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An Exercise on Real Elementary Functions in the Book "Symbolic Integration I" (second edition)

Shaoshi Chen, Ruyong Feng, Ziming Li and Huaifu Wang
Key Laboratory of Mathematics Mechanization
Institute of Systems Science, AMSS, Academia Sinica
Beijing 100080, China
{schen, ryfeng}@amss.ac.cn, {zmli,wanghf}@mmrc.iss.ac.cn

Abstract. In this note, we present an answer to Exercise 9.3 in the Book *Symbolic Integration I* (second edition) by M. Bronstein, under an additional assumption that the real elementary extension in the exercise is purely transcendental. Our answer is based on a rather technical lemma derived from a naive attempt to do the exercise inductively.

1. Introduction

Exercise 9.3 in [1, Chapter 9] states that

Let C be a field of characteristic 0, x be transcendental over C, and (K, D) be a real elementary extension of $(C(x), d/dx)$ with $\mathrm{Const}_D(K) = C$. Suppose that there are a, b in K such that $b^2 + 1 \neq 0$, Da/a is not the derivative on an element of K, and

$$\frac{Da}{a} = \frac{Db}{b^2+1}. \tag{1}$$

Show that $\sqrt{-1} \in C$. Conclude that if C is a real field, then the index sets $L_{K/C(x)}$ and $A_{K/C(x)}$ are disjoint.

We now explain the terminologies appearing in the exercise. Let (F, D) be a differential field of characteristic zero, and (E, D) a differential extension of (F, D). An element $t \in E$ is said to be *real elementary* over F if either t is algebraic over F, or there exists an element $a \in F$ such that one of the following conditions is fulfilled:

(i) $D(t) = D(a)t$, in which case we say that t is *exponential* over F, and write $t = \exp(a)$;

(ii) $D(t) = \frac{D(a)}{a}$ with $a \neq 0$, in which case we say that t is *logarithmic* over F, and write $t = \log(a)$;

(iii) $D(t) = D(a)(t^2+1)$, in which case we say that t is a *tangent* over F, and write $t = \tan(a)$;

(iv) $D(t) = \frac{D(a)}{a^2+1}$ with $a^2 + 1 \neq 0$, in which case we say that t is an *arc-tangent* over F, and write $t = \arctan(a)$.

We may write the real elementary extension in Exercise 9.3 as

$$K = C(x)(t_1, t_2, \ldots, t_n),$$

where t_i is real elementary over $C(x)(t_1, \ldots, t_{i-1})$ for all i with $1 \leq i \leq n$. Put $K_0 = C(x)$ and $K_i = C(x)(t_1, \ldots t_i)$ for i with $1 \leq i \leq n$. We define the following four index sets:

(a) $E_{K/K_0} = \{i \in \{1, \ldots, n\} \,|\, t_i$ is transendental and exponential over $K_{i-1}\}$,

(b) $L_{K/K_0} = \{i \in \{1, \ldots, n\} \,|\, t_i$ is transendental and logarithmic over $K_{i-1}\}$,

(c) $T_{K/K_0} = \{i \in \{1, \ldots, n\} \,|\, t_i$ is transendental and tangent over $K_{i-1}\}$, and

(d) $A_{K/K_0} = \{i \in \{1, \ldots, n\} \,|\, t_i$ is transendental and arc-tangent over $K_{i-1}\}$.

In this note we do this exercise under the additional assumption that t_i is transcendental over K_{i-1} for all i with $1 \leq i \leq n$. We are not yet able to complete the exercise when some t_i is algebraic over K_{i-1}.

2. Preliminaries

For brevity, $\sqrt{-1}$ is denoted by $\mathbf{i}$ in the sequel. Let (F, D) be a differential field and t an indeterminate over F. We extend D to $F[t]$ by defining $D(t)$ to be an element of $F[t]$ (see [1, Theorem 3.2.2]). Then D can be further extended uniquely to $F(t)$. Such a field extension is called a *monomial extension* of F (see [1, Deinition 3.4.1].) A polynomial $f \in F[t]$ is said to be *special* with respect to D if $\gcd(f, D(f)) = f$, while f is said to be *normal* if $\gcd(f, D(f)) = 1$. Note that a special polynomial is also called a *Darboux polynomial* in $F[t]$.

We are interested in the set of special polynomials in $F[t]$, where t is a transcendental and real elementary over F with $\mathrm{Const}(F(t)) = \mathrm{Const}(F)$. By a straightforward calculation, we have the following table, in which MISP is the abbreviation for monic, irreducible and special polynomials.

	t is exp	t is log	t is tan	t is arctan
MISP ($\mathbf{i} \notin F$)	$1, t$	1	$1, t^2 + 1$	1
MISP ($\mathbf{i} \in F$)	$1, t$	1	$1, t - \mathbf{i}, t + \mathbf{i}$	1

Every element of F is special. The product of special polynomials is special, and every factor of a special polynomial is special (see [1, Theorem 3.4.1]). All special polynomials in $F[t]$ can be obtained from the monic and irreducible ones. By the definition of special (normal) polynomials, we see that if p is special (normal) in $F[t]$, it is special (normal) in $E[t]$, where E is an algebraic extension of F.

For $p \in F[t]$ with $\deg p > 0$, we define a map ν_p from $F[t]$ to $\mathbb{N} \cup \infty$ by sending 0 to ∞, and a nonzero element a to $\max\{n \in \mathbb{N} \,|\, p^n | a\}$. Extend ν_p to $F(t)$ by sending $\frac{a}{b}$ to $\nu_p(a) - \nu_p(b)$, where $a, b \in F[t]$. For $f \in F(t)$, the value $\nu_p(f)$ is called the *order of f at p*. The following lemma will be frequently used in the sequel.

Lemma 1 *Let $F(t)$ be a monomial extension over F, and p an irreducible polynomial in $F[t] \setminus F$. Then the following statements hold.*

1. $\nu_p\left(\frac{D(f)}{f})\right) \geq -1$ *for all nonzero elements f in $F(t)$.*

2. *If p is normal, then $\nu_p(Df) = \nu_p(f) - 1$ for every $f \in F(t)$ with $\nu_p(f) \neq 0$, and $\nu_p(Df) \geq 0$ for every $f \in F(t)$ with $\nu_p(f) = 0$. In particular, we have $\nu_p(f) \neq -1$ for all $f \in F(t)$.*

3. *If t is real elementary over F and if p is special, then $\nu_p(Df) = \nu_p(f)$ for every $f \in F(t)$ with $\nu_p(f) \neq 0$.*

The first assertion of the lemma is given in [1, Corollary 4.4.2]. The rest is a special case in [1, Theorem 4.4.2]. The correctness of the last assertion is due to the fact that a special polynomial in a transcendental and real elementary extension over F is always of the first kind. Of course, one can prove this lemma by direct calculations using the above table.

Next, we recall the notion of residues at a normal and irreducible polynomial p in $F[t]$. The valuation ring of ν_p is

$$O_p = \{f \in F(t) \,|\, \nu_p(f) \geq 0\}.$$

The unique maximal ideal of O_p is

$$p\,O_p = \{f \in F(t) \,|\, \nu_p(f) \geq 1\}.$$

Denote by π_p the canonical projection from O_p to $O_p/p\,O_p$, and let

$$R_p = \{f \in F(t) \,|\, \nu_p(f) \geq -1\},$$

which is a vector space over F. For every $f \in R_p$, the product $fp/D(p)$ is an element in O_p, because, $\nu_p(fp)$ is greater than or equal to zero and $D(p)$ is co-prime with p. The residue at p is defined to be the map

$$\begin{array}{rccc} \rho_p: & R_p & \longrightarrow & O_p/p\,O_p \\ & f & \mapsto & \pi_p\left(f\frac{p}{D(p)}\right). \end{array}$$

Note that the field $O_p/p\,O_p$ is isomorphic to $F[t]/(p)$ (see [1, Theorem 4.2.1]). The following two properties are useful.

1. The residue map ρ_p is F-linear.

2. $\rho_p\left(\frac{Df}{f}\right) = \nu_p(f)$ for every nonzero element f in $F(t)$

These two assertions are special cases in Theorem 4.4.1 and Corollary 4.4.2 in [1]. Again, one can verify their correctness directly.

Lemma 2 *Let $F(t)$ be a monomial extension of (F, D) and p a normal and irreducible polynomial in $F[t] \setminus F$. If $f \in F[t]$ is special, then, for every $g \in F[t]$, the residue $\rho_p\left(\frac{g}{f}\right)$ is well-defined and equal to zero.*

Proof. Since f is special, so are its factors. Therefore, p is not a factor of f. It follows that $\frac{g}{f}\frac{p}{Dp}$ is in $p\,O_p$. □

3. Results

In this section, we do Exercise 9.3 in [1] under the additional assumption that the real elementary extension is purely transcendental. For an element f in $F(t)$, we denote the numerator and denominator of f by $\mathrm{num}(f)$ and $\mathrm{den}(f)$, respectively. The denominator of an element of $F(t)$ is normalized to be monic. If $\mathbf{i}$ is not in a field F and f is in $F(\mathbf{i})$, we denote the conjugate of f by $\bar{f}$.

Lemma 3 *Let $(k,\, D)$ be a differential field of characteristic 0, and C_k be the field of constants in k. Assume that $\mathbf{i}$ is not in k. Let $k(t)$ be a monomial extension of k. Assume further that*

$$D(v) + \frac{D(a)}{a} + \sum_{j=1}^{\lambda} c_j \frac{D(f_j)}{f_j} = \frac{D(b)}{b^2+1} + \sum_{\ell=1}^{\mu} d_\ell \frac{D(g_\ell)}{g_\ell^2+1}, \tag{2}$$

in which v, a, f_j, b, g_ℓ are in $k(t)$ and c_j, d_ℓ in C_k. Then the following assertions are true.

(i) *In the field $k(\mathbf{i})[t]$, (2) can be rewritten as*

$$\begin{aligned} & D(v) + \tfrac{D(\mathrm{num}(a))}{\mathrm{num}(a)} - \tfrac{D(\mathrm{den}(a))}{\mathrm{den}(a)} + \textstyle\sum_{j=1}^{\lambda} c_j \left(\tfrac{D(\mathrm{num}(f_j))}{\mathrm{num}(f_j)} - \tfrac{D(\mathrm{den}(f_j))}{\mathrm{den}(f_j)}\right) \\ = \ & \tfrac{1}{2\mathbf{i}}\left(\tfrac{DB}{B} - \tfrac{D\overline{B}}{\overline{B}}\right) + \tfrac{1}{2\mathbf{i}} \textstyle\sum_{\ell=1}^{\mu} d_\ell \left(\tfrac{D(G_\ell)}{G_\ell} - \tfrac{D(\overline{G}_\ell)}{\overline{G}_\ell}\right) \end{aligned} \tag{3}$$

where $B = \mathrm{num}(b) - \mathbf{i}\cdot\mathrm{den}(b)$ and $G_\ell = \mathrm{num}(g_\ell) - \mathbf{i}\cdot\mathrm{den}(g_\ell)$. In addition, both $\gcd(B,\, \overline{B})$ *and* $\gcd(G_\ell,\, \overline{G}_\ell)$ *are equal to one.*

(ii) *If both $\{1, c_1, \cdots, c_\lambda\}$ and $\{1, d_1, \cdots, d_\mu\}$ are linearly independent over $\mathbb{Q}$, then* $\mathrm{den}(v)$, $\mathrm{num}(a)$, $\mathrm{den}(a)$, $\mathrm{num}(f_j)$ *and* $\mathrm{den}(f_j)$ *are special in $k[t]$, and, moreover, B, $\overline{B}$, G_ℓ and $\overline{G}_\ell$ are special in $k(\mathbf{i})[t]$.*

Proof. Applying the logarithmic derivative identity yields

$$\frac{Da}{a} = \frac{D(\mathrm{num}(a))}{\mathrm{num}(a)} - \frac{D(\mathrm{den}(a))}{\mathrm{den}(a)}.$$

The same holds when a is replaced by f_j. A straightforward calculation yields

$$\frac{Db}{b^2+1} = \left(\frac{D(B)}{B} - \frac{D(\overline{B})}{\overline{B}}\right).$$

The same holds when b and B are replaced by g_ℓ and G_ℓ, respectively. So (2) is rewritten as (3) in $K(\mathbf{i})$.

Since $\gcd(\mathrm{num}(b), \mathrm{den}(b)) = 1$, $\gcd(B + \overline{B}, B - \overline{B}) = 1$. Consequently, $\gcd(B, \overline{B}) = 1$. In the same vein, we derive that $\gcd(G_\ell, \overline{G}_\ell) = 1$. The first assertion is proved.

To prove the second assertion, we regard all polynomials appearing in (3) as elements in $k(\mathbf{i})[t]$.

First, we show that $\mathrm{den}(v)$ is special. Suppose the contrary that $\mathrm{den}(v)$ is not special. Then $\mathrm{den}(v)$ has a factor p, which is irreducible and normal in $k(\mathbf{i})[t]$. Since $\nu_p(v)$ is less than zero, $\nu_p(Dv)$ is less than -1 by the second assertion of Lemma 1, while the order of a logarithmic derivative at p is at least -1 by the first assertion of Lemma 1. So the order of the left hand-side of (3) at p is less than -1. But that of the right hand-side of (3) is at least -1, a contradiction.

Next, we show that $\mathrm{num}(a)$ is special. Suppose on the contrary that $\mathrm{num}(a)$ is not special. Then $\mathrm{num}(a)$ has a factor p, which is irreducible and normal in $k(\mathbf{i})[t]$. By Lemma 2, the residue $\rho_p(v) = 0$ since $\gcd(\mathrm{den}(v), p) = 1$, and, moreover $\rho_p(\mathrm{den}(a)) = 0$ since $\gcd(\mathrm{den}(a), \mathrm{num}(a)) = 1$. By the second property of ρ_p listed in Section 2.,

$$\rho_p\left(\frac{D(\mathrm{num}(a))}{\mathrm{num}(a)}\right) = \nu_p(\mathrm{num}(a)) \in \mathbb{Z}.$$

In the same way, we have, for all j with $1 \le j \le \lambda$,

$$\rho_p\left(\frac{D(\mathrm{num}(f_j))}{\mathrm{num}(f_j)} - \frac{D(\mathrm{den}(f_j))}{\mathrm{den}(f_j)}\right) = \rho_p\left(\frac{D(f_j)}{f_j}\right) = \nu_p(f_j) \in \mathbb{Z}.$$

Therefore, the residue of the left hand-side of (3) is equal to

$$r_1 = \nu_p(\mathrm{num}(a)) + \sum_{j=1}^{\lambda} c_j \nu_p(f_j) \in C_k.$$

To compute the residue of the right hand-side of (3), we get

$$\begin{aligned} r_2 &= \rho_p\left(\tfrac{1}{2\mathbf{i}}\left(\left(\tfrac{DB}{B} - \tfrac{D\overline{B}}{\overline{B}}\right) + \textstyle\sum_{\ell=1}^{\mu} d_\ell \left(\tfrac{D(G_\ell)}{G_\ell} - \tfrac{D(\overline{G}_\ell)}{\overline{G}_\ell}\right)\right)\right) \\ &= \tfrac{1}{2\mathbf{i}}\rho_p\left(\tfrac{DB}{B} - \tfrac{D\overline{B}}{\overline{B}}\right) + \tfrac{1}{2\mathbf{i}}\textstyle\sum_{\ell=1}^{\mu} d_\ell \rho_p\left(\tfrac{D(G_\ell)}{G_\ell} - \tfrac{D(\overline{G}_\ell)}{\overline{G}_\ell}\right) \\ &= \tfrac{1}{2\mathbf{i}}\left(\nu_p(B) - \nu_p(\overline{B})\right) + \tfrac{1}{2\mathbf{i}}\textstyle\sum_{\ell=1}^{\mu} d_\ell \left(\nu_p(G_\ell) - \nu_p(\overline{G}_\ell)\right) \end{aligned} \tag{4}$$

which is in $\frac{1}{2\mathbf{i}}C_k$. Since $r_1 = r_2$ and $\mathbf{i} \notin C_k$, r_1 has to be zero. Note that the order of a at p is nonzero, since p divides $\mathrm{num}(a)$. Hence $r_1 = 0$ would imply that $1, c_1, \cdots, c_\lambda$ are $\mathbb{Q}$-linearly dependent, a contradiction. It follows that $\mathrm{num}(a)$ is special. In the same vein, one sees that $\mathrm{den}(a)$, $\mathrm{num}(f_j)$ and $\mathrm{den}(f_j)$ are all special for all j with $1 \le j \le \lambda$.

At last, we show that B, $\overline{B}$, G_ℓ, and $\overline{G}_\ell$, are special in $k(\mathbf{i})[t]$. Suppose that B is not special. Then there exists a normal and irreducible polynomial q in $k(\mathbf{i})[t] \setminus k(\mathbf{i})$ dividing B. Since $\gcd(B, \overline{B}) = 1$, $\nu_p(\overline{B})$ equals zero. Thus, the difference $\nu_p(B) - \nu_p(\overline{B})$ is nonzero. Since $\mathrm{den}(v)$ is special, $\rho_p(v) = 0$ by Lemma 2. Similarly, we have

$$\rho_p(\mathrm{num}(f_j)) = \rho_p\mathrm{den}(f_j)) = \rho_q(\mathrm{num}(a)) = \rho_q(\mathrm{den}(a)) = 0.$$

Thus r_1 is equal to zero, and so is r_2. By (4),

$$\nu_p(B) - \nu_p(\overline{B}) + \sum_{\ell=1}^{\mu} d_\ell \left(\nu_p(G_\ell) - \nu_p(\overline{G}_\ell)\right) = 0.$$

Since $\nu_q(B) - \nu_q(\overline{B})$ is nonzero, the elements $1, d_1, \ldots, d_\mu$ are $\mathbb{Q}$-linearly dependent, a contradiction. $\square$

The next lemma is the main result of this note.

Lemma 4 *Let C be a field of characteristic zero with $\mathbf{i} \notin C$, and x an indeterminate over C. Let $K = C(x)(t_1, t_2, ..., t_n)$ be a differential extension of $\left(C(x), \frac{d}{dx}\right)$, in which t_i is transcendental and real elementary over the subfield $C(x)(t_1, t_2, ..., t_{i-1})$, $i = 1, \ldots, n$. Assume that C is the constant field of K, and denote by D the derivation operator on K. Assume further that*

$$D(v) + \frac{D(a)}{a} + \sum_{j=1}^{\lambda} c_j \frac{D(f_j)}{f_j} = \frac{D(b)}{b^2+1} + \sum_{\ell=1}^{\mu} d_\ell \frac{D(g_\ell)}{g_\ell^2+1}, \tag{5}$$

where v, a, f_j, b, g_ℓ are in K, and c_j, d_ℓ are in C, $j = 1, ..., \lambda$, and $\ell = 1, ..., \mu$. If both $1, c_1, \ldots, c_\lambda$ and $1, d_1, \ldots, d_\mu$ are $\mathbb{Q}$-linearly independent, then $\frac{Da}{a}$ is a derivative of some element of K.

Proof. We proceed by induction on n. If $n = 0$, then the special polynomials in $C[x]$ are precisely the elements of C. By the second assertion of Lemma 3, both $\mathrm{num}(a)$ and $\mathrm{den}(a)$ are in C, so is a itself. Thus $\frac{D(a)}{a} = 0 = \frac{D(1)}{1}$.

Assume $n > 0$, and put $k = C(x)(t_1, \cdots, t_{n-1})$ and $t = t_n$. In particular, $k = C(x)$ when $n = 1$. We assume that the lemma holds when $K = k$ and prove that it holds when $K = k(t)$ by a case distinction.

Before completing our induction, we recall a useful identity. Let (F, D) be a differential field, and p, q in F. If $p + q \neq 0$, then

$$\frac{D(p)}{p^2+1} + \frac{D(q)}{q^2+1} = \frac{D(r)}{r^2+1}, \qquad \text{where } r = \tfrac{pq-1}{p+q},$$

which, together with the obvious identity $-\frac{D(p)}{p^2+1} = \frac{D(-p)}{(-p)^2+1}$, implies that, for any $m_i \in \mathbb{Z}$ and $h_i \in F$, there exists $h \in F$ such that

$$\sum_i m_i \frac{D(h_i)}{h_i^2+1} = \frac{D(h)}{h^2+1}. \tag{6}$$

Logarithmic Case: Suppose that $D(t) = \frac{D(u)}{u}$ for some $u \in k$. Note that $D(u)$ is nonzero, because $\mathrm{Const}(K) = C$ and $t \notin k$. The special polynomials in $k[t]$ are the elements of k (see the table in Section 2.). By the second assertion of Lemma 3, we conclude that

$$a, f_1, \ldots, f_\lambda, b, g_1, \ldots, g_\mu \in k \tag{7}$$

and that v is in $k[t]$. Moreover, $D(v)$ is in k by (5). Therefore v is of the form $ct + v_0$, where c is in C and v_0 is in k. It follows that

$$D(v) = c\frac{D(u)}{u} + D(v_0). \tag{8}$$

So (5) can be rewritten as

$$D(v_0) + \frac{D(a)}{a} + c\frac{D(u)}{u} + \sum_{j=1}^{\lambda} c_j \frac{D(f_j)}{f_j} = \frac{D(b)}{b^2+1} + \sum_{\ell=1}^{\mu} d_\ell \frac{D(g_\ell)}{g_\ell^2+1}. \tag{9}$$

If $c, 1, c_1, \ldots, c_\lambda$ are $\mathbb{Q}$-linear independent, then, by (7), the induction hypothesis can be directly applied to (9), which yields that $\frac{D(a)}{a}$ is a derivative of some element of k. If c is $\mathbb{Q}$-linearly dependent on $1, c_1, \ldots, c_\lambda$, then there exist ξ, ξ_j and nonzero η in $\mathbb{Z}$ such that

$$c = \frac{\xi}{\eta} + \sum_{j=1}^{\lambda} \frac{\xi_j}{\eta} c_j. \tag{10}$$

Substituting the right hand-side of (10) for c into (9) yields

$$D(\eta v_0) + \frac{D(au^\xi)}{au^\xi} + \sum_{j=1}^{\lambda} c_j \frac{D(f_j u^{\xi_j})}{f_j u^{\xi_j}} = \eta \left(\frac{D(b)}{b^2+1} + \sum_{\ell=1}^{\mu} d_\ell \frac{D(g_\ell)}{g_\ell^2+1} \right).$$

By (6), the above equation implies

$$D(\eta v_0) + \frac{D(au^\xi)}{au^\xi} + \sum_{j=1}^{\lambda} c_j \frac{D(f_j u^{\xi_j})}{f_j u^{\xi_j}} = \left(\frac{D(\tilde{b})}{\tilde{b}^2+1} + \sum_{\ell=1}^{\mu} d_\ell \frac{D(\tilde{g}_\ell)}{\tilde{g}_\ell^2+1} \right)$$

for some $\tilde{b}, \tilde{g}_1, \ldots, \tilde{g}_\mu \in k$. Applying the induction hypothesis to this equation yields that $\frac{D(au^\xi)}{au^\xi}$ is a derivative of some element in k. Observe that

$$\frac{D(au^\xi)}{au^\xi} = \frac{D(a)}{a} + \xi \frac{D(u)}{u} = \frac{D(a)}{a} + D(\xi t).$$

So $\frac{D(a)}{a}$ is a derivative of some element in K. This completes the induction for the logarithmic case.

Arc-tangent Case: Suppose that $D(t) = \frac{D(u)}{u^2+1}$ for some $u \in k$ with $u \neq 0$. The special polynomials in $k[t]$ are the elements of k (see the table in Section 2.). By the second assertion of Lemma 3, we conclude that

$$a, f_1, \ldots, f_\lambda, b, g_1, \ldots, g_\mu \in k \tag{11}$$

and that v is in $k[t]$. Moreover, $D(v)$ is in k by (5). Therefore v is of the form $-dt + v_0$, where d is in C and v_0 is in k. It follows that

$$D(v) = -d \frac{D(u)}{u^2+1} + D(v_0). \tag{12}$$

So (5) can be rewritten as

$$D(v_0) + \frac{D(a)}{a} + \sum_{j=1}^{\lambda} c_j \frac{D(f_j)}{f_j} = \frac{D(b)}{b^2+1} + d \frac{D(u)}{u^2+1} + \sum_{\ell=1}^{\mu} d_\ell \frac{D(g_\ell)}{g_\ell^2+1}. \tag{13}$$

If $d, 1, d_1, \ldots, d_\mu$ are $\mathbb{Q}$-linear independent, then, by (11), the induction hypothesis can be directly applied to (13), which yields that $\frac{D(a)}{a}$ is a derivative of some element of k. If d is $\mathbb{Q}$-linearly dependent on $1, d_1, \ldots, d_\mu$, then there exist ξ, ξ_ℓ and nonzero η in $\mathbb{Z}$ such that

$$d = \frac{\xi}{\eta} + \sum_{\ell=1}^{\mu} \frac{\xi_\ell}{\eta} d_\ell. \tag{14}$$

Substituting the right hand-side of (14) for d into (13) yields

$$D(\eta v_0) + \frac{D(a^\eta)}{a^\eta} + \sum_{j=1}^{\lambda} c_j \frac{D(f_j^\eta)}{f_j^\eta} = \eta \frac{D(b)}{b^2+1} + \xi \frac{D(u)}{u^2+1} + \sum_{\ell=1}^{\mu} d_\ell \left(\xi_\ell \frac{D(u)}{u^2+1} + \frac{D(g_\ell)}{g_\ell^2+1} \right).$$

Again, (6) allows us to derive from the above equation that

$$D(\eta v_0) + \frac{D(a^\eta)}{a^\eta} + \sum_{j=1}^{\lambda} c_j \frac{D(f_j^\eta)}{f_j^\eta} = \frac{D(p)}{p^2+1} + \sum_{\ell=1}^{\mu} d_\ell \frac{D(q_\ell)}{q_\ell^2+1}.$$

for some $p, q_1, \ldots, q_\mu \in k$. Applying the induction hypothesis to this equation yields that $\frac{D(a^\eta)}{a^\eta}$ is a derivative of some element in k, and so is $\frac{D(a)}{a}$. This completes the induction for the arc-tangent case.

Exponential Case: Suppose that $D(t) = D(u)t$ for some $u \in k$ with $u \neq 0$. The special polynomials in $k[t]$ are either elements in k or monomials in t (see the table in Section 2.). By the second assertion of Lemma 3, both $\mathrm{num}(a)$ and $\mathrm{den}(a)$ are special. Thus $a = \tilde{a}t^\alpha$ with $\tilde{a} \in k$ and $\alpha \in \mathbb{Z}$. It follows that

$$\frac{D(a)}{a} = \frac{D(\tilde{a})}{\tilde{a}} + \alpha \frac{D(t)}{t} = \frac{D(\tilde{a})}{\tilde{a}} + \alpha D(u). \tag{15}$$

In the same vein, for $j = 1, \ldots, \mu$,

$$\frac{D(f_j)}{f_j} = \frac{D(\tilde{f}_j)}{\tilde{f}_j} + \alpha_j D(u) \quad \text{for some } \tilde{f}_j \in k \text{ and } \alpha_j \in \mathbb{Z}. \tag{16}$$

Since u is in k, (15) and (16) imply that

$$\frac{D(a)}{a}, \frac{D(f_1)}{f_1}, \ldots, \frac{D(f_\lambda)}{f_\lambda} \in k. \tag{17}$$

We are going to show that

$$b,\, g_1, \ldots, g_\mu \in k. \tag{18}$$

By the first assertion of Lemma 3, (5) implies (3). The second assertion then implies that $B = \mathrm{num}(b) - \mathbf{i} \cdot \mathrm{den}(b)$ is special in $k(\mathbf{i})[t]$. Thus, $B = (h_1 - \mathbf{i}h_2)t^\beta$ for some $h_1, h_2 \in k$ and $\beta \in \mathbb{Z}$. Since $\mathbf{i}$ is not in $k[t]$, $\mathrm{num}(b)$ and $\mathrm{den}(b)$ are equal to $h_1 t^\beta$ and $h_2 t^\beta$, respectively. Since $\mathrm{num}(b)$ and $\mathrm{den}(b)$) are co-prime, β is zero, and thus b is in k. The same argument concludes that the g_ℓ are all in k.

Now, we show that v is in k. It follows from (5), (17) and (18) that $D(v)$ is in k. The second assertion of Lemma 3 implies that $\mathrm{den}(v) = t^\gamma$ for some $\gamma \in \mathbb{N}$. If γ is nonzero, then $\nu_t(v)$ less than zero, and so is $\nu_t(D(v))$ by the last assertion of Lemma 1, which contradicts with the fact that $D(v)$ is in k. Thus v is in $k[t]$. Suppose that the degree ϵ of v is greater than zero. Write

$$v = rt^\epsilon + \text{terms in which } t \text{ has degree less than } \epsilon,$$

with $r \in k$ and $r \neq 0$. Then

$$D(v) = (D(r) + \epsilon r D(u))t^{\epsilon} + \text{terms in which } t \text{ has degree less than } \epsilon.$$

It follows from the fact $D(v) \in k$ that

$$D(r) + r\epsilon D(u) = D(r) + \epsilon r \frac{D(t)}{t} = r \cdot \frac{D\left(rt^{\epsilon}\right)}{rt^{\epsilon}} = 0.$$

Consequently, rt^{ϵ} is in C, and, thus, t is algebraic over k, a contradiction. This proves that v is in k.

Set $w = v + \left(\alpha + \sum_{j=1}^{\lambda} \alpha_j c_j\right) u$. From (15) and (16) we rewrite (5) as

$$D(w) + \frac{D(\tilde{a})}{\tilde{a}} + \sum_{j=1}^{\lambda} c_j \frac{D(\tilde{f}_j)}{\tilde{f}_j} = \frac{D(b)}{b^2+1} + \sum_{\ell=1}^{\mu} d_\ell \frac{D(g_\ell)}{g_\ell^2+1}.$$

By (15), (16), (17) and (18) and the definition of w, we see that w, $\tilde{a}$, $\tilde{b}$, the $\tilde{f}_j$ and the $\tilde{g}_\ell$ are all in k. By the induction hypothesis there exists $\tilde{r}$ in k such that $\frac{D(\tilde{a})}{a} = D(\tilde{r})$. It follows from (15) that

$$\frac{D(a)}{a} = D(\tilde{r}) + \alpha D(u) = D(\tilde{r} + \alpha u).$$

This completes our induction for the exponential case.

Tangent Case: Suppose that $D(t) = D(u)(t^2+1)$ for some $u \in k$ with $D(u) \neq 0$. We proceed along in the same line as in the exponential case. The set of the monic and irreducible special polynomials in $k[t]$ is $\{1, t^2+1\}$ (see the table in Section 2.). By the second assertion of Lemma 3, both $\mathrm{num}(a)$ and $\mathrm{den}(a)$ are special in $k[t]$. Thus $a = \tilde{a}(t^2+1)^{\alpha}$ with $\tilde{a} \in k$ and $\alpha \in \mathbb{Z}$. It follows that

$$\frac{D(a)}{a} = \frac{D(\tilde{a})}{\tilde{a}} + 2\alpha \cdot \frac{D(t)}{t^2+1} \cdot t = \frac{D(\tilde{a})}{\tilde{a}} + 2\alpha D(u)t. \tag{19}$$

In the same vein, for $j = 1, \ldots, \mu$,

$$\frac{D(f_j)}{f_j} = \frac{D(\tilde{f}_j)}{\tilde{f}_j} + 2\alpha_j D(u)t \quad \text{for some } \tilde{f}_j \in k \text{ and } \alpha_j \in \mathbb{Z}. \tag{20}$$

Since u is in k, (19) and (20) imply that

$$\frac{D(a)}{a}, \frac{D(f_1)}{f_1}, \ldots, \frac{D(f_\lambda)}{f_\lambda} \in k[t]. \tag{21}$$

We are going to show that the right hand-side of (5) is an element in k. Recall that the special polynomials in $k(\mathbf{i})[t]$ are either elements of $k(\mathbf{i})$ or monomials in $t-\mathbf{i}$ and $t+\mathbf{i}$ over $k(\mathbf{i})$. (see the table in Section 2.). Let $B = \mathrm{num}(b) - \mathbf{i} \cdot \mathrm{den}(b)$ and $\bar{B} = \mathrm{num}(b) + \mathbf{i} \cdot \mathrm{den}(b)$. By the second assertion of Lemma 3 both B and $\bar{B}$ are special. Since B and $\bar{B}$ are co-prime (see the first assertion of Lemma 3), So we may assume w.l.o.g. that $B = z(t-\mathbf{i})^\beta$ and $\bar{B} = \bar{z}(t+\mathbf{i})^\beta$, where $z = z_1 + z_2\mathbf{i}$ and $z_1, z_2 \in k$. We compute

$$\begin{aligned}
\tfrac{D(B)}{B} - \tfrac{D(\bar{B})}{\bar{B}} &= \tfrac{D(z)}{z} + \beta\tfrac{D(t-\mathbf{i})}{t-\mathbf{i}} - \tfrac{D(\bar{z})}{\bar{z}} - \beta\tfrac{D(t+\mathbf{i})}{t+\mathbf{i}} \\
&= \tfrac{D(z)}{z} + \beta\tfrac{D(u)(t^2+1)}{t-\mathbf{i}} - \tfrac{D(\bar{z})}{\bar{z}} - \beta\tfrac{D(u)(t^2+1)}{t+\mathbf{i}} \\
&= \tfrac{D(z)}{z} - \tfrac{D(\bar{z})}{\bar{z}} + 2\mathbf{i}\beta D(u) \\
&= 2\mathbf{i}\tfrac{D(y)}{y^2+1} + 2\mathbf{i}\beta D(u), \quad \text{where } y = \begin{cases} z_1/z_2 & \text{if } z_2 \neq 0 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Thus

$$\frac{1}{2\mathbf{i}}\left(\frac{D(B)}{B} - \frac{D(\bar{B})}{\bar{B}}\right) = \frac{D(y)}{y^2+1} + \beta D(u) \quad \text{for some } y \in k. \tag{22}$$

Similarly, we have, for all ℓ with $1 \leq \ell \leq \mu$,

$$\frac{1}{2\mathbf{i}}\left(\frac{D(G_\ell)}{G_\ell} - \frac{D(\bar{G}_\ell)}{\bar{G}_\ell}\right) = \frac{D(y_\ell)}{y_\ell^2+1} + \beta_\ell D(u) \quad \text{for some } y_\ell \in k. \tag{23}$$

Thus, both $\frac{1}{2\mathbf{i}}\left(\frac{D(B)}{B} - \frac{D(\bar{B})}{\bar{B}}\right)$ and $\frac{1}{2\mathbf{i}}\left(\frac{D(G_\ell)}{G_\ell} - \frac{D(\bar{G}_\ell)}{\bar{G}_\ell}\right)$ belong to k. Consequently, the right hand-side of (3) is an element in k, so is that of (5).

Next, we show that v is in k. By (5), (21) and the conclusion made in the preceding paragraph, $D(v)$ is in $k[t]$. Since $\operatorname{den}(v)$ is special by Lemma 3, $\operatorname{den}(v)$ equals $(t^2+1)^\gamma$ for some $\gamma \in \mathbb{N}$. If γ is nonzero, then $\nu_{t^2+1}(v)$ is less than zero, and so is $\nu_{t^2+1}(D(v))$ by the last assertion of Lemma 1, which contradicts with the fact that $D(v)$ is in $k[t]$. Thus v is also in $k[t]$. If the degree of v is greater than zero, then that of $D(v)$ is greater than one by a straightforward calculation. On the other hand, (19) and (20) implies that the degree of $D(v)$ is at most one, a contradiction.

By the conclusions made in the preceding two paragraphs, (5) implies that

$$\frac{D(a)}{a} + \sum_{j=1}^{\lambda} c_j \frac{D(f_j)}{f_j}$$

belongs to k. It follows from (19) and (20) that $D(u)\left(\alpha + \sum_{j=1}^{\lambda} c_j\alpha_j\right) = 0$. Since $D(u) \neq 0$, we have that $\alpha + \sum_{j=1}^{\lambda} c_j\alpha_j = 0$. Thus $\alpha = \alpha_1 = \cdots = \alpha_\lambda = 0$, because the constants $1, c_1, \ldots, c_\lambda$ are linearly independent over $\mathbb{Q}$. Accordingly, (19) and (20) become

$$\frac{D(a)}{a} = \frac{D(\tilde{a})}{\tilde{a}} \quad \text{and} \quad \frac{D(f_j)}{f_j} = \frac{D(\tilde{f}_j)}{\tilde{f}_j}, \tag{24}$$

respectively. By these two equations, (22) and (23) we rewrite (5) as

$$D\left(\underbrace{v - \beta u - \sum_{\ell=1}^{\mu} \beta_\ell u}_{q}\right) + \frac{D(\tilde{a})}{\tilde{a}} + \sum_{j=1}^{\lambda} \frac{D(\tilde{f}_j)}{\tilde{f}_j} = \frac{D(y)}{y^2+1} + \sum_{\ell=1}^{\mu} \frac{D(y_\ell)}{y_\ell^2+1}.$$

Since $q, \tilde{a}, \tilde{f}_j, y, y_\ell$ are all in k, we can apply the induction hypothesis to the above equation to conclude that $\frac{D(\tilde{a})}{\tilde{a}}$ is a derivative of some element in k, and so is $\frac{D(a)}{a}$ by (24). This completes the induction for the tangent case. □

We are ready to complete Exercise 9.3 under the additional assumption that

$$K = C(x)(t_1, t_2, \ldots, t_n),$$

where t_i is real elementary and *transcendental* over $C(x)(t_1, \ldots, t_{i-1})$ for all i with $1 \leq i \leq n$.

Set $v = 1$ and $\lambda = \mu = 0$. Then (5) in Lemma 4 becomes (1). Suppose on the contrary that $\mathbf{i}$ is not in C. Then, by Lemma 4, $\frac{D(a)}{a}$ would be a derivative of some element in K, contradiction. Suppose that there exists an integer m in the intersection of $L_{K/C(x)}$ and $A_{K/(C(x)}$. Then

$$t_m = \frac{D(a)}{a} \quad \text{and} \quad t_m = \frac{D(b)}{b^2 + 1.}$$

It follows that (1) holds, so $\mathbf{i}$ is in C, that is, C is not a real field.

References

[1] M. Bronstein. *Symbolic Integration I: Transcendental Functions*, 2nd Edition, Springer, 2004.