Rationality Theorems on D-finite Power Series

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June 15, 2018, Shanghai, China

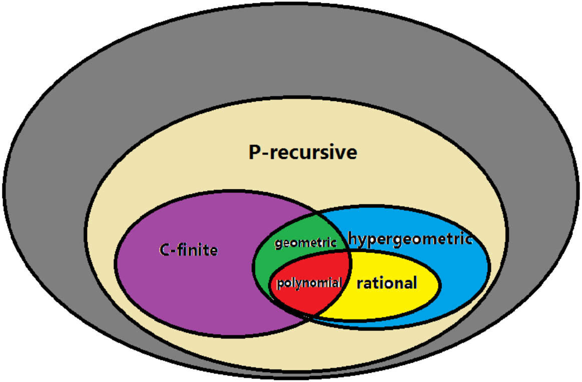
joint work with Jason P. Bell (University of Waterloo)

Sequences

Sequences are fundamental objects in mathematics:

$$s : \mathbb{N} \to \mathbb{K}, \quad \text{where } \mathbb{K} \text{ is any ring or field.}$$

Examples:



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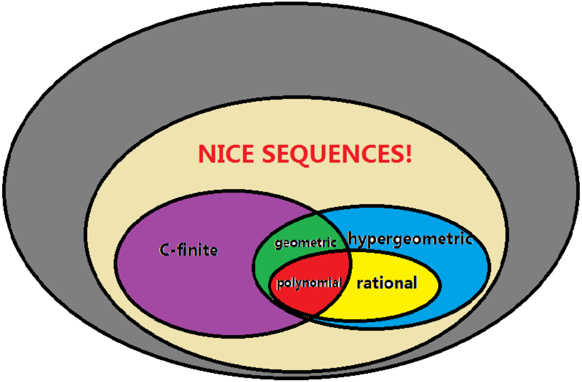

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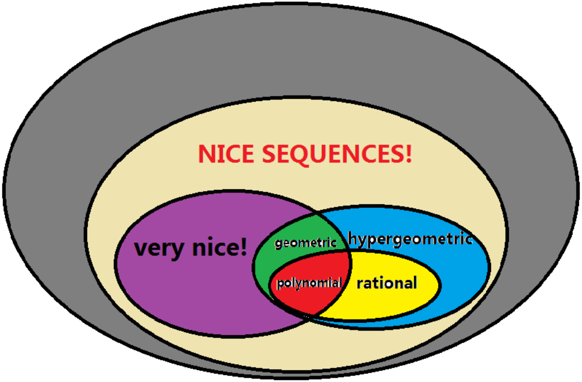

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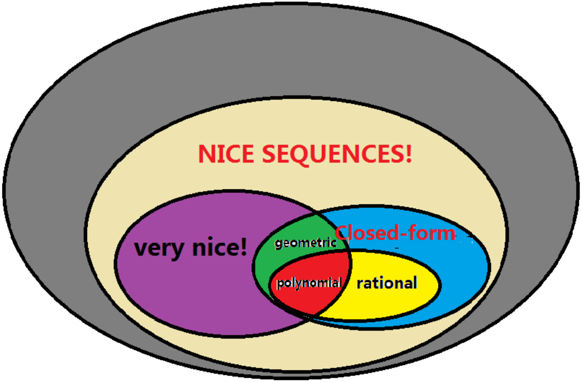

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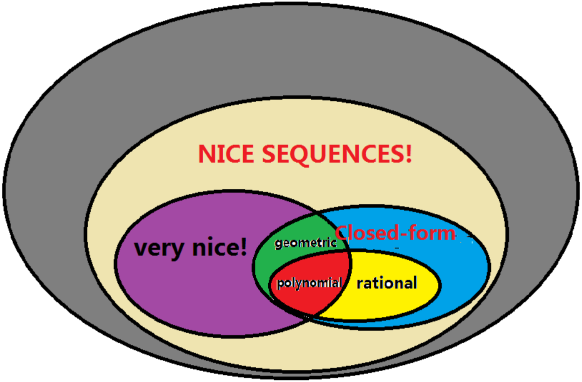

OEIS. The On-Line Encyclopedia of Integer Sequences (N.J.A. Sloan)

Power series as generating functions

The generating function of a sequence a_n is

$$f(x) := \sum_{n=0}^{+\infty} a_n x^n \in \mathbb{K}[[x]].$$

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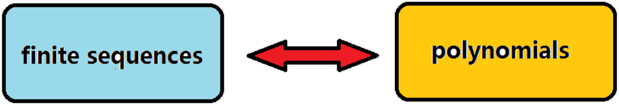

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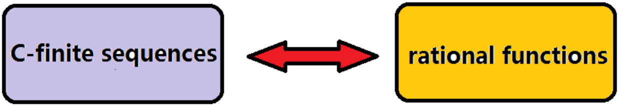

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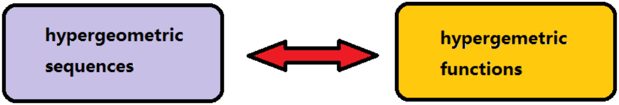

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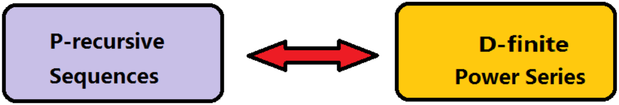

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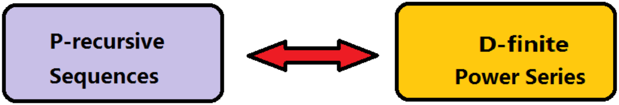

Example. Bell numbers b_n count the number of partitions of a set

$1, 1, 2, 5, 15, 52, 203, 877, 4140, \ldots$ (sequence A000110 in OEIS)

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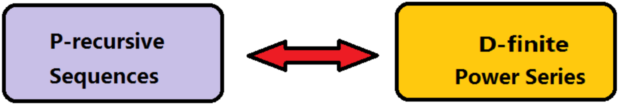

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$$\sum_{n=0}^{+\infty} \frac{b_n x^n}{n!} = \exp(\exp(x) - 1).$$

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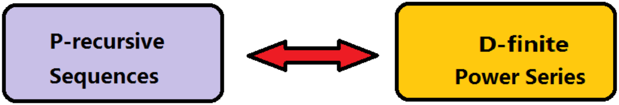

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$$\exp(\exp(x) - 1) \text{ is not D-finite!}$$

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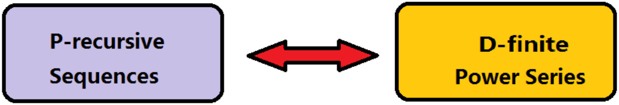

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b_n is not P-recursive!

Power series as generating functions

Let $a_{n_1,\ldots,n_d} : \mathbb{N}^d \to \mathbb{K}$ be a sequence. Then the formal power series

$$f(x_1,\ldots,x_d) := \sum_{n_1,\ldots,n_d=0}^{+\infty} a_{n_1,\ldots,n_d} x_1^{n_1} \cdots x_d^{n_d} \in \mathbb{K}[[x_1,\ldots x_d]]$$

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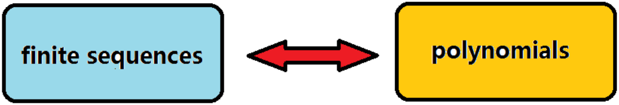

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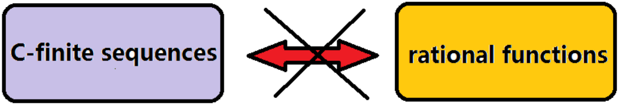

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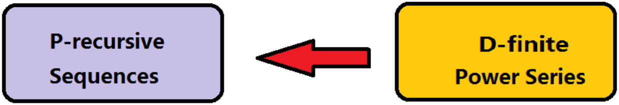
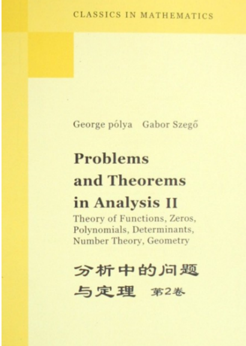


Arithmetic aspects of power series

Problem. Decide whether a given power series is rational, algebraic, or D-finite?



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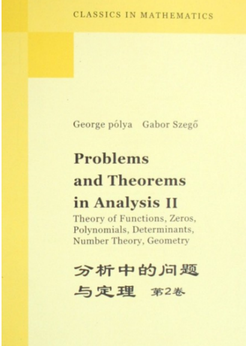

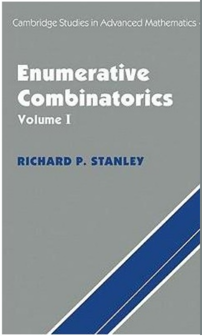

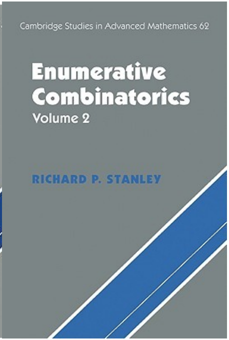

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G. Eisenstein, Uber eine allgemeine Eigenschaft der Reihenentwicklungen aller algebraischen Funcktionen, Belin, Sitzber, 441-443, 1852

关于代数函数幂级数展开的系数一般特征

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关于代数函数幂级数展开的系数一般特征

Theorem (Eisenstein 1852, Heine 1853). If $f(x)$ represents an algebraic function over $\mathbb{Q}(x)$, then $\exists\ T \in \mathbb{Z}$, s.t.

$$\sum_{n \geq 0} a_n T^n x^n \in \mathbb{Z}[[x]].$$

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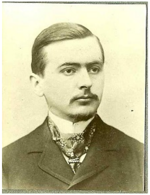

Pierre Fatou (1878-1929)

Pierre Fatou, *Séries trigonométriques et séries de Taylor*, Acta Math. **30** (1906), no. 1, 335–400.

Fatou's lemma. If $f(x)$ represents a rational function, then

$$f(x) = \frac{P(x)}{Q(x)}, \quad \text{where } P, Q \in \mathbb{Z}[x] \text{ and } Q(0) = 1.$$

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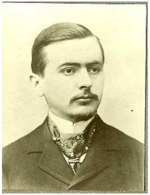

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Theorem. If $f(x)$ converges inside the unit disk, then it is either rational or transcendental over $\mathbb{Q}(x)$.

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Pólya-Carlson Theorem. If $f(x)$ converges inside the unit disk, then either it is rational or has the unit circle as natural boundary.

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Corollary. If $f(x)$ is algebraic or D-finite, then it is rational.

Power series with integral coefficients (the multivariate case)

Multivariate extensions of the Pólya-Carlson Theorem:

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- André Martineau, *Extension en n-variables d'un théorème de Pólya-Carlson concernant les séries de puissances à coefficients entiers*, C. R. Acad. Sci. Paris Sér. A-B **273** (1971), A1127–A1129. MR 0291495

- V. P. Šeĭnov, *Transfinite diameter and certain theorems of Pólya in the case of several complex variables*, Sibirsk. Mat. Ž. **12** (1971), 1382–1389.

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Theorem (BellChen, 2016) If the multivariate power series

$$F = \sum f(n_1, \ldots, n_d) x_1^{n_1} \cdots x_d^{n_d} \in \mathbb{Z}[[x_1, \ldots, x_d]]$$

is D-finite and converges on the unit polydisc, then it is rational.

Power series with finitely distinct coefficients

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Gábor Szegő (1895-1985)

From 1917 to 1922, there are four papers with the same title:

Über Potenzreihen mit endlich vielen verschiedenen Koeffizienten.

Power Series with Finitely Distinct Coefficients

1. G. Polya in 1917, Math. Ann.
2. R. Jentzsch in 1918, Math. Ann.
3. F. Carlson in 1919, Math. Ann.
4. G. Szego in 1922, Math Ann.

Szegö's Theorem (1922)
A power series with finitely distinct coefficients in $\mathbb{C}$ is either rational or has the unit circle as its natural boundary.

Power series with finitely distinct coefficients (the multivariate case)

Theorem (van der Poorten & Shparlinsky, 1994).

Let $a_n : \mathbb{N} \to \Delta$, where $|\Delta|$ is a finite subset of $\mathbb{Q}$. If the generating function $f(x) = \sum_n a_n x^n$ is D-finite, then it is rational.

Remark. This follows from Szegö's theorem by the fact that a D-finite power series can only have finitely many singularities.

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Theorem (BellChen, 2016). Let $a_{n_1,\ldots,n_d} : \mathbb{N}^d \to \Delta$, where $|\Delta|$ is a finite subset of $\mathbb{Q}$. If the generating function

$$f(x_1,\ldots,x_d) = \sum a_{n_1,\ldots,n_d} x_1^{n_1} \cdots x_d^{n_d}$$

is D-finite, then it is rational.

Nonnegative integer points on algebraic varieties

Let V be an algebraic variety over an algebraically closed field K of characteristic zero. We define the listing generating function

$$F_V(x_1,\ldots,x_d) := \sum_{(n_1,\ldots,n_d)\in V\cap\mathbb{N}^d} x_1^{n_1}\cdots x_d^{n_d}$$

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We may ask the following questions:

When F_V is zero?

Remark. This is Hilbert Tenth Problem when K is $\mathbb{Q}$. In 1970, Matiyasevich proved that this problem is undecidable.

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We may ask the following questions:

When F_V is a polynomial?

Remark. In 1929, Siegel proved that a smooth algebraic curve C of genus $g \geq 1$ has only finitely many integer points over a number field K.

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We may ask the following questions:

When F_V is a rational function?

Remark. If V is defined by linear polynomials over $\mathbb{Q}$, then F_V is rational.

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We may ask the following questions:

When F_V is a D-finite function?

Corollary.

$$F_V \text{ is } D\text{-finite} \quad \Leftrightarrow \quad F_V \text{ is rational}.$$

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Theorem.

The problem of testing whether F_V is rational is undecidable!

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We may ask the following questions:

When F_V is a differentially algebraic function?

Definition. $F \in K[[x_1,\ldots,x_d]]$ is differentially algebraic if the transcendence degree of the filed generated by the derivatives $D_{x_1}^{i_1}\cdots D_{x_d}^{i_d}(F)$ with $i_j \in \mathbb{N}$ over $K(x_1,\ldots,x_d)$ is finite.

Nonnegative integer points on algebraic curves

Theorem. Let $p(x,y) \in \mathbb{C}[x,y]$. If the generating function

$$F_p(x,y) := \sum_{(n,m)\in V(p)\cap\mathbb{N}^2} x^n y^m$$

is rational. Then $p = f \cdot g$, where $f, g \in \mathbb{C}[x,y]$ s.t.

$$f = \prod_i (s_i \cdot x + t_i \cdot y + c_i) \quad \text{with } s_i, t_i \in \mathbb{Z} \text{ and } c_i \in \mathbb{C}$$

and g has only finite zeros in $\mathbb{N}^2$.

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and g has only finite zeros in $\mathbb{N}^2$.

Example. Let $p = x^2 - y$. Since p is not a product of integer-linear polynomials, the power series $F_p(x,y)$ is not D-finite.

Conjecture: *differential algebraic* $\Leftrightarrow$ *rational*

Conjecture. Let V be an algebraic variety over $\mathbb{C}$. Then the power series

$$\sum_{(n_1,\ldots,n_d)\in V\cap\mathbb{N}^d} x_1^{n_1}\cdots x_d^{n_d}$$

is differentially algebraic if and only if it is rational.

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Example. Let $p = x^2 - y$. Then the power series

$$F_p(x,y) := \sum_{m\geq 0} x^m y^{m^2}$$

is not differentially algebraic, otherwise, $F_p(x,2) = \sum 2^{m^2} x^m$ is differentially algebraic. By Mahler's lemma, we get a contradiction

$$2^{m^2} \ll (m!)^c \quad \text{for any positive constant } c.$$

Summary

Multivariate Pólya-Carlson Theorem If the power series

$$F = \sum f(n_1, \ldots, n_d) x_1^{n_1} \cdots x_d^{n_d} \in \mathbb{Z}[[x_1, \ldots, x_d]]$$

is D-finite and converges on the unit polydisc, then it is rational.

Multivariate Szëgo Theorem If the power series

$$f(x_1, \ldots, x_d) = \sum a_{n_1,\ldots,n_d} x_1^{n_1} \cdots x_d^{n_d}, \quad a_{n_1,\ldots,a_d} \in \Delta \text{ with } |\Delta| < +\infty$$

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Thank you!