A Reduction Approach to Creative Telescoping*

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ABSTRACT

Creative telescoping is the core in the algorithmic proof theory of combinatorial identities developed by Wilf and Zeilberger in the early 1990s. For multivariate functions, the process of creative telescoping constructs linear differential or recurrence operators in one variable. Such operators are called telescopers. Four classes of algorithms have been developed for creative telescoping according to different algorithmic techniques that they are based on. The fourth and most recent one is the reduction-based telescoping algorithms that are based on the Ostrogradsky-Hermite reduction and its variants. Algorithms in this class share the common feature that they separate the computation of telescopers from the costly computation of certificates. This idea was first worked out for bivariate rational functions in 2010. It has since been extended to more general classes of functions, such as hyperexponential functions, hypergeometric terms, algebraic functions and most recently D-finite functions. In this tutorial, we will overview several reduction algorithms in symbolic integration and summation, explain the idea of creative telescoping via reductions, and present intriguing applications of this new approach.

CCS CONCEPTS

• **Computing methodologies → Algebraic algorithms**.

KEYWORDS

Abramov's reduction, D-finite function, Hyperexponential function, Hypergeometric term, Ostrogradsky-Hermite reduction, Telescoper, Zeilberger's algorithm

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1 CREATIVE TELESCOPING

In the 1990s, Zeilberger formulated the method of creative telescoping as an algorithmic tool for proving special-function identities [67–69]. A large class of special-function identities in mathematical handbooks [8, 55] involve integrals or sums with free parameters. The main idea of proving such identities is showing that both sides of such an identity satisfy the same linear differential or recurrence equations with respect to the parameters and some initial conditions. Algorithms for creative telescoping have been widely used for finding linear differential or recurrence equations satisfied by parameterized definite integrals and sums [56]. Linear differential and recurrence equations provide a nice data structure for representing and manipulating functions and sequences [59].

Creative telescoping is an algorithmic process that constructs, for a given function $f(x, y_1, \ldots, y_n)$, a nonzero linear differential or recurrence operator L in x such that

$$L(f) = \partial_{y_1}(g_1) + \cdots + \partial_{y_n}(g_n),$$

where ∂_{y_i} can be the derivation or difference operator in the variable y_i and the g_i's are in the same class of functions as f. The operator L is called a *telescoper* for f, and the g_i's are called the *certificates* of L. The extensive work overviewed in the surveys [35, 48] on creative telescoping mainly concerns the existence and construction problems of telescopers. The existence problem asks for a decision procedure for checking whether a given function has a telescoper or not. If telescopers exist for a given function, the construction problem asks for efficient algorithms for computing telescopers. In this tutorial, we will explain how reduction algorithms of the next section play a crucial role in solving the existence and construction problems of telescopers.

2 REDUCTION ALGORITHMS

The first reduction algorithm was presented by Ostrogradsky [51] in 1845 and later by Hermite [43] in 1872, which is now a classical technique in symbolic integration [17]. Let $\mathbb{F}$ be a field of characteristic zero and $\mathbb{F}(y)$ be the field of rational functions in y over $\mathbb{F}$, on which the usual derivation in y is denoted by D_y. For a given rational function $f \in \mathbb{F}(y)$, the Ostrogradsky-Hermite reduction, also called *rational reduction* below, decomposes f into the form

$$f = D_y(g) + r \quad \text{with } r = \frac{a}{b}, \tag{2.1}$$

where $g \in \mathbb{F}(y)$ and $a, b \in \mathbb{F}[y]$ are such that $\deg_y(a) < \deg_y(b)$ and b is squrefree in y. The remainder r obtained by the reduction satisfies two properties: firstly, it is *minimal* in the sense that b has the smallest degree in y among all possible such decompositions; secondly, it is a *normal form* for the quotient space $\mathbb{F}(y)/D_y(\mathbb{F}(y))$ since $f \in D_y(\mathbb{F}(y))$ if and only if $a = 0$. The Ostrogradsky-Hermite reduction has been generalized in different directions.

(1) From rational functions to elementary functions. The case of transcendental elementary extensions was studied by Risch [57], Rothstein [58], and Davenport [38]. Trager extended the rational reduction to the case of algebraic functions [63], which was refined in [27] with the normal form property. The general case of elementary extension was studied by Bronstein [16, 17]. Recently, we improved the reduction algorithm from [17] with the normal form property for special primitive extensions in [19]

(2) From rational functions to D-finite functions. The case of hyperexponential functions was studied by Davenport [38] and by Geddes, Le and Li in [42] with a refined version in [12] satisfying the normal-form property. Trager's reduction for algebraic was extended to fuchsian D-finite functions in [27, 31]. Recently, the case of general D-finite functions that satisfy arbitrary order linear differential equations with polynomial coefficients was handled in [13, 64].

(3) From univariate to multivariate. The rational reduction was extended to the bivariate case in [28] via residues that is equivalent to the univariate algebraic case and then to the multivariate case in [15, 49] using the Griffiths–Dwork method.

(4) From continuous to discrete. The discrete analogue for rational functions was presented by Abramov in [1, 2] and also by Paule via greatest factorial factorizations in [52]; Abramov's redution has been extended to the bivariate case in [21]; the hypergeometric case was studied by Abramov and Petkovšek in [5, 6] and modified in [23] with the normal-form property. The q-analogue of the modified Abramov-Petkovšek reduction was presented in [40]. The general P-recursive case (in terms of linear difference systems) has been given in [65].

3 EXISTENCE VIA REDUCTION

Zeilberger's algorithm [69] is the first fast algorithm for creative telescoping, which has been implemented in most of computer algebra systems. The termination problem of Zeilberger's algorithm is equivalent to the existence problem of telescopers.

The first celebrated result on the existence of telescopers is Zeilberger's theorem that telescopers always exist for holonomic functions using Bernstein's theory of algebraic D-modules [68]. With an elementary dimensional counting, Wilf and Zeilberger in [67] proved that telescopers also exist for proper hypergeometric terms that are products of polynomials, geometric sequences and factorials. Actually, holonomicity is equivalent to properness for hypergeometric terms via the Wilf and Zeilberger conjecture, which has been proved [7, 29, 44, 53]. The above work only provides sufficient conditions for the existence of telescopers.

Telescopers may still exist for non-holonomic functions or nonproper terms [33, 36]. So holonomicity and properness are not necessary conditions. The first necessary and sufficient condition on the existence of telescopers was given by Abramov and Le [4] for rational functions in two discrete variables. In 2003, Abramov presented the existence criterion for the bivariate hypergeometric case [3]. Abramov's criterion was soon extended to the q-hypergeometric case in [32], and more recently to the mixed rational and hypergeometric case in [18, 30]. In the bivariate case, all of the existence criteria state that a bivariate (q-)hypergeometric term or mixed hypergeometric term has a telescoper if and only if the remainder in the additive decomposition obtained by reduction as in (2.1) is proper. But this pattern is not preserved when one go beyond the bivariate case in which the situation becomes more complicated. The existence problem of telescopers for rational functions in three variables was studied in [20, 22] using variants of the Ostrogradsky–Hermite reduction.

4 CONSTRUCTION VIA REDUCTION

The available algorithms for constructing telescopers can be divided into four generations. Algorithms of the first generation are based on the noncommutative elimination theory for operator ideals [37, 41, 54, 61, 62, 66]. Zeilberger's algorithm [68] and its generalizations [9, 34, 46, 60] form the second generation. The third generation is inspired by complexity analysis of creative telescoping algorithms with the first work by Apagodu and Zeilberger [10, 50] with generalizations in [24–26, 47]. The fourth and most recent generation of creative telescoping algorithms are called reduction-based algorithms. They were first introduced in 2010 for bivariate rational functions using the Ostrogradsky–Hermite reduction [11]. The basic idea is explained as follows. Let $f \in \mathbb{K}(x, y)$ with $\mathbb{K}$ being a field of characteristic zero. Applying the Ostrogradsky–Hermite reduction to the successive derivatives $D_x^i(f)$ yields

$$D_x^i(f) = D_y(g_i) + r_i \quad \text{with } r_i = \frac{a_i}{b_i}$$

where $g_i \in \mathbb{K}(x, y)$ and $a_i, b_i \in \mathbb{K}[x, y]$ with $\deg_y(a_i) < \deg_y(b_i)$ and $\gcd(a_i, b_i) = 1$. Moreover, all of the b_i's are squarefree and divide the squarefree part $b \in \mathbb{K}[x, y]$ of the denominator of f as polynomials in y over $\mathbb{K}(x)$. So we can write $r_i = u_i/b$ with $\deg_y(u_i) < \deg_y(b)$, which implies that the $\mathbb{K}(x)$-subspace of $\mathbb{K}(x, y)$ spanned by the remainders $r_0, r_1, \ldots$ is of finite dimension. Assume that d is the dimension. Then we can find $c_0, \ldots, c_d \in \mathbb{K}(x)$, not all zero, such that $c_0 r_0 + \cdots + c_d r_d = 0$. For these $c_0, \ldots, c_d$, we then have

$$c_0 f + c_1 D_x(f) + \cdots + c_d D_x^d(f) = D_y(g_0 + g_1 + \cdots + g_d),$$

this means that $L = c_0 + c_1 D_x + \cdots c_d D_x^d$ is a telescoper for f. The first linear dependency between the r_i's leads to the telescoper for f of minimal order due to the normal-form property.

The approach is not limited to bivariate rational functions and has been generalized to bivariate hyperexponential functions [12], hypergeometric terms [23, 45], algebraic functions [27], and fuchsian D-finite functions [31]. It has also been worked out for the mixed hyperexponential-hypergeometric case in [14], and it is being worked out in the ongoing work [39] in the q-case. The most general D-finite case has been also studied in [64, 65]. Beyond the bivariate case, the first extension is for the differential case in [15, 49] and then recently for the trivariate rational case in [21]. The significate advantage of this approach is that these algorithms separate the computation of telescopers from that of certificates which is often more costly and not needed for many applications.

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