$ holds, we will say that P and Q are incomparable in rank and write $P \sim Q$.*

Remark 2.5. A differential polynomial which effectively involves indeterminates will have higher rank than one which does not.

For differential polynomials P and Q in $\mathscr{F}\{X\}$, P is partially reduced with respect to Q if there is no $\theta \in \Theta$ with $\mathrm{ord}(\theta) > 0$ such that $\theta(L_Q)$ appears in P. P is reduced with respect to Q if P is partially reduced with respect to Q and $\deg(P, L_Q) < \deg(Q, L_Q)$. A differential polynomial P is said to be (partially) reduced with respect to a set of differential polynomials $\mathbb{P}$ if it is (partially) reduced with respect to each differential polynomial in $\mathbb{P}$. A subset $\mathbb{P}$ of $\mathscr{F}\{X\}$ is called an ascending set if either $\mathbb{P} = \{f\}$ where $f \in \mathscr{F}$ or no elements of $\mathbb{P}$ belong to

$\mathscr{F}$ and the elements in $\mathbb{P}$ can be arranged as a finite sequence in increasing rank such that each differential polynomial is reduced with respect to the preceding ones.

Definition 2.6. *Let $\mathbb{A} = \{A_1, A_2, \dots, A_r\}$ and $\mathbb{B} = \{B_1, B_2, \dots, B_s\}$ be two ascending sets in $\mathscr{F}\{X\}$.*

$$A_1 \prec A_2 \prec \cdots \prec A_r; \quad B_1 \prec B_2 \prec \cdots \prec B_s.$$

$\mathbb{A}$ is said to be of lower rank than $\mathbb{B}$ if one of the following conditions is satisfied:

(1) *There exists $k \leqslant \min\{r, s\}$ such that $A_i \sim B_i$ for $1 \leqslant i \leqslant k-1$ and $A_k \prec B_k$;*

(2) *$r > s$ and $A_i \sim B_i$ for all $i \leqslant s$.*

An element in $\mathscr{F}$ is considered to be of the lowest rank. It is easy to see that the above definition really introduces a partial ordering among the ascending sets.

Proposition 2.7 (See [15, p. 77], [16, p. 3])**.** *Any sequence of ascending sets decreasing in order is finite.*

Proposition 2.7 guarantees that for a differential polynomial set $\mathbb{P} \subset \mathscr{F}\{X\}$, we can always find an ascending set $\mathbb{A}$ in $\mathbb{P}$ which is not higher than any other ascending set in $\mathbb{P}$ in finite steps. Such an ascending set is called a characteristic set of $\mathbb{P}$. The algorithms on how to find a characteristic set of partial differential polynomials can be found in [2, 12]. We will not give the details here. In the following, we will give the remainder formula which is very important in characteristic set method (cf. [2, 7, 12], [16, p. 165]).

Theorem 2.8. *Let $\mathbb{A} : A_1 \prec A_2 \prec \cdots \prec A_r$ be an ascending set. For any differential polynomial P in $\mathscr{F}\{X\}$ which is not in $\mathscr{F}$, there are the integers s_k, t_k $(k = 1, \dots, r)$ and partial derivatives $\partial_{\tau_{i,j}} \in \Theta$ and differential polynomials $C_{\tau_{i,j}}$ such that*

$$R = S_1^{s_1} \cdots S_r^{s_r} I_1^{t_1} \cdots I_r^{t_r} P - \sum_{i,j} C_{\tau_{i,j}} \partial_{\tau_{i,j}} A_i \tag{1}$$

is reduced with respect to $\mathbb{A}$, where S_i and I_i are respectively separant and initial of A_i.

The above formula is called the Remainder Formula of P with respect to $\mathbb{A}$. We denote R in (1) by $\mathrm{Prem}(P, \mathbb{A})$.

Example 2.9. Let $P = x\partial_{1,1}y - y$ be a differential polynomial in $\mathscr{F}\{x, y\}$, where $\mathscr{F} = \mathbb{R}(u, v)$. Let $y > x$ and the ranking of the derivatives be in Remark 2.2. The set $\mathscr{A} = \{y\partial_{0,1}y - x,\ x\partial_{1,0}y + y\}$ is an ascending set in $\mathscr{F}\{x, y\}$. We write $A_1 = x\partial_{1,0}y + y$, $A_2 = y\partial_{0,1}y - x$. Since

$$\begin{aligned}
&\partial_{0,1}A_1 = \partial_{0,1}x\partial_{1,0}y + x\partial_{1,1}y + \partial_{0,1}y, \\
&P = x\partial_{1,1}y - y = \partial_{0,1}A_1 - \partial_{0,1}x\partial_{1,0}y - \partial_{0,1}y - y.
\end{aligned}$$

Then we have

$$\begin{aligned}
xP &= x\partial_{0,1}A_1 - x\partial_{0,1}x\partial_{1,0}y - x\partial_{0,1}y - xy \\
&= x\partial_{0,1}A_1 - (A_1 - y)\partial_{0,1}x - x\partial_{0,1}y - xy \\
&= x\partial_{0,1}A_1 - A_1\partial_{0,1}x + y\partial_{0,1}x - x\partial_{0,1}y - xy.
\end{aligned}$$

Because $A_2 = y\partial_{0,1}y - x$, we have

$$\begin{aligned} xyP &= xy\partial_{0,1}A_1 - yA_1\partial_{0,1}x + y^2\partial_{0,1}x - xy\partial_{0,1}y - xy^2 \\ &= xy\partial_{0,1}A_1 - yA_1\partial_{0,1}x + y^2\partial_{0,1}x - x(A_2 + x) - xy^2 \\ &= xy\partial_{0,1}A_1 - yA_1\partial_{0,1}x + y^2\partial_{0,1}x - xA_2 - x^2 - xy^2 \\ &= xy\partial_{0,1}A_1 - yA_1\partial_{0,1}x - xA_2 + y^2\partial_{0,1}x - x^2 - xy^2. \end{aligned}$$

Hence

$$y^2\partial_{0,1}x - x^2 - xy^2 = xyP - xy\partial_{0,1}A_1 + yA_1\partial_{0,1}x + xA_2$$

is the remainder of P with respect to $\mathbb{A}$, i.e., $\mathrm{Prem}(P, \mathbb{A}) = y^2\partial_{0,1}x - x^2 - xy^2$.

For any differential polynomial sets $\mathbb{P}$, $\mathbb{Q} \subset \mathscr{F}\{X\} = \mathscr{F}\{x_1, \ldots, x_n\}$, let $\mathrm{Zero}(\mathbb{P}) = \{z \in \mathscr{E}^n | \forall P \in \mathbb{P}, P(z) = 0\}$ be the set of all zeros of the differential polynomials of $\mathbb{P}$ in $\mathscr{E}^n$, where $\mathscr{E}$ is an extension field of $\mathscr{F}$. Note that $\mathrm{Zero}(\mathbb{P}) = \mathrm{Zero}(\mathrm{Ideal}(\mathbb{P})) = \mathrm{Zero}(\{\mathbb{P}\})$. Let $\mathrm{Zero}(\mathbb{P}/\mathbb{Q}) = \mathrm{Zero}(\mathbb{P}) \setminus \bigcup_{Q\in\mathbb{Q}} \mathrm{Zero}(Q)$.

We will need the following Wu-Ritt's decomposition theorem in this paper (cf. [4]).

Theorem 2.10. *For any finite differential polynomial sets* $\mathbb{A} \in \mathscr{F}\{X\}$*, there is an algorithm which will give a finite set of ascending sets* $\mathbb{C}_i$ *in a finite steps such that* $\mathrm{Zero}(\mathbb{A}) = \bigcup_{i=0}^{s} \mathrm{Zero}(\mathbb{C}_i/\mathrm{IS}_i)$*, where* IS_i *are the products of the initials and separants of* $\mathbb{C}_i$*.*

There are some implementations of characteristic set method in case of partial differential polynomials such as Wsolve, diffalg package in Maple, Aldor, etc. In this paper, we will use diffalg to compute the characteristic set of partial differential polynomials. The command "Rosenfeld_Groebner" in diffalg decomposes a radical differential ideal into an intersection of characterizable differential ideals which are represented by characteristic sets (cf. [17]). The command "reduced_form($P, \{\mathbb{P}\}$)" computes the reduced form of P module radical ideal $\{\mathbb{P}\}$. We will use reduced_form to compute the remainder.

The following algorithm will show us the principle of mechanical theorem proving in differential geometry.

Definition 2.11. *A formula like*

$$\forall\ x_1, \ldots, x_n, \quad [(H_1 = 0 \wedge \cdots \wedge H_r = 0 \wedge D_1 \neq 0 \wedge \cdots \wedge D_t \neq 0) \Rightarrow \mathbb{S} = 0]$$

is said to be a statement of equality type, where $H_i\ (i = 1, \ldots, r), D_i\ (i = 1, \ldots, t)$ *are differential polynomials in* $\mathscr{F}\{X\}$ *and* $\mathbb{S}$ *is a finite subset of* $\mathscr{F}\{X\}$*. We denote* $\mathbb{H} = \{H_1, \ldots, H_r\}$*, which is the hypothesis differential polynomial set, and denote* $\mathbb{D} = \{D_1, \ldots, D_t\}$ *which is used to describe degenerate conditions. Here* $\mathbb{S} = 0$ *means that all polynomials in* $\mathbb{S}$ *vanish on* $x_1, \ldots, x_n$ *which satisfy* $H_i(x_1, \ldots, x_n) = 0$ *and* $D_j(x_1, \ldots, x_n) \neq 0$ *for* $i = 1, \ldots, r; j = 1, \ldots, t$*.*

Definition 2.12. *A statement of equality type is said to be true in any extension field* $\mathscr{E}$ *of* $\mathscr{F}$*, if*

$$\forall\, (x_1, x_2, \ldots, x_n) \in \mathscr{E}^n, \quad [(H_1 = 0 \wedge H_r = 0 \wedge D_1 \neq 0 \wedge \cdots \wedge D_t \neq 0) \Rightarrow \mathbb{S} = 0].$$

A statement is called universally true if it is true in any extension field of $\mathscr{F}$*. Here* $\mathbb{S} = 0$ *means that all polynomials in* $\mathbb{S}$ *vanish on* $\mathrm{Zero}(\mathbb{H}/\mathbb{D})$*, that is,* $\mathrm{Zero}(\mathbb{H}/\mathbb{D}) \subseteq \mathrm{Zero}(\mathbb{S})$*.*

Note that for a characteristic set $\mathbb{C}$ and a differential polynomial P, if $Prem(P,\mathbb{C})=0$, then $Zero(\mathbb{C}/IS_i)\subseteq Zero(P)$. Hence we have the following algorithm.

Algorithm 2.13. *Decide whether a statement of equality type is universally true.*

1. By Theorem 2.10, we can compute

$$Zero(\mathbb{H}/\mathbb{D})=\bigcup_{i=0}^{s} Zero(\mathbb{C}_i/\{\mathrm{IS}_i,\mathbb{D}\}),$$

where IS_i are the products of the initials and sperants of $\mathbb{C}_i$ and $\mathbb{D}$ is a set of differential polynomials in $\mathscr{F}\{X\}$.

2. If $s=0$ or $Prem(\mathbb{S},\mathbb{C}_i)=0$ for $i=1,\ldots,s$, then the statement of equality type is universally true.

3 A modified Wronskian determinant to test linear dependence

In this section, we will improve the result given by Kolchin in [15, p. 86, Theorem 1]. Let $\Theta(s)=\{\theta\in\Theta, ord(\theta)\leqslant s\}$. Denote the field of constants of $\mathscr{F}$ by $\mathscr{C}$. Kolchin proved the following theorem, which generalized the well-known classical result on Wronskian determinants (cf. [15, p. 86, Theorem 1]).

Theorem 3.1. *Let $\eta_j=(\eta_{j,1},\ldots,\eta_{j,r}), 1\leqslant j\leqslant n$ be the elements of $\mathscr{F}^r$. If they are linearly dependent over $\mathscr{C}$, then*

$$\det(\theta_i\eta_{j,k(i)})_{1\leqslant i\leqslant n,1\leqslant j\leqslant n}=0, \tag{2}$$

for all choices of $\theta_1,\ldots,\theta_n\in\Theta$ and all choices of the indices $k(1),\ldots,k(n)\in\{1,2,\ldots,n\}$. Conversely, if (2) holds for all choices of $\theta_1,\ldots,\theta_n$ with $\theta_i\in\Theta(i-1)\,(1\leqslant i\leqslant n)$ and all choices of $k(1),\ldots,k(n)$, then $\eta_1,\ldots,\eta_n$ are linearly dependent over $\mathscr{C}$.

In the case $r=1$ and two derivation operations, $|\Theta(s-1)|=\frac{s(s+1)}{2}$. Then the number of determinants in (2) is $n!(n+1)!/2^n$. In fact, in (2), there are many determinants which automatically vanish and we do not need to calculate these determinants. The corollary below improves Theorem 3.1.

Corollary 3.2. *Let $\mathscr{F}$ be a differential field with two derivative operators δ_1, δ_2, and $\mathscr{C}$ be its constant field. Θ is the commutative semigroup generated by δ_1 and δ_2, $x_1,\ldots,x_n\in\mathscr{F}$. If there exist $\tau_i\in\Theta(i-1)$ for $i=1,\ldots,n-1$ such that $\gamma_i=(\tau_ix_1,\ldots,\tau_ix_n)$, $1\leqslant i\leqslant n-1$ are linearly independent over $\mathscr{F}$, then $x_1,\ldots,x_n$ are linearly dependent over $\mathscr{C}$ if and only if the following $2(n-1)$ determinants*

$$\begin{vmatrix} \tau_1x_1 & \tau_1x_2 & \cdots & \tau_1x_n \\ \vdots & \vdots & \ddots & \vdots \\ \tau_{n-2}x_1 & \tau_{n-2}x_2 & \cdots & \tau_{n-2}x_n \\ \tau_{n-1}x_1 & \tau_{n-1}x_2 & \cdots & \tau_{n-1}x_n \\ \delta_1\tau_ix_1 & \delta_1\tau_ix_2 & \cdots & \delta_1\tau_ix_n \end{vmatrix}, \quad \begin{vmatrix} \tau_1x_1 & \tau_1x_2 & \cdots & \tau_1x_n \\ \vdots & \vdots & \ddots & \vdots \\ \tau_{n-2}x_1 & \tau_{n-2}x_2 & \cdots & \tau_{n-2}x_n \\ \tau_{n-1}x_1 & \tau_{n-1}x_2 & \cdots & \tau_{n-1}x_n \\ \delta_2\tau_ix_1 & \delta_2\tau_ix_2 & \cdots & \delta_2\tau_ix_n \end{vmatrix}, \quad 1\leqslant i\leqslant n-1$$

are all zero. Note that $\tau_1=1$.

Proof. ($\Rightarrow$) It is obvious by Theorem 3.1.

($\Leftarrow$) In Theorem 3.1, let $r = 1$ and $\eta_i = \tau_1 x_i = x_i$. Then by Theorem 3.1, we only need to prove that $\det(\theta_i x_j)_{1\leqslant i\leqslant n, 1\leqslant j\leqslant n} = 0$ for all choices of $\theta_1, \ldots, \theta_n$ with $\theta_i \in \Theta(i-1)$. Suppose that $\gamma_1, \ldots, \gamma_{n-1}$ are linearly independent over $\mathscr{F}$. Since the $2(n-1)$ determinants are all equal to zero, then $\delta_1\gamma_i = (\delta_1\tau_i x_1, \ldots, \delta_1\tau_i x_n)$, $\delta_2\gamma_i = (\delta_2\tau_i x_1, \ldots, \delta_2\tau_i x_n)$ are linear combinations of $\gamma_1, \ldots, \gamma_{n-1}$ for each $i \in \{1, 2, \ldots, n\}$. Because there are only two derivative operators δ_1 and δ_2 which are commutative, any $\theta \in \Theta$ can be written as $\delta_1^k\delta_2^l$. By the induction on k and l, we can easily prove that $\delta_1^k\delta_2^l\gamma_1$ are the linear combination of $\gamma_1, \ldots, \gamma_{n-1}$ for any non-negative integers k, l. Since $\gamma_1 = (x_1, x_2, \ldots, x_n)$, $\det(\theta_i x_j)_{1\leqslant i\leqslant n, 1\leqslant j\leqslant n} = 0$ for all choices of $\theta_1, \ldots, \theta_n$ with $\theta_i \in \Theta(i-1)$.

From the above corollary, we can decide whether $x_1, \ldots, x_n$ are linearly dependent as follows: First, let

$$A = \begin{pmatrix} x_1 & x_2 \\ \delta_1 x_1 & \delta_1 x_2 \end{pmatrix}, \quad B = \begin{pmatrix} x_1 & x_2 \\ \delta_2 x_1 & \delta_2 x_2 \end{pmatrix}.$$

If $\det(A) = \det(B) = 0$, then x_1, x_2 are linearly dependent by Corollary 3.2 and we have that $x_1, \ldots, x_n$ are linearly dependent. Otherwise, without loss of generality, assume that $\det(A) \neq 0$ then (x_1, x_2) and $(\delta_1 x_1, \delta_1 x_2)$ are linearly independent. Let matrixes M_1, M_2, M_3, M_4 respectively be

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ \delta_1 x_1 & \delta_1 x_2 & \delta_1 x_3 \\ \delta_1 x_1 & \delta_1 x_2 & \delta_1 x_3 \end{pmatrix}, \quad \begin{pmatrix} x_1 & x_2 & x_3 \\ \delta_1 x_1 & \delta_1 x_2 & \delta_1 x_3 \\ \delta_1\delta_1 x_1 & \delta_1\delta_1 x_2 & \delta_1\delta_1 x_3 \end{pmatrix},$$

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ \delta_1 x_1 & \delta_1 x_2 & \delta_1 x_3 \\ \delta_2 x_1 & \delta_2 x_2 & \delta_2 x_3 \end{pmatrix}, \quad \begin{pmatrix} x_1 & x_2 & x_3 \\ \delta_1 x_1 & \delta_1 x_2 & \delta_1 x_3 \\ \delta_2\delta_1 x_1 & \delta_2\delta_1 x_2 & \delta_2\delta_1 x_3 \end{pmatrix}.$$

If $\det(M_1) = \det(M_2) = \det(M_3) = \det(M_4) = 0$ then x_1, x_2, x_3 are linearly dependent by Corollary 3.2, and then $x_1, \ldots, x_n$ are linearly dependent. Otherwise, x_1, x_2, x_3 are linearly independent. Repeat the process for x_1, x_2, x_3, x_4 as above. Then we need to compute $n(n-1)$ determinants matrixes at most. In fact, because τ_2 in Corollary 3.2 must be δ_1 or δ_2, we have $\det(C) = 0$ or $\det(D) = 0$ where

$$C = \begin{pmatrix} \tau_1 x_1 & \tau_1 x_2 & \cdots & \tau_1 x_n \\ \tau_2 x_1 & \tau_2 x_2 & \cdots & \tau_2 x_n \\ \vdots & \vdots & \vdots & \vdots \\ \tau_{n-1} x_1 & \tau_{n-1} x_2 & \cdots & \tau_{n-1} x_n \\ \delta_1 x_1 & \delta_1 x_2 & \cdots & \delta_1 x_n \end{pmatrix}, \quad D = \begin{pmatrix} \tau_1 x_1 & \tau_1 x_2 & \cdots & \tau_1 x_n \\ \tau_2 x_1 & \tau_2 x_2 & \cdots & \tau_2 x_n \\ \vdots & \vdots & \vdots & \vdots \\ \tau_{n-1} x_1 & \tau_{n-1} x_2 & \cdots & \tau_{n-1} x_n \\ \delta_2 x_1 & \delta_2 x_2 & \cdots & \delta_2 x_n \end{pmatrix}.$$

Hence we only need to compute $n^2 - 2n + 2 (= n(n-1) - n + 2)$ determinants to test whether $x_1, x_2, \ldots, x_n$ are linearly dependent.

Remark 3.3. In the sections below, for the convenience, we will use $\mathrm{WR}(x_1, \ldots, x_n)$ to denote the set of determinants in Theorem 3.1 and $\mathrm{WR}(x_1, \ldots, x_n) = 0$ means that all of the determinants in $\mathrm{WR}(x_1, \ldots, x_n)$ vanish. Then if $\mathrm{WR}(x_1, \ldots, x_n) = 0$, $x_1, x_2, \ldots, x_n$ are linearly dependent over the constant field of $\mathscr{F}$; otherwise $x_1, x_2, \dots, x_n$ are linearly independent over the constant field of $\mathscr{F}$.

4 Mechanical theorem proving for local theory of surfaces

4.1 Local theory of surfaces

Next we will recall the basic fact of the local theory of surfaces. Let (u, v), (x, y, z) respectively be the coordinates of the points of Euclidean space of dimensions 2 and 3. The parametric representation of a surface can be $r = r(u, v) = (x(u, v), y(u, v), z(u, v))$, in which $x(u, v)$, $y(u, v)$, and $z(u, v)$ are functions in u and v. For the simplicity, we use $x_{i,j}$ to denote $\frac{\partial^{i+j}x}{\partial^i u \partial^j v}$ and $y_{i,j}, z_{i,j}, n_{k_{i,j}}$ are the same sense.

Let $p = (x, y, z)$, $n = (n_1, n_2, n_3)$, where $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$. View x, y, z, n_i as the differential indeterminates. Then some basic concept in the local theory of surfaces can be described as the following differential equations. These equations are necessary for characterizing the surfaces in algebraic language.

(1) The curvature of the surface r at point $p(x, y, z)$ is k:

$$k^2 - (x_{1,0}y_{0,1} - y_{1,0}x_{0,1})^2 - (y_{1,0}z_{0,1} - z_{1,0}y_{0,1})^2 - (z_{1,0}x_{0,1} - x_{1,0}z_{0,1})^2 = 0.$$

(2) The unit normal vector of the surface r at point $p(x, y, z)$ is $n = (n_1, n_2, n_3)$:

$$\begin{aligned} &kn_1 - y_{1,0}z_{0,1} + y_{0,1}z_{1,0} = 0\wedge \\ &kn_2 - z_{1,0}x_{0,1} + z_{0,1}x_{1,0} = 0\wedge \\ &kn_3 - x_{1,0}y_{0,1} + x_{0,1}y_{1,0} = 0. \end{aligned}$$

(3) The first fundamental form of the surface r is $I = Edu^2 + 2Fdudv + Gdv^2$:

$$\begin{aligned} &E - x_{1,0}^2 - y_{1,0}^2 - z_{1,0}^2 = 0\wedge \\ &F - x_{1,0}x_{0,1} - y_{1,0}y_{0,1} - z_{1,0}z_{0,1} = 0\wedge \\ &G - x_{0,1}^2 - y_{0,1}^2 - z_{0,1}^2 = 0. \end{aligned} \tag{3}$$

(4) The second fundamental form of the surface r is $II = Ldu^2 + 2Mdudv + Ndv^2$:

$$\begin{aligned} &L + n_{1_{1,0}}x_{1,0} + n_{2_{1,0}}y_{1,0} + n_{3_{1,0}}z_{1,0} = 0\wedge \\ &M + n_{1_{0,1}}x_{1,0} + n_{2_{0,1}}y_{1,0} + n_{3_{0,1}}z_{1,0} = 0\wedge \\ &N + n_{1_{0,1}}x_{0,1} + n_{2_{0,1}}y_{0,1} + n_{3_{0,1}}z_{0,1} = 0. \end{aligned} \tag{4}$$

4.2 Some basic languages

Use the same notation as in Subsection 4.1, that is, the point $p = (x, y, z)$ and the normal vector $n = (n_1, n_2, n_3)$. By the modified Wronskian determinant, we can translate the following predicates for surfaces into algebraic language. Note that the equations below are only the sufficient conditions of these predicates but not necessary conditions.

1. The planes passing through p and with n as their normal vectors pass through a fixed point. Its equations are

$$\mathrm{WR}(n_1, n_2, n_3, n_1x + n_2y + n_3z) = 0 \wedge \mathrm{WR}(n_1, n_2, n_3) \neq 0.$$

We know that equations of the planes are $n_1(X - x) + n_2(Y - y) + n_3(Z - z) = 0$. Under the above conditions, there are the constants c_1, c_2, c_3 and $c_4 \neq 0$ such that

$$c_1n_1 + c_2n_2 + c_3n_3 + c_4(n_1x + n_2y + n_3z) = 0.$$

Then we have

$$\frac{c_1}{c_4}n_1 + \frac{c_2}{c_4}n_2 + \frac{c_3}{c_4}n_3 + n_1X + n_2Y + n_3Z = 0.$$

Hence the planes pass through the fixed point $(-\frac{c_1}{c_4}, -\frac{c_2}{c_4}, -\frac{c_3}{c_4})$.

2. The planes passing through p and with n as their normal vector pass through a fixed line. Its equations are

$$\begin{aligned}&(\mathrm{WR}(n_1,n_2,n_3) = \mathrm{WR}(n_1,n_2,n_1x+n_2y+n_3z) = 0 \wedge \mathrm{WR}(n_1,n_2) \neq 0)\vee\\ &(\mathrm{WR}(n_1,n_2,n_3) = \mathrm{WR}(n_1,n_3,n_1x+n_2y+n_3z) = 0 \wedge \mathrm{WR}(n_1,n_3) \neq 0)\vee\\ &(\mathrm{WR}(n_1,n_2,n_3) = \mathrm{WR}(n_2,n_3,n_1x+n_2y+n_3z) = 0 \wedge \mathrm{WR}(n_2,n_3) \neq 0).\end{aligned}$$

In the first case, since $\mathrm{WR}(n_1,n_2,n_1x+n_2y+n_3z) = 0 \wedge \mathrm{WR}(n_1,n_2) \neq 0$, the planes pass through a fixed point $(a_1,a_2,0)$ where $a_1n_1+a_2n_2 = n_1x+n_2y+n_3z$. By $\mathrm{WR}(n_1,n_2,n_3) = 0$, the planes are perpendicular to a fixed surface $c_1X+c_2Y+c_3Z = 0$, where $c_1n_1+c_2n_2+c_3n_3 = 0$. Therefore, all planes include the fixed line which passes through the point $(a_1,a_2,0)$ and perpendicular to the plane $c_1X + c_2Y + c_3Z = 0$ or a fixed line perpendicular to the plane $c_1X + c_2Y + c_3Z = 0$ at $(0,0,0)$ when $a_1 = a_2 = 0$. In the other cases, we can get the conclusion by the same way.

3. The lines passing through p and parallel to n pass through a fixed point. Its equations are

$$\begin{aligned}&\mathrm{WR}(n_1n_3, n_1n_2, n_2n_3) \neq 0\wedge\\ &\mathrm{WR}(n_1,n_2,n_1y-n_2x) = 0\wedge\\ &\mathrm{WR}(n_1,n_3,n_1z-n_3x) = 0\wedge\\ &\mathrm{WR}(n_2,n_3,n_2z-n_3y) = 0.\end{aligned}$$

From $\mathrm{WR}(n_1n_3,n_1n_2,n_2n_3) \neq 0$, we have that $\mathrm{WR}(n_i,n_j) \neq 0$, where $i,j = 1,2,3$ and $i \neq j$. Hence there are the constants a_i, b_i such that

$$n_1y - n_2x = a_1n_1 + b_1n_2; \tag{5}$$

$$n_1z - n_3x = a_2n_1 + b_2n_3; \tag{6}$$

$$n_2z - n_3y = a_3n_2 + b_3n_3. \tag{7}$$

From $(5) * n_3 - (6) * n_2 + (7) * n_1$, we get

$$(a_1+b_3)n_1n_3 + (a_3-a_2)n_1n_2 + (b_1-b_2)n_2n_3 = 0.$$

By $\mathrm{WR}(n_1n_3,n_1n_2,n_2n_3) \neq 0$ again, we have that $a_1+b_3 = 0;\ a_3-a_1 = 0;\ b_1-b_2 = 0$. Hence by (5)–(7),

$$n_1(y-a_1) = n_2(x+b_1); \quad n_3(x+b_1) = n_1(z-a_2); \quad n_2(z-a_2) = n_3(y-a_1).$$

Since $\mathrm{WR}(n_1n_3,n_1n_2,n_2n_3) \neq 0$, there are at least two of n_1,n_2,n_3 which are not always equal to zero, without lost of generality, assume that $n_1n_2 \neq 0$. Let $u = \frac{x+b_1}{n_1}$. We have that $x - un_1 = -b_1, y - un_2 = a_1$. If $n_3 \equiv 0$, then $z \equiv a_2$, otherwise $z - un_3 = a_2$. The equations of the lines are $X = x+tn_1$, $Y = y+tn_2$, $Z = z+tn_3$. Therefore all the lines pass through a fixed point $(-b_1,a_1,a_2)$.

5 Examples

In this section, we will present some examples. In these examples, we will use Algorithm 2.13 to prove that the statements in the examples are universally true.

Example 5.1. Show that all normal lines of a sphere pass through a fixed point.

Without lost of generality, we can assume that the equation of this sphere is $x^2+y^2+z^2-r^2=0$, where x,y,z are differential indeterminates over differential field $\mathbb{R}(u,v)$ and r is the constant. Then the normal lines are the lines passing through (x,y,z) and parallel to (n_1,n_2,n_3). Using the definition equations of the normal vectors in Subsection 4.1, we have the hypothesis set $\mathbb{H}$ which includes the following differential polynomials:

$$H_1 = x^2+y^2+z^2-r^2;$$
$$H_2 = k^2-(x_{1,0}y_{0,1}-y_{1,0}x_{0,1})^2-(y_{1,0}z_{0,1}-z_{1,0}y_{0,1})^2-(z_{1,0}x_{0,1}-x_{1,0}z_{0,1})^2;$$
$$H_3 = kn_1-y_{1,0}z_{0,1}+y_{0,1}z_{1,0};$$
$$H_4 = kn_2-z_{1,0}x_{0,1}+z_{0,1}x_{1,0};$$
$$H_5 = kn_3-x_{1,0}y_{0,1}+x_{0,1}y_{1,0}.$$

Then we need to prove that the third predicate in Subsection 4.2 is true under the hypothesis set $\mathbb{H}$. Hence the conclusion set $\mathbb{S}$ contains fifteen differential polynomials. Let $n_3>n_2>n_1>z>y>x>k$ and use the ranking in Remark 2.2. Here, we take r as the parameter. If $k=0$, then $r=0$, i.e. the sphere degenerates to a point. Hence we will consider $\mathrm{Zero}(\{H_1,\ldots,H_5\}/k)$. Using the package diffalg in Maple, we can get

$$\mathrm{Zero}(\{H_1,\ldots,H_5\}/k)=\mathrm{Zero}(\mathbb{C}_1/\{I_1,k\})\bigcup\mathrm{Zero}(\mathbb{C}_2/\{I_2,k\}),$$

where

$$\begin{aligned}
\mathbb{C}_1 =\,& kn_3-x_{1,0}y_{0,1}+y_{1,0}x_{0,1},\\
& kn_2(x^2+y^2-r^2)+zx_{1,0}yy_{0,1}-zx_{0,1}yy_{1,0},\\
& kn_1(x^2+y^2-r^2)-zy_{1,0}xx_{0,1}+zy_{0,1}xx_{1,0},\\
& z^2+x^2+y^2-r^2,\\
& y_{1,0}^2x_{0,1}^2r^2+x_{1,0}^2r^2y_{0,1}^2-2x_{1,0}y_{1,0}x_{0,1}r^2y_{0,1}+k^2y^2+k^2x^2-k^2r^2;\\
\mathbb{C}_2 =\,& kn_3-x_{1,0}y_{0,1},\\
& kn_2(x^2+y^2-r^2)+zx_{1,0}yy_{0,1},\\
& kn_1(x^2+y^2-r^2)+zy_{0,1}xx_{1,0},\\
& z^2+x^2+y^2-r^2,\\
& x_{1,0}^2r^2y_{0,1}^2+k^2x^2-k^2r^2+k^2y^2,\\
& x_{0,1};
\end{aligned}$$

and $\mathrm{IS}_1,\mathrm{IS}_2$ are the products of the initials and separants of $\mathbb{C}_1,\mathbb{C}_2$ respectively. Let

$$W=\begin{vmatrix} n_1n_2 & n_1n_3 & n_2n_3\\ n_{11,0}n_2+n_1n_{21,0} & n_{11,0}n_3+n_1n_{31,0} & n_{21,0}n_3+n_2n_{31,0}\\ n_{10,1}n_2+n_1n_{20,1} & n_{10,1}n_3+n_1n_{30,1} & n_{20,1}n_3+n_2n_{30,1}\end{vmatrix}.$$

We have that $\text{Prem}(W, \mathbb{C}_1) \neq 0$. Hence $\text{WR}(n_1n_2, n_1n_3, n_2n_3) \neq 0$. Consider the following four determinants:

$$D_i = \begin{vmatrix} n_1 & n_2 & n_1y - n_2x \\ n_{1_{0,1}} & n_{2_{0,1}} & n_{1_{0,1}}y - n_{2_{0,1}}x + n_1y_{0,1} - n_2x_{0,1} \\ \theta_i n_1 & \theta_i n_2 & \theta_i(n_1y - n_2x) \end{vmatrix}, \quad i = 1, 2, 3, 4,$$

where $\theta_1 = \partial_{1,0}, \theta_2 = \partial_{0,2}, \theta_3 = \partial_{1,1}, \theta_4 = \partial_{2,0}$. By the computation process, we have that $\text{Prem}(D_i, \mathbb{C}_1) = \text{Prem}(D_i, \mathbb{C}_2) = 0$ for $i = 1, 2, 3, 4$. By Corollary 3.2, $\text{WR}(n_1, n_2, n_1y - n_2x) = 0$. By the same way, we have that $\text{WR}(n_1, n_3, n_1z - n_3x) = 0$ and $\text{WR}(n_2, n_3, n_2z - n_3y) = 0$. Therefore the statement is universally true.

Example 5.2. Show that if a regular surface is a sphere, then there exists a nonzero constant c such that $(E, F, G) = c(L, M, N)$, where E, F, G are as in (3) and L, M, N are as in (4).

Using the notation and results in Example 5.1, we have had the characteristic sets $\mathbb{C}_1$ and $\mathbb{C}_2$. Since $\text{Prem}(E, \mathbb{C}_1) \neq 0$ and

$$\text{Prem}(EL_{1,0} - E_{1,0}L, \mathbb{C}_1) = \text{Prem}(EL_{1,0} - E_{1,0}L, \mathbb{C}_2) = 0,$$
$$\text{Prem}(EL_{0,1} - E_{0,1}L, \mathbb{C}_1) = \text{Prem}(EL_{0,1} - E_{0,1}L, \mathbb{C}_2) = 0,$$

E and L are linearly dependent over the constant field $\mathbb{R}$, that is, there exists a nonzero constant c such that $E = cL$. Because

$$\text{Prem}(F, \mathbb{C}_1) \neq 0, \text{Prem}(EM - LF, \mathbb{C}_1) = \text{Prem}(EM - LF, \mathbb{C}_2) = 0,$$
$$\text{Prem}(G, \mathbb{C}_1) \neq 0, \text{Prem}(EN - GL, \mathbb{C}_1) = \text{Prem}(EN - GL, \mathbb{C}_2) = 0,$$

(E, F, G) and (L, M, N) are parallel. Hence $(E, F, G) = c(L, M, N)$.

Example 5.3. Prove that the first fundamental form of the surface will not change when the surface does rigid motions in $\mathbb{R}^3$.

Let $(a_{i,j})_{1 \leqslant i \leqslant 3, 1 \leqslant j \leqslant 3}$ be the matrix of the rotation and $(b_1, b_2, b_3)^t$ be the translation. Let (e_i, f_i, g_i) be the first fundamental forms corresponding to the surfaces (x_i, y_i, z_i) respectively, where $i = 1, 2$. Then by the definition equations in Subsection 4.1, we have the hypothesis set $\mathbb{H}$ which includes:

$$H_1 = x_1 - b_1 - a_{1,1}x_2 - a_{1,2}y_2 - a_{1,3}z_2;$$
$$H_2 = y_1 - b_2 - a_{2,1}x_2 - a_{2,2}y_2 - a_{2,3}z_2;$$
$$H_3 = z_1 - b_3 - a_{3,1}x_2 - a_{3,2}y_2 - a_{3,3}z_2;$$
$$H_4 = a_{1,1}^2 + a_{2,1}^2 + a_{3,1}^2 - 1;$$
$$H_5 = a_{1,2}^2 + a_{2,2}^2 + a_{3,2}^2 - 1;$$
$$H_6 = a_{1,3}^2 + a_{2,3}^2 + a_{3,3}^2 - 1;$$
$$H_7 = a_{1,1}a_{1,2} + a_{2,1}a_{2,2} + a_{3,1}a_{3,2};$$
$$H_8 = a_{1,1}a_{1,3} + a_{2,1}a_{2,3} + a_{3,1}a_{3,3};$$
$$H_9 = a_{1,2}a_{1,3} + a_{2,2}a_{2,3} + a_{3,2}a_{3,3};$$
$$H_{10} = e_1 - x_{1_{1,0}}^2 - y_{1_{1,0}}^2 - z_{1_{1,0}}^2;$$
$$H_{11} = e_2 - x_{2_{1,0}}^2 - y_{2_{1,0}}^2 - z_{2_{1,0}}^2;$$

$$H_{12} = f_1 - x_{11,0}x_{10,1} - y_{11,0}y_{10,1} - z_{11,0}z_{10,1};$$
$$H_{13} = f_2 - x_{21,0}x_{20,1} - y_{21,0}y_{20,1} - z_{21,0}z_{20,1};$$
$$H_{14} = g_1 - x1_{0,1}^2 - y1_{0,1}^2 - z1_{0,1}^2;$$
$$H_{15} = g_2 - x2_{0,1}^2 - y2_{0,1}^2 - z2_{0,1}^2.$$

Since $a_{i,j}$ are constants where $i, j = 1, 2, 3$, the equations below need to be included in $\mathbb{H}$: $H_{i,j} = a_{i,j_{1,0}}$ and $G_{i,j} = a_{i,j_{0,1}}$. The conclusion set is $\mathbb{S} = \{-e_2 + e_1, f_2 - f_1, g_1 - g_2\}$. Let $e_1 > e_2 > g_1 > g_2 > f_1 > f_2 > z_1 > y_1 > x_1 > z_2 > y_2 > x_2 > a_{1,1} > a_{1,2} > \cdots > a_{3,3}$ and use the ranking in Remark 2.2. Take b_1, b_2, b_3 as the parameters. We can get $\mathrm{Zero}(\mathbb{H}) = \prod_{i=1}^{10} \mathrm{Zero}(\mathbb{C}_i/\mathrm{IS}_i)$ where $\mathbb{C}_i$ are the ascending sets and IS_i are the products of the initials and separants of $\mathbb{C}_i$. Each ascending set includes about 20 differential polynomials. Since $\mathrm{Prem}(\mathbb{S}, \mathbb{C}_i) = 0$ for $i = 1, 2, \ldots, 10$, the conclusion is true.

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