Creative Telescoping

Theory and Algorithms

Shaoshi Chen

Chinese Academy of Sciences
University of Waterloo

Joint Lab Meeting, Western University, CA
March 18, 2016

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joint works with A. Bostan, F. Chyzak, R. Feng, G. Fu, Q. Hou, H. Huang
M. Kauers, C. Koutschan, G. Labahn, Z. Li, M. Singer,
R. Wang, G. Xin

Zeilberger's method

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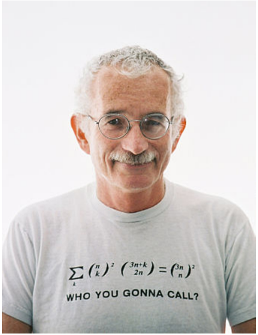

D. Zeilberger

Telescoping

Problem. For a sequence $f(k)$ in some class $\mathfrak{S}(k)$, decide whether there exists $g(k) \in \mathfrak{S}(k)$ s.t.

$$f(k) = g(k+1) - g(k)$$

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Examples.

- Rational sums

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \Delta_k\left(-\frac{1}{k}\right) = 1 - \frac{1}{n+1}$$

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- Hypergeometric sums

$$\sum_{k=0}^{n} \frac{\binom{2k}{k}^2}{(k+1)4^{2k}} = \sum_{k=0}^{n} \Delta_k\left(\frac{4k\binom{2k}{k}^2}{4^{2k}}\right) = \frac{4(n+1)\binom{2n+2}{n+1}^2}{4^{2n+2}}$$

Creative telescoping

Problem. For a sequence $f(n,k)$ in some class $\mathfrak{S}(n,k)$, find a linear recurrence operator $L \in \mathbb{F}[n,S_n]$ and $g \in \mathfrak{S}(n,k)$ s.t.

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Call g the certificate for L.

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Example. Let $f(n,k) = \binom{n}{k}^2$. Then a telescoper for f and its certificate are

$$L = (n+1)S_n - 4n - 2 \quad \text{and} \quad g = \frac{(2k-3n-3)k^2\binom{n}{k}^2}{(k-n-1)^2}$$

Proving identities

$$F(n) := \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

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Creative telescoping for $f = \binom{n}{k}^2$: $L(f) = \Delta_k(g)$, where

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Since $f(n,k) = 0$ when $k < 0$ or $k > n$, we have

$$\sum_{k=-\infty}^{+\infty} \binom{n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}^2$$

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Taking sums on both sides of $L(f) = \Delta_k(g)$:

$$\sum_{k=-\infty}^{+\infty} L(f) = L\left(\sum_{k=-\infty}^{+\infty} f\right) = g(n,+\infty) - g(n,-\infty) = 0$$

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The sequence $F(n)$ satisfies

$$(n+1)F(n+1) - (4n+2)F(n) = 0$$

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Verify the initial condition:

$$F(1) = 2 = \binom{2}{1}$$

Then the identity is proved!

Handbooks of identities

Dixon's identity

$$\sum_{k=-a}^{a}(-1)^k\binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+a}{c+k}=\frac{(a+b+c)!}{a!b!c!}$$

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Hille-Hardy's identity

$$\sum_{n=0}^{\infty}\sum_{k_1}\sum_{k_2}\frac{u^n n!}{(a+1)_n}\binom{n+a}{n-k_1}\frac{(-x)^{k_1}}{k_1!}\binom{n+a}{n-k_2}\frac{(-y)^{k_2}}{k_2!}$$
$$=(1-u)^{-a-1}\exp\left\{-\frac{(x+y)^u}{1-u}\right\}\sum_n\frac{1}{n!(a+1)_n}\left(\frac{xyu}{(1-u)^2}\right)^n$$

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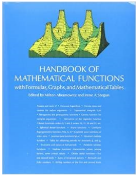

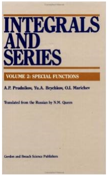

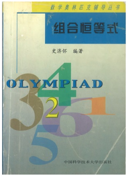

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Fundamental problems

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$$\underbrace{L(x, D_x)}_{\text{Telescoper}}(f(x,y)) = D_y(g(x,y))$$

Fundamental problems

Creative telescoping

$$\int_{-\infty}^{+\infty} \exp(-x^2/y^2 - y^2)dy = \sqrt{\pi}\exp(-2x)$$

Fundamental problems

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$$\underbrace{L(x, D_x)}_{\text{Telescoper}}(f(x,k)) = \Delta_k(g(x,k))$$

Fundamental problems

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$$\sum_{k=0}^{+\infty} \binom{2k}{k} x^k = \frac{1}{\sqrt{1-4x}}$$

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$$\underbrace{L(x, \partial_x)}_{\text{Telescoper}} (f(x, y_1, \ldots, y_m)) = \sum_{i=1}^{m} \partial_{y_i}(g_i(x, y_1, \ldots, y_m))$$

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For a function $f(n,k)$, decide whether telescopers exist?

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Tools:

- Algebraic analysis (holonomic D-modules)
- Differential and difference algebra
- Non-commutative rings (Ore polynomials)
- Computational algebraic geometry
- ...

Existence of telescopers

Timeline of works on existence problem

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Timeline of works on existence problem

1990: Zeilberger proved that telescopers always exist for holonomic functions:

Journal of Computational and Applied Mathematics 32 (1990) 321–368 321
North-Holland

A holonomic systems approach to special functions identities *

Doron ZEILBERGER
Department of Mathematics, Temple University, Philadelphia, PA 19122, USA

Existence of telescopers

Timeline of works on existence problem

1992: Wilf and Zeilberger proved that telescopers always exist for proper hypergeometric terms:

Invent. math. 108: 575–633 (1992)

Inventiones mathematicae

An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities

Herbert S. Wilf* and Doron Zeilberger**

Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104, USA
Department of Mathematics, Temple University, Philadelphia, PA 19122, USA

Existence of telescopers

Timeline of works on existence problem

2002: Abramov and Le solved the existence problem for rational functions in two discrete variables:

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Discrete Mathematics 259 (2002) 1–17

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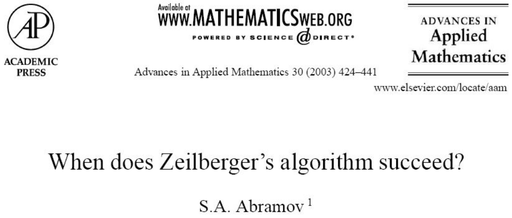
A criterion for the applicability of Zeilberger's algorithm to rational functions ☆

S.A. Abramov[a], H.Q. Le[b,*]

Existence of telescopers

Timeline of works on existence problem

2003: Abramov solved the existence problem for bivariate hypergeometric terms: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Advances in Applied Mathematics 30 (2003) 424–441

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When does Zeilberger's algorithm succeed?

S.A. Abramov [1]

Existence of telescopers

Timeline of works on existence problem

2005: W.Y.C. Chen, Hou and Mu solved the existence problem for bivariate q-hypergeometric terms:

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Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Symbolic Computation 39 (2005) 155–170
Journal of Symbolic Computation
www.elsevier.com/locate/js

Applicability of the q-analogue of Zeilberger's algorithm

William Y.C. Chen*, Qing-Hu Hou, Yan-Ping Mu

Center for Combinatorics, LPMC, Nankai University, Tianjin 300071, PR China

Existence of telescopers

Timeline of works on existence problem

2012: S. Chen and Singer solved the existence problem for bivariate rational functions in the mixed cases:

Advances in Applied Mathematics 49 (2012) 111–133

Contents lists available at SciVerse ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Residues and telescopers for bivariate rational functions ☆

Shaoshi Chen, Michael F. Singer *

Department of Mathematics, North Carolina State University, Box 8205, Raleigh, NC 27695-8205, USA

Existence of telescopers

Timeline of works on existence problem

2015: Chen et al. solved the existence problem for bivariate mixed hypergeometric terms:

Journal of Symbolic Computation 68 (2015) 1–26

Contents lists available at ScienceDirect

Journal of Symbolic Computation

www.elsevier.com/locate/jsc

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On the existence of telescopers for mixed hypergeometric terms ☆

CrossMark

Shaoshi Chen[a], Frédéric Chyzak[b], Ruyong Feng[a], Guofeng Fu[a], Ziming Li[a]

Existence of telescopers

Timeline of works on existence problem

2016: Chen et al. solved the existence problem for rational functions in three discrete variables:

Existence Problem of Telescopers: Beyond the Bivariate Case *

Shaoshi Chen[1,2], Qing-Hu Hou[3], George Labahn[2], Rong-Hua Wang[4]

Mixed hypergeometric terms

Let $\mathbb{F}$ be a field of char. zero and algebraically closed.

$$\mathbf{t} = (t_1, \ldots, t_m), \qquad \mathbf{x} = (x_1, \ldots, x_n)$$

$$D_i : \underbrace{\partial/\partial t_i}_{\text{derivations}}, \qquad S_j : \underbrace{x_j \to x_j + 1}_{\text{shifts}}$$

Definition. $h(\mathbf{t}, \mathbf{x})$ is mixed hypergeometric over $\mathbb{F}(\mathbf{t}, \mathbf{x})$ if

$$\text{all} \quad \frac{D_i(h)}{h} \quad \text{and} \quad \frac{S_j(h)}{h} \quad \text{are rational functions in } \mathbb{F}(\mathbf{t}, \mathbf{x}).$$

Remark. Mixed hypergeometric terms are solutions of systems of first-order homogeneous differential and difference equations.

Examples

- Rational functions:

$$t_1 + t_2 + x_1, \quad \frac{1}{(t_1 + t_2)}, \quad \frac{t_1 + x_1 + 1}{t_1 + t_2 + x_1^2 + 3}, \quad \dots$$

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- Hyperexponential functions:

$$\exp(t_1 + t_2^2), \quad (t_1^2 + t_2 + 1)^{\sqrt{5}}, \quad \exp\left(\int \frac{1}{t_1 + t_2}\right), \quad \ldots$$

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- Symbolic powers:

$$t_1^{x_1}, \quad (t_1 + t_2)^{x_1} \cdot (t_2 + t_3^2)^{x_2}, \quad \ldots$$

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- Hypergeometric terms:

$$2^{x_1}, \quad x_1!, \quad (x_1 + 2x_2 + \sqrt{3})!, \quad \ldots$$

Structure theorem

Theorem. Any mixed hypergeometric term $h(\mathbf{t},\mathbf{x})$ is of the form

$$f(\mathbf{t},\mathbf{x}) \cdot \prod_{j=1}^{n} \beta_j(\mathbf{t})^{x_j} \cdot \exp(g_0(\mathbf{t})) \cdot \prod_{\ell=1}^{L} g_\ell(\mathbf{t})^{c_\ell} \cdot \prod_{\lambda} (\mathbf{v}_\lambda \cdot \mathbf{x} + p_\lambda)!^{e_\lambda}$$

where f is a rational function in $\mathbb{F}(\mathbf{t},\mathbf{x})$.

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where f is a rational function in $\mathbb{F}(\mathbf{t},\mathbf{x})$.

Proper terms. A mixed hypergeometric term $h(\mathbf{t},\mathbf{x})$ is proper if it is of the form

$$P(\mathbf{t},\mathbf{x})\cdot\prod_{j=1}^{n}\beta_j(\mathbf{t})^{x_j}\cdot\exp(g_0(\mathbf{t}))\cdot\prod_{\ell=1}^{L}g_\ell(\mathbf{t})^{c_\ell}\cdot\prod_{\lambda}(\mathbf{v}_\lambda\cdot\mathbf{x}+p_\lambda)!^{e_\lambda}$$

where P is a polynomial in $\mathbb{F}[\mathbf{t},\mathbf{x}]$.

Holonomic terms

Let $H(\mathbf{z})$ be a function of continuous variables $\mathbf{z} = (z_1, \ldots, z_s)$.
Notation: $\mathscr{A}_s := \mathbb{F}[z_1, \ldots, z_s]\langle D_{z_1}, \ldots, D_{z_s} \rangle$, and

$$\mathrm{ann}_{\mathscr{A}_s}(H(\mathbf{z})) := \{L \in \mathscr{A}_s \mid L(H) = 0\}.$$

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Definition.

- $H(\mathbf{z})$ is holonomic if the Hilbert dimension of $\mathrm{ann}_{\mathscr{A}_s}(H(\mathbf{z}))$ as a left ideal of $\mathscr{A}_s$ is s.
- A function $h(\mathbf{t}, \mathbf{x})$ is holonomic if the generating function

$$H(\mathbf{t}, \mathbf{z}) = \sum_{x_1, \ldots, x_n \geq 0} h(\mathbf{t}, \mathbf{x}) z_1^{x_1} \cdots z_n^{x_n}$$

  is holonomic over $\mathscr{A}_{m+n} := \mathbb{F}(\mathbf{t}, \mathbf{z})\langle D_{t_1}, \ldots, D_{t_m}, D_{z_1}, \ldots, D_{z_n} \rangle$.

Holonomic terms

Let $H(\mathbf{z})$ be a function of continuous variables $\mathbf{z} = (z_1, \ldots, z_s)$.
Notation: $\mathscr{A}_s := \mathbb{F}[z_1, \ldots, z_s]\langle D_{z_1}, \ldots, D_{z_s}\rangle$, and

$$\operatorname{ann}_{\mathscr{A}_s}(H(\mathbf{z})) := \{L \in \mathscr{A}_s \mid L(H) = 0\}.$$

Definition.

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Remark. No algorithm for verifying holonomicity:-(

Wilf–Zeilberger conjecture: Holonomic $\Leftrightarrow$ Proper

In the fundamental paper by Wilf and Zeilberger:

Invent. math. 108: 575–633 (1992)

Inventiones mathematicae

An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities

Herbert S. Wilf* and Doron Zeilberger**

Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104, USA
Department of Mathematics, Temple University, Philadelphia, PA 19122, USA

Wilf–Zeilberger conjecture: Holonomic $\Leftrightarrow$ Proper

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Inventiones mathematicae
© Springer-Verlag 1992

An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities

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Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104, USA
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In Page 585, they said:

Our examples are all proper-hypergeometric. We conjecture that a hypergeometric term is proper-hypergeometric if and only if it is holonomic.

Wilf–Zeilberger conjecture: Holonomic $\Leftrightarrow$ Proper

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Inventiones mathematicae

An algorithmic proof theory for hypergeometric (ordinary and "*q*") multisum/integral identities

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Chen and Koutschan recently proved the conjecture:

Proof of the Wilf–Zeilberger Conjecture for Mixed Hypergeometric Terms

Shaoshi Chen[a,b], Christoph Koutschan[c]

Existence criteria: bivariate case

(L, ∂_x)	D_y	Δ_y	$\Delta_{q,y}$
$\mathbb{F}(t)\langle D_x\rangle$			
$\mathbb{F}(t)\langle S_x\rangle$			
$\mathbb{F}(t)\langle S_{q,x}\rangle$			

9 types of telescopers for bivariate mixed hypergeometric terms

Existence criteria: bivariate case

(L, ∂_x)	D_y	Δ_y	$\Delta_{q,y}$
$\mathbb{F}(t)\langle D_x\rangle$	Zeilberger1990		
$\mathbb{F}(t)\langle S_x\rangle$			
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9 types of telescopers for bivariate mixed hypergeometric terms

- In the pure continuous case, telescopers always exists since h is holonomic.

Existence criteria: bivariate case

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$\mathbb{F}(t)\langle D_x\rangle$	Zeilberger1990		
$\mathbb{F}(t)\langle S_x\rangle$		Abramov2003	
$\mathbb{F}(t)\langle S_{q,x}\rangle$			

9 types of telescopers for bivariate mixed hypergeometric terms

- In the pure continuous case, telescopers always exists since h is holonomic.
- In the pure discrete case, Abramov proved that

$$h \text{ has a telescoper} \quad \Leftrightarrow \quad h = \Delta_y(g) + r, \text{where } r \text{ is proper}$$

Existence criteria: bivariate case

(L, ∂_x)	D_y	Δ_y	$\Delta_{q,y}$
$\mathbb{F}(t)\langle D_x \rangle$	Zeilberger1990		
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- In the pure q-discrete case, Chen, Hou and Mu proved a q-anlogue of Abramov's criterion.

Existence criteria: bivariate case

(L, ∂_x)	D_y	Δ_y	$\Delta_{q,y}$
$\mathbb{F}(t)\langle D_x \rangle$	Zeilberger1990	$\surd$	$\surd$
$\mathbb{F}(t)\langle S_x \rangle$	$\surd$	Abramov2003	$\surd$
$\mathbb{F}(t)\langle S_{q,x} \rangle$	$\surd$	$\surd$	ChenHouMu2005

9 types of telescopers for bivariate mixed hypergeometric terms

- In the pure continuous case, telescopers always exists since h is holonomic.
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- In the pure q-discrete case, Chen, Hou and Mu proved a q-anlogue of Abramov's criterion.
- In 2015, Chen et al. solved the other 6 mixed cases.

Beyond the bivariate case

All existence criteria in the bivariate case show that

$h(x,y)$ has a telescoper $\quad \Leftrightarrow \quad h = \Delta_y(g) + r$, where r is proper

Beyond the bivariate case

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$h(x,y)$ has a telescoper $\quad\Leftrightarrow\quad h=\Delta_y(g)+r$, where r is proper

Example in the discrete case:

$$\frac{1}{x^2+y^2} \text{ has no telescoper of type } (S_x,\Delta_y)!$$

Beyond the bivariate case

All existence criteria in the bivariate case show that

$$h(x,y) \text{ has a telescoper} \quad \Leftrightarrow \quad h = \Delta_y(g) + r, \text{ where } r \text{ is proper}$$

Example in the mixed case:

$$\frac{1}{x+y} \text{ has no telescoper of type } (S_x, D_y)!$$

Beyond the bivariate case

All existence criteria in the bivariate case show that

$h(x,y)$ has a telescoper $\quad \Leftrightarrow \quad h = \Delta_y(g) + r$, where r is proper

Is this pattern still true in the trivariate case?

Beyond the bivariate case

All existence criteria in the bivariate case show that

$$h(x,y) \text{ has a telescoper} \quad \Leftrightarrow \quad h = \Delta_y(g) + r, \text{ where } r \text{ is proper}$$

$\dfrac{1}{x+y+z^2}$ is not proper but it still has a telescoper!

Beyond the bivariate case

All existence criteria in the bivariate case show that

$$h(x,y) \text{ has a telescoper} \quad \Leftrightarrow \quad h = \Delta_y(g) + r, \text{ where } r \text{ is proper}$$

$$\frac{1}{x+y+z^2} \text{ is not proper but it still has a telescoper!}$$

Existence Problem of Telescopers: Beyond the Bivariate Case *

Shaoshi Chen[1,2], Qing-Hu Hou[3], George Labahn[2], Rong-Hua Wang[4]

[1]KLMM, AMSS, Chinese Academy of Sciences, Beijing, 100190, (China)

[2]Symbolic Computation Group, University of Waterloo, Ontario, N2L3G1, (Canada)

[3]Center for Applied Mathematics, Tianjin University, Tianjin, 300072, (China)

[4]Center for Combinatorics, Nankai University, Tianjin, 300071, (China)

schen@amss.ac.cn, qh_hou@tju.edu.cn

glabahn@uwaterloo.ca, wangwang@mail.nankai.edu.cn

Construction of telescopers

Four approaches:

Construction of telescopers

Four approaches:

1902 -- 2012	1947 -- 1998	1990 -- 2010	2010 -- 2016
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1902: Picard proved the existence of Picard-Fuchs equations for parameterized integrals of algebraic functions:

ÉMILE PICARD

Sur les périodes des intégrales doubles dans la théorie des fonctions algébriques de deux variables

Annales scientifiques de l'É.N.S. 3e série, tome 19 (1902), p. 65-73.

Construction of telescopers

Four approaches:

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....	....	....	- ChenKaKoutschan 2016

1958: Manin gave a constructive method for finding Picard-Fuchs equations:

ALGEBRAIC CURVES OVER FIELDS WITH DIFFERENTIATION

Ju. I. MANIN

A differential-algebraic homomorphism is constructed from the group of divisor classes of degree zero on a curve defined over a constant field with differentiation into the additive group of a finite-dimensional vector space over the constant field. A partial study of the kernel of this homomorphism is made.

Construction of telescopers

Four approaches:

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1958: Manin gave a constructive method for finding Picard-Fuchs equations:

$$\alpha(x) = \oint_{\Gamma} \frac{dy}{\sqrt{y(y-1)(y-x)}} \quad \rightsquigarrow \quad y'' + \frac{2x-1}{x(x-1)}y' + \frac{1}{4x(x-1)}y = 0$$

Construction of telescopers

Four approaches:

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1969: Griffiths developed the Dwork-Griffiths reduction, which later is used to compute telescopers for multivariate rational functions:

Annals of Mathematics

On the Periods of Certain Rational Integrals: I
Author(s): Philip A. Griffiths
Source: *Annals of Mathematics*, Second Series, Vol. 90, No. 3 (Nov., 1969), pp. 460-495

Construction of telescopers

Four approaches:

1902 -- 2012	1947 -- 1998	1990 -- 2010	2010 -- 2016
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....	....	....	- ChenKaKoutschan 2016

2012: Chen, Kauers and Singer gave a method for computing telescopers for algebraic functions via residues:

Telescopers for Rational and Algebraic Functions via Residues

Shaoshi Chen* Manuel Kauers† Michael F. Singer*

Construction of telescopers

Four approaches:

1902 -- 2012	1947 -- 1998	1990 -- 2010	2010 -- 2016
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1947: Fasenmyer gave a method, so-called Sister Celine's method, to find recurrence relations satisfied by hypergeometric sums:

SOME GENERALIZED HYPERGEOMETRIC POLYNOMIALS

SISTER MARY CELINE FASENMYER

1. Introduction. We shall obtain some basic formal properties of the hypergeometric polynomials

$$f_n(a_i; b_j; x) \equiv f_n(a_1, a_2, \cdots, a_p; b_1, b_2, \cdots, b_q; x)$$

(1)

Construction of telescopers

Four approaches:

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1990: Zeilberger's algorithm for computing telescopers for holonomic functions via non-commutative elimination in Weyl algebra:

Journal of Computational and Applied Mathematics 32 (1990) 321–368 321
North-Holland

A holonomic systems approach to special functions identities *

Doron ZEILBERGER
Department of Mathematics, Temple University, Philadelphia, PA 19122, USA

Construction of telescopers

Four approaches:

1902 -- 2012	1947 -- 1998	1990 -- 2010	2010 -- 2016
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1990: Zeilberger's algorithm for computing telescopers for holonomic functions via non-commutative elimination in Weyl algebra:

$$\begin{cases} P(x,y,D_x)(h)=0 \\ Q(x,y,D_y)(h)=0 \end{cases} \rightsquigarrow A(x,D_x,D_y)(h)=0 \rightsquigarrow A(x,D_x,0) \text{ is telescoper}$$

Construction of telescopers

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1992: Takayama improved the non-commutative elimination in Weyl algebra by Groebner bases computation:

J. Symbolic Computation (1992) **14**, 265–282

An Approach to the Zero Recognition Problem by Buchberger Algorithm

NOBUKI TAKAYAMA

Department of Mathematics, Kobe University, Rokko, Kobe, 657, Japan

Construction of telescopers

Four approaches:

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1998: Chyzak and Salvy applied non-commutative elimination in Ore algebra to identities proofs :

J. Symbolic Computation (1998) **26**, 187–227
Article No. sy980207

Non-commutative Elimination in Ore Algebras Proves Multivariate Identities

FRÉDÉRIC CHYZAK† AND BRUNO SALVY‡

INRIA-Rocquencourt and École polytechnique, France

Construction of telescopers

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....	....	....	- ChenKaKoutschan 2016

1990: Based on Gosper's algorithm, Zeilberger developed an algorithm for computing telescoping for bivariate hypergeometric terms:

Discrete Mathematics 80 (1990) 207–211
North-Holland

COMMUNICATION

A FAST ALGORITHM FOR PROVING TERMINATING HYPERGEOMETRIC IDENTITIES

Doron ZEILBERGER*
Department of Mathematics, Drexel University, Philadelphia, PA 19104, USA

Construction of telescopers

Four approaches:

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....	....	....	- ChenKaKoutschan 2016

1990: Almkvist and Zeilberger extends Zeilberger's algorithm to the hyperexponential case:

J. Symbolic Computation (1990) **10**, 571–591

The Method of Differentiating under the Integral Sign

GERT ALMKVIST[1] AND DORON ZEILBERGER[2]†

Construction of telescopers

Four approaches:

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2000: Chyzak extends Zeilberger's algorithm to the high-order case:

N·H
ELSEVIER
Discrete Mathematics 217 (2000) 115–134
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

An extension of Zeilberger's fast algorithm to general holonomic functions☆

Frédéric Chyzak

Construction of telescopers

Four approaches:

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2010: Koutschan improved Chyzak's algorithm via advanced ansatz and applied to solve many conjectures in combinatorics:

Math.Comput.Sci. (2010) 4:259–266
DOI 10.1007/s11786-010-0055-0

Mathematics in Computer Science

A Fast Approach to Creative Telescoping

Christoph Koutschan

Construction of telescopers

Four approaches:

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....	....	....	- ChenKaKoutschan 2016

2010: Bostan et al. design a fast algorithm for creative telescoping for bivariate rational functions using classical Hermite reduction:

Complexity of Creative Telescoping for Bivariate Rational Functions*

Alin Bostan, Shaoshi Chen, Frédéric Chyzak
Algorithms Project-Team, INRIA
Paris-Rocquencourt
78153 Le Chesnay (France)

Ziming Li
Key Laboratory of Mathematics Mechanization,
Academy of Mathematics and System Sciences
100190 Beijing (China)

Construction of telescopers

Four approaches:

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2010: Bostan et al. design a fast algorithm for creative telescoping for bivariate rational functions using classical Hermite reduction:

$$f(x) = D_x(g) + \frac{p}{q}$$

where $p, q \in \mathbb{F}[x]$ with q squarefree and $\deg_x(p) < \deg_x(q)$.

Construction of telescopers

Four approaches:

1902 -- 2012	1947 -- 1998	1990 -- 2010	2010 -- 2016
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$$\int f(x)dx = \text{rational part} + \text{logarithmic part}$$

Construction of telescopers

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2013: Bostan et al. generalize the Hermite reduction to hyperexponential case and design a reduction-based telescoping algorithm:

Hermite Reduction and Creative Telescoping for Hyperexponential Functions*

Alin Bostan[1], Shaoshi Chen[2], Frédéric Chyzak[1], Ziming Li[3], Guoce Xin[4]

Construction of telescopers

Four approaches:

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2013: Bostan, Lairez and Salvy design a telescoping algorithm for multivariate rational function based on Dwork-Griffiths reduction:

Creative Telescoping for Rational Functions Using the Griffiths–Dwork Method*

Alin Bostan
Inria (France)
alin.bostan@inria.fr

Pierre Lairez
Inria (France)
pierre.lairez@inria.fr

Bruno Salvy
Inria (France)
bruno.salvy@inria.fr

Construction of telescopers

Four approaches:

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2015: Chen et al. design a telescoping algorithm for bivariate hypergeometric terms based on modified Abramov-Petkovsek reduction:

A Modified Abramov-Petkovšek Reduction and Creative Telescoping for Hypergeometric Terms*

Shaoshi Chen[1], Hui Huang[1,2], Manuel Kauers[2], Ziming Li[1]

Construction of telescopers

Four approaches:

1902 -- 2012	1947 -- 1998	1990 -- 2010	2010 -- 2016
Algebraic-Geometry Approach	Elimination-Based Approach	Gosper-Based Approach	Redution-Based Approach
- Picard 1902	- Fasenmyer 1947	- Zeilberger 1990	- BostanChenChyzakLi 2010
- Manin 1958	- Zeilberger 1990	- AlmkvistZeilberger 1990	- BoChChLiXin 2013
- Griffiths 1969	- Takayama 1992	- Chyzak 2000	- BoLairezSalvy 2013
- ChKauersSinger 2012	- ChyzakSalvy 1998	- Koutschan 2010	- ChHuangKaLi 2015
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2016: Chen, Kauers and Koutschan design a telescoping algorithm for bivariate algebraic functions based on Trager's reduction and polynomial reduction:

Reduction-Based Creative Telescoping for Algebraic Functions*

Shaoshi Chen[1,2], Manuel Kauers[3], Christoph Koutschan[4]

Gosper's algorithm

In 1978, Gosper solved the telescoping problem for hypergeometric terms.

Proc. Natl. Acad. Sci. USA
Vol. 75, No. 1, pp. 40–42, January 1978
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Decision procedure for indefinite hypergeometric summation

(algorithm/binomial coefficient identities/closed form/symbolic computation/linear recurrences)

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Example. $k! = \Delta_k(\text{No solution!})$

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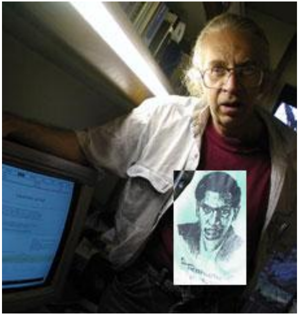

B. Gosper

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Input: A proper hypergeometric term $H(n,k)$
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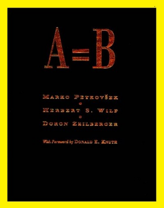

Petkovsek, Wilf & Zeilberger

Telescoper

Example.

$$H = \frac{k^{10}}{n+k}$$

The telescoper of minimal order L for H is

$$L = n^{10} S_n - (n+1)^{10}$$

Telescoper

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The telescoper of minimal order L for H is

$$L = n^{10} S_n - (n+1)^{10}$$

Guess the certificate of L?

Certificate

$$\frac{1}{2520(n+k)}(2100k^8n^2 - 84n^3 - 68460k^6n^4 - 840n^4 - 3720n^5 + 140700k^4n^6 - 9480n^6 - 15024n^7 - 10500k^2n^8 - 14808n^8 - 8400n^9 - 79590n^2k^7 + 284235n^4k^5 - 143640n^6k^3 + 210nk^8 - 26250n^3k^6 + 133035n^5k^4 - 35700n^7k^2 + 252k^{11} + 18900k^9n - 213780k^7n^3 + 368340k^5n^5 - 110460k^3n^7 - 2100n^{10} + 1890k^9 - 1764k^7 + 1260k^5 - 378k^3 - 1260k^{10} - 294nk^2 + 700nk^4 - 588nk^6 + 63504k^{11}n^5 + 52920k^{11}n^4 + 30240k^{11}n^3 + 11340k^{11}n^2 - 2940n^2k^2 - 13080n^3k^2 - 33780n^4k^2 - 55116n^5k^2 - 57348n^6k^2 - 17360k^3n^2 - 48860k^3n^3 - 94920k^3n^4 - 135156k^3n^5 - 55440k^3n^8 - 13860k^3n^9 - 3780k^3n + 7000n^2k^4 + 31185n^3k^4 + 80850n^4k^4 + 90090n^7k^4 + 27720n^8k^4 + 57141k^5n^2 + 155610k^5n^3 + 347886k^5n^6 + 238392k^5n^7 + 110880k^5n^8 + 27720k^5n^9 + 12600k^5n - 5880n^2k^6 - 114114n^5k^6 - 123816n^6k^6 - 83160n^7k^6 - 27720n^8k^6 - 379830k^7n^4 - 469128k^7n^5 - 411840k^7n^6 - 257400k^7n^7 - 110880k^7n^8 - 27720k^7n^9 - 17640k^7n + 9405n^3k^8 + 24750n^4k^8 + 42075n^5k^8 + 47520n^6k^8 + 34650n^7k^8 + 13860n^8k^8 + 85085k^9n^2 + 398475k^9n^4 + 23100k^9n^9 + 480480k^9n^5 + 92400k^9n^8 + 235620k^9n^7 + 227150k^9n^3 + 404250k^9n^6 - 12628k^{10}n - 13860k^{10}n^9 - 152460k^{10}n^3 - 60060k^{10}n^8 - 267960k^{10}n^4 - 157080k^{10}n^7 - 271656k^{10}n^6 - 56980k^{10}n^2 - 323400k^{10}n^5 + 2520k^{11}n + 2520k^{11}n^9 + 11340k^{11}n^8 + 30240k^{11}n^7 + 52920k^{11}n^6)$$

Certificate

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Very often, certificates are not needed!

Telescoping without certificates

Problem. Can we compute the telescopers without also computing the certifiates?

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Algorithms: $L(x,\partial_x)(f) = \partial_{y_1}(g_1) + \cdots + \partial_{y_m}(g_m)$

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  Abramov-Petkovsek reduction + polynomial reduction

- Bivariate algebraic case:

  Trager's reduction + polynomial reduction

Reduction: the rational case

Fact:

$$\frac{1}{k+1} = \Delta_k\left(\frac{1}{k}\right) + \frac{1}{k}$$

Reduction: the rational case

Fact:

$$\frac{1}{k+2} = \Delta_k\left(\frac{1}{k+1} + \frac{1}{k}\right) + \frac{1}{k}$$

Reduction: the rational case

Fact:

$$\frac{1}{k+s} = \Delta_k \left(\frac{1}{k+s-1} + \cdots + \frac{1}{k} \right) + \frac{1}{k}$$

Abramov's reduction: For any $f \in F(k)$,

$$f(k) = \Delta_k(g) + \frac{a}{b},$$

where $f \in F(k), \deg_k(a) < \deg_k(b)$ and b is shift-free, i.e., the distance of any two roots of b is not integer.

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Remark. The decomposition above is not unique, e.g.

$$\frac{2k+1}{k(k+1)} = \Delta_k\left(\frac{1}{k}\right)+\frac{2}{k} = \Delta_k\left(-\frac{1}{k}\right)+\frac{2}{k+1}$$

Telescoping via reduction

$$f = \frac{1}{(k+n)(2k+3n)} = \Delta_k(\cdots) + \underbrace{\frac{1}{(k+n)(2k+3n)}}_{r_0}$$

$$S_n(f) = \frac{1}{(k+n+1)(2k+3n+3)} = \Delta_k(\cdots) + \underbrace{\frac{n+3}{(n+1)(k+n)(2k+3n+3)}}_{r_1}$$

$$S_n^2(f) = \frac{1}{(k+n+2)(2k+3n+6)} = \Delta_k(\cdots) + \underbrace{\frac{n}{(n+2)(k+n)(2k+3n)}}_{r_2}$$

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Note that

$$n \cdot r_0 + 0 \cdot r_1 - (n+2) \cdot r_2 = 0$$

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$$\Downarrow$$

$L = -(n+2)S_n^2 + n$ is a telescoper for f.

Reduction: the hypergeometric case

Let T be hypergeometric w.r.t. k with $f = S_k(T)/T \in F(k)$.

$$f = \frac{S_k(r)}{r} \cdot K \quad \leftrightsquigarrow \quad T = r \cdot H \quad \text{with } \frac{S_k(H)}{H} = K.$$

We can have $K = c/d$ satisfying $\gcd(c, S_k^i(d)) = 1$ for all $i \in \mathbb{Z}$

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Modified Abramov-Petkovšek's reduction:

$$T = \Delta_k(\cdots) + \left(\frac{a}{b} + \frac{p}{d}\right) \cdot H,$$

where $a, b, p \in F[k]$ with $\deg_k(a) < \deg_k(b)$, b shift-free and p in a f.d. vector space V_K.

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Proposition.

$$T = \Delta_k(T') \quad \Leftrightarrow \quad a = 0 \text{ and } p = 0$$

Telescoping via reductions

Consider

$$T = \frac{1}{n+k} \cdot k!$$

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$$\begin{aligned} T &= \Delta_k(g_0) + \frac{1}{n+k} H \\ S_n(T) &= \Delta_k(\cdots) + \frac{1}{(n+k+1)^2} H \\ &= \Delta_k(g_1) + \left(-\frac{1/n}{n+k} + \frac{1}{n} \right) H \end{aligned}$$

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$$S_n^2(T) = \Delta_k(\cdots) + \left(-\frac{1/(n+1)}{n+k+1} + \frac{1}{n+1} \right) H$$

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&= \Delta_k\left(g_1\right) + \left(-\frac{1/n}{n+k} + \frac{1}{n}\right)H \\
S_n^2(T) &= \Delta_k\left(\cdots\right) + \left(-\frac{1/(n+1)}{n+k+1} + \frac{1}{n+1}\right)H \\
&= \Delta_k\left(g_2\right) + \left(-\frac{1/(n(n+1))}{n+k} + \frac{n-1}{n(n+1)}\right)H
\end{aligned}$$

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Telescoping via reductions

Consider

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$$c_0(n) \cdot \frac{1}{n+k}$$

$$+ c_1(n) \cdot \left(-\frac{1/n}{n+k} + \frac{1}{n} \right)$$

$$+ c_2(n) \cdot \left(-\frac{1/(n(n+1))}{n+k} + \frac{n-1}{n(n+1)} \right)$$

$$= 0$$

Telescoping via reductions

Consider

$$T = \frac{1}{n+k} \cdot k!$$

$$-\,1 \cdot \frac{1}{n+k}$$

$$+\,(1-n) \cdot \left(-\frac{1/n}{n+k} + \frac{1}{n}\right)$$

$$+\,(n+1) \cdot \left(-\frac{1/(n(n+1))}{n+k} + \frac{n-1}{n(n+1)}\right)$$

$$= 0$$

Telescoping via reductions

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$$\begin{aligned} G &= (n+1) \cdot g_2 - (n-1) \cdot g_1 - 1 \cdot g_0 \\ &= \frac{k!}{(n+k)(n+k+1)} \end{aligned}$$

Timings (in seconds)

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$$T = \frac{f(n,k)}{g_1(n+k)g_2(2n+k)} \frac{\Gamma(2\alpha n+k)}{\Gamma(n+\alpha k)}$$

with

- $g_i(z) = p_i(z)p_i(z+\lambda)p_i(z+\mu)$, $\alpha, \lambda, \mu \in \mathbb{N}$,
- $\deg(p_1) = \deg(p_2) = m$ and $\deg(f) = n$.

(m,n,α,λ,μ)	Zeilberger	RCT+cert	RCT	order
$(2,0,1,5,10)$	354.46	58.01	4.93	4
$(2,0,2,5,10)$	576.31	363.25	53.15	6
$(2,0,3,5,10)$	2989.18	1076.50	197.75	7
$(2,3,3,5,10)$	3074.08	1119.26	223.41	7
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- Maxima: `Zeilberger` by Fabrizio Caruso
- Reduce: `zeilberg` by Wolfram Koepf
- ...

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Thank you!