RATIONAL DYNAMICAL SYSTEMS, S-UNITS, AND D-FINITE POWER SERIES

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Abstract. Let K be an algebraically closed field of characteristic zero, let G be a finitely generated subgroup of the multiplicative group of K, and let X be a quasiprojective variety defined over K. We consider K-valued sequences of the form $a_n := f(\varphi^n(x_0))$, where $\varphi\colon X \dashrightarrow X$ and $f\colon X \dashrightarrow \mathbb{P}^1$ are rational maps defined over K and $x_0 \in X$ is a point whose forward orbit avoids the indeterminacy loci of φ and f. Many classical sequences from number theory and algebraic combinatorics fall under this dynamical framework, and we show that the set of n for which $a_n \in G$ is a finite union of arithmetic progressions along with a set of upper Banach density zero. In addition, we show that if $a_n \in G$ for every n and X is irreducible and the φ orbit of x is Zariski dense in X then there is a multiplicative torus $\mathbb{G}_m^d$ and maps $\Psi : \mathbb{G}_m^d \to \mathbb{G}_m^d$ and $g : \mathbb{G}_m^d \to \mathbb{G}_m$ such that $a_n = g \circ \Psi^n(y)$ for some $y \in \mathbb{G}_m^d$. We then obtain results about the coefficients of D-finite power series using these facts.

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1. Introduction

A *rational dynamical system* is a pair (X, φ), where X is a quasiprojective variety defined over a field K, and $\varphi : X \dashrightarrow X$ is a rational map. The forward φ-orbit of a point $x_0 \in X$ is given by

$$O_\varphi(x_0) := \{x_0, \varphi(x_0), \varphi^2(x_0), \ldots\}$$

as long as this orbit is defined (*i.e.*, x_0 is outside the indeterminacy locus of φ^n for every $n \geq 0$).

In [BGS] and [BHS], the authors develop a broad dynamical framework giving rise to many classical sequences from number theory and algebraic combinatorics, by considering *dynamical sequences*, which are sequences of the form $f \circ \varphi^n(x_0)$, where (X, φ) is a rational dynamical system, $f : X \dashrightarrow \mathbb{P}^1$

is a rational map, and $x_0 \in X$. In particular, the class of dynamical sequences includes all sequences whose generating functions are D-finite, i.e., those satisfying homogeneous linear differential equations with rational function coefficients. This is an important class of power series since it appears ubiquitously in algebra, combinatorics, and number theory. In particular, this class contains:

- all hypergeometric series (see, for example, [Gar09, WZ92]);
- generating functions for many classes of lattice walks [DHRS18];
- diagonals of rational functions [Lip88];
- power series expansions of algebraic functions [Sta96, Chapter 6];
- generating series for the cogrowth of many finitely presented groups [GP17];
- many classical combinatorial sequences (see Stanley [Sta96, Chapter 6] and Mishna [Mis20, Chapter 5] for more examples).

The D-finiteness of generating functions reflects the complexity of combinatorial classes [Pak18]. Since this class is closed under addition, multiplication, and the process of taking diagonals, it has become a useful data structure for the manipulation of special functions in symbolic computation [Sal17]. The main goal of this paper is to study D-finite power series in the framework of rational dynamical systems.

The study of power series with coefficients in a finitely generated subgroup G of the multiplicative group of a field enjoys a long history, going back at least to the early 1920s, with the pioneering work of Pólya [Pól21], who characterized rational functions whose Taylor expansions at the origin have coefficients lying in a finitely generated multiplicative subgroup of $\mathbb{Z}$. Pólya's results were later extended to D-finite power series by Bézivin [Béz86]. From this point of view, it is then natural to consider when the dynamical sequences we study take values in a finitely generated multiplicative group. This question and related questions have already been considered in the case of self-maps of $\mathbb{P}^1$ in [BOSS, OSSZ19].

Conventions. Throughout, $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. If R is a ring then R^* is its multiplicative group of units. An arithmetic progression is a set of the form $\{a + bn\}_{n \geq 0} \subseteq \mathbb{N}_0$ where $a, b \in \mathbb{N}_0$. A singleton counts as an arithmetic progression with $b = 0$. A subset $N \subseteq \mathbb{N}_0$ is called *eventually periodic* if it is a union of finitely many arithmetic progressions.

We show that, with the above notation, the set of n for which $f \circ \varphi^n(x_0) \in G$ is well-behaved. To make this precise, we recall that the *upper Banach density* of a subset $S \subseteq \mathbb{N}_0$ is

$$\delta(S) := \limsup_{|I| \to \infty} \frac{|S \cap I|}{|I|},$$

where I ranges over all non-empty intervals of $\mathbb{N}_0$ (see [Fur81, Definition 3.7]). Then our main result is the following.

Theorem 1.1. *Let X be a quasiprojective variety over a field K of characteristic zero, let $\varphi : X \dashrightarrow X$ be a rational map, let $f : X \dashrightarrow \mathbb{P}^1$ be a rational* *function, and let $G \le K^*$ be a finitely generated group. If $x_0 \in X$ is a point with well-defined forward φ-orbit that also avoids the indeterminacy locus of f, then the set*

$$N := \{n \in \mathbb{N}_0 : f(\varphi^n(x_0)) \in G\}$$

is a finite union of arithmetic progressions along with a set of upper Banach density zero.

In the case when G is the trivial group, Theorem 1.1 follows independently from the work of several authors [BGT15, Gig14, Pet15], and was originally conjectured by Denis [Den94], and is now considered as a relaxed version of the so-called Dynamical Mordell-Lang Conjecture (see [BGT16] for further background). There is also some overlap between our result and that of Ghioca and Nguyen [GN17]: in particular, when G is an infinite cyclic group and f is a linear map, one gets a special case of their result.

The zero density sets in Theorem 1.1 are not necessarily finite. For a simple example, consider the map $\varphi : \mathbb{A}^1 \to \mathbb{A}^1$ defined by $\varphi(x) = x + 1$ and the rational function $f(x) = x$. Let $G = \langle 2 \rangle$. Then for the initial point $x_0 = 1$ the set $\{n \in \mathbb{N}_0 : f(\varphi^n(x_0)) \in G\} = \{2^n - 1 \mid n \in \mathbb{N}_0\}$ is an infinite set of upper Banach density zero.

The case $N = \mathbb{N}_0$ can be easily achieved: let $T := \mathbb{G}_m^d$ be a d-dimensional multiplicative torus. Then an endomorphism φ of T is a map of the form

$$(x_1, \ldots, x_d) \mapsto \left(c_1 \prod_j x_j^{a_{1,j}}, \ldots, c_d \prod_j x_j^{a_{d,j}} \right).$$

Now if we begin with a point $x_0 = (\beta_1, \ldots, \beta_d)$, then every point in the orbit of x_0 has coordinates in the multiplicative group G generated by

$$c_1, \ldots, c_d, \beta_1, \ldots, \beta_d.$$

In particular, if $f : T \to \mathbb{P}^1$ is a map of the form $(x_1, \ldots, x_d) \mapsto \kappa x_1^{p_1} \cdots x_d^{p_d}$ with $\kappa \in G$ then $f \circ \varphi^n(x_0) \in G$ for every $n \ge 0$.

In fact, we show that in characteristic zero any dynamical system (X, φ) with $N = \mathbb{N}_0$ is "controlled" by one of this form, in the following sense.

Theorem 1.2. *Let K be an algebraically closed field of characteristic zero and let X be an irreducible quasiprojective variety with a dominant self-map $\varphi : X \dashrightarrow X$ and let $f : X \dashrightarrow \mathbb{P}^1$ be a dominant rational map, all defined over K. Suppose that $x_0 \in X$ has the following properties:*

1. *every point in the orbit of x_0 under φ avoids the indeterminacy loci of φ and f;*
2. *$O_\varphi(x_0)$ is Zariski dense;*
3. *there is a finitely generated multiplicative subgroup G of K^* such that $f \circ \varphi^n(x_0) \in G$ for every $n \in \mathbb{N}_0$.*

Then there exists a dominant rational map $\Theta : X \dashrightarrow \mathbb{G}_m^d$ for some nonnegative integer d, and a dominant endomorphism $\Phi : \mathbb{G}_m^d \to \mathbb{G}_m^d$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, dashed, "\varphi"] \arrow[d, dashed, "\Theta"'] & X \arrow[d, dashed, "\Theta"] \\ \mathbb{G}_m^d \arrow[r, "\Phi"'] & \mathbb{G}_m^d. \end{tikzcd}$$

Moreover, $O_\varphi(x_0)$ avoids the indeterminacy locus of Θ and $f = g \circ \Theta$, where $g : \mathbb{G}_m^d \to \mathbb{G}_m$ is a map of the form

$$g(t_1, \ldots, t_d) = Ct_1^{i_1} \cdots t_d^{i_d}$$

for some $i_1, \ldots, i_d \in \mathbb{Z}$ and some $C \in G$.

One can interpret the above theorem as saying that if the entire orbit of a point under a self-map has some "coordinate" that lies in a finitely generated multiplicative group then there must be a compelling geometric reason causing this to occur: in this case, it is that the dynamical behaviour of the orbit is in some sense determined by the behaviour of a related system associated with a multiplicative torus. In fact, we prove a more general version of this result involving semigroups of maps (see Corollary 3.5). We remark that the situation in positive characteristic is more subtle and the conclusion to the statement of Theorem 1.2 fails (see Example 3.7).

As a consequence of Theorem 1.2, we get the following characterization of orbits whose values lie in a group of S-units, which shows that on arithmetic progressions they are well-behaved.

Corollary 1.3. *Let K be an algebraically closed field of characteristic zero and let X be an irreducible quasiprojective variety with a dominant self-map $\varphi : X \dashrightarrow X$ and let $f : \mathbb{X} \dashrightarrow \mathbb{P}^1$ be a dominant rational map, all defined over K. Suppose that $x_0 \in X$ has the following properties:*

(1) *every point in the orbit of x_0 under φ avoids the indeterminacy loci of φ and f;*
(2) *there is a finitely generated multiplicative subgroup G of K^* such that $f \circ \varphi^n(x_0) \in G$ for every $n \in \mathbb{N}_0$.*

Then there are integers p and L with $p \geq 0$ and $L > 0$ such that if $h_1, \ldots, h_m$ generate G then there are integer valued linear recurrences $b_{j,1}(n), \ldots, b_{j,m}(n)$ for $j \in \{0, \ldots, L-1\}$ such that

$$f \circ \varphi^{Ln+j}(x_0) = \prod_{i=1}^{m} h_i^{b_{j,i}(n)}$$

for $n \geq p$.

Finally, we apply our results to D-finite power series, which, as stated above, are the generating functions of sequences that fall under the dynamical framework we study. We use Theorem 1.1 to prove the following result.

Theorem 1.4. *Let $F(x) = \sum_{n \geq 0} a_n x^n$ be a D-finite power series defined over a field K of characteristic zero. Consider the sets*

$$N := \{n \geq 0 : a_n \in G\} \qquad \text{and} \qquad N_0 := \{n \geq 0 : a_n \in G \cup \{0\}\},$$

where $G \leq K^*$ is a finitely generated group. Then N and N_0 are both expressible as a union of finitely many infinite arithmetic progressions along with a set of upper Banach density zero.

When $G = \{1\}$, this recovers a result of Methfessel [Met00] and Bézivin [Béz89]. We conclude by revisiting the earlier works of Pólya [Pól21] and Bézivin [Béz86] in terms of the dynamical results obtained in this paper. The conclusion to the statement to Theorem 1.4 does not hold if one instead works with a power series $F(x)$ that is differentiably algebraic (cf. Pak [Pak18]).

Organization. In §2 we develop a general theory of recurrences for sequences indexed by semigroups, which will be needed in proving our main results. In §3 we prove a semigroup version of Theorem 1.2. In §4, we give the proof of Theorem 1.1, and in §5 we give results on the heights of points in orbits for dynamical systems defined over $\bar{\mathbb{Q}}$. Finally, §6 gives applications of our results to D-finite power series.

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2. Linear recurrences in abelian groups

In this section we develop the necessary background on general recurrences in abelian groups. Because we will ultimately prove a result about a semigroup of morphisms, we will work with sequences indexed by monoids in this section. The proofs of these results become significantly simpler when the underlying monoid is just $(\mathbb{N}_0, +)$, which is the key case needed for dealing with a single map.

Let $(A, +)$ be an abelian group, let S be a finitely generated monoid with identity $1 = 1_S$, and let Z be a set upon which S acts. Then the space of sequences

$$A^Z := \{u : Z \to A\}$$

is an abelian group, and for $u \in A^Z$ and $z \in Z$, we use the notations $u_z = u(z)$ interchangeably. Given a ring R, we let $R[S]$ denote the semigroup algebra of S with coefficient ring R; that is, the elements of $R[S]$ are formal R-linear combinations of elements of S, where we multiply via the rule $(rs) \cdot (r's') = rr'ss'$ for $r, r' \in R$ and $s, s' \in S$ and we extend this multiplication bilinearly. Notice that A^Z has a natural $\mathbb{Z}[S]$-module structure given by the rule:

$$(ms \cdot u)_z = mu_{sz} \qquad \text{for } m \in \mathbb{Z}, s \in S, \text{and } z \in Z.$$

We call the set of $f \in \mathbb{Z}[S]$ such that $f \cdot u = 0$ the *annihilator* of u. It is not hard to check that the annihilator of u is a two-sided ideal of $\mathbb{Z}[S]$.

For us, the abelian groups we work with will often be subgroups of the multiplicative group of a field K, in which case it makes sense to use multiplication rather than addition for the group operation. In the case when A is an abelian group under multiplication, we can still endow A^Z with a $\mathbb{Z}[S]$-module structure and we now have

$$(ms \cdot u)_z = u_{sz}^m \qquad \text{for } m \in \mathbb{Z}, s \in S, \text{and } z \in Z.$$

Definition 2.1. Let A be an abelian group, let S be a finitely generated monoid that acts on a set Z, and let $u \in A^Z$ be a sequence. Then u satisfies an *S-linear recurrence* if the annihilating ideal I of u has the property that $\mathbb{Z}[S]/I$ is a finitely generated $\mathbb{Z}$-module. We say that the sequence u satisfies an *S-quasilinear recurrence* if there is a set of generators $s_1, \ldots, s_d$ of S and a natural number M such that whenever $s_{i_1} \cdots s_{i_M}$ is an element of S that is a product of M elements of $s_1, \ldots, s_d$, there is an element in I of the form

$$\sum_{j=1}^{M} c_j s_{i_j} \cdots s_{i_M}$$

with $c_1, \ldots, c_M \in \mathbb{Z}$ satisfying that $\gcd(c_1, \ldots, c_M) = 1$.

Observe that the given definition works for abelian groups under addition and multiplication, once we use the relevant $\mathbb{Z}[S]$-module structure on A^Z provided earlier. In the case when the abelian group A is written multiplicatively, we will use the terms *multiplicative linear recurrence* and *multiplicative quasilinear recurrence* when considering recurrences in the multiplicative setting. The reason for introducing the notion of quasilinear recurrences is for later convenience, as it is often easier to demonstrate that a quasilinear recurrence holds.

Example 2.2. In general, a quasilinear recurrence may not be linear. To see this, let $S = \mathbb{N}_0$ and let A be the additive group $(\mathbb{Q}, +)$. Then if we consider the sequence $a_n = 1/2^n$ and identify $\mathbb{Z}[S]$ with $\mathbb{Z}[x]$, then this sequence is annihilated by the primitive polynomial $f(x) = 2x - 1$, but it does not satisfy an S-linear recurrence since a_{n+1} is never in the additive group generated by the initial terms $a_1, \ldots, a_n$.

We will make use of the following well-known facts throughout this section.

Lemma 2.3. *Let T be a commutative noetherian integral domain and let R be a finitely generated associative (but not necessarily commutative) T-algebra and suppose that I and J are two ideals of R such that both R/I and R/J are finitely generated T-modules. Then the following hold:*

(a) *R/IJ is also a finitely generated T-module;*
(b) *I and J are finitely generated as left ideals of R.*

Proof. We first prove (a). Let $U = \{u_1, \ldots, u_d\}$ be elements of R with $u_1 = 1$ whose images span both R/I and R/J as T-modules and that generate R as a T-algebra. We prove that every finite product of elements from $u_1, \ldots, u_d$ is congruent to a T-linear combination of elements of the form $u_i u_j u_k u_\ell$ modulo

IJ. (Since $u_1 = 1$, this includes products of smaller length.) We prove this by induction on the length of the product, with the case for products of length at most four following by construction. Suppose now that the result holds for all products of elements from $u_1, \ldots, u_d$ of length less than M with $M \geq 5$, and consider a product $u_{i_1} \cdots u_{i_M}$. Then by our choice of U we have

$$u_{i_1} \cdots u_{i_{M-2}} \equiv \sum a_i u_i \pmod{I}$$

and

$$u_{i_{M-1}} u_{i_M} \equiv \sum b_i u_i \pmod{J}$$

for $a_i, b_i \in T$. Hence

$$\left(u_{i_1} \cdots u_{i_{M-2}} - \sum a_i u_i\right) \left(u_{i_{M-1}} u_{i_M} - \sum b_i u_i\right) \in IJ.$$

Then expanding the product, we see that $u_{i_1} \cdots u_{i_M}$ is congruent to a T-linear combination of products of $u_1, \ldots, u_d$ of length at most $\max(M-1, 3) = M-1$, and so by the induction hypothesis it is in the span of products of length at most 4. Thus (a) now follows by induction.

For part (b), it suffices to prove that I is finitely generated as a left ideal. Then since U spans R/I as a T-module, there exist elements $c_{i,j,k} \in T$ such that

$$\alpha_{i,j} := u_i u_j - \sum_k c_{i,j,k} u_k \in I \text{ for } 1 \leq i, j, k \leq d.$$

Next, consider the submodule M of T^d defined by

$$M := \left\{(t_1, \ldots, t_d) \in T^d \colon \sum t_i u_i \in I\right\}.$$

Then since T is noetherian, M is finitely generated as a T-module and we pick elements

$$\beta_k = \sum t_{i,k} u_i \text{ for } k = 1, \ldots, s$$

such that $(t_{1,k}, \ldots, t_{d,k})$ with $k = 1, \ldots, s$ generate M.

Let L denote the finitely generated left ideal in R generated by the $\alpha_{i,j}$ and the β_k. By construction $L \subseteq I$ and so to complete the proof of (b) it suffices to show that $I \subseteq L$. We first claim that every non-trivial finite product of elements from $\{u_1, \ldots, u_d\}$ is congruent modulo L to a T-linear combination of $u_1, \ldots, u_d$. This is immediate for products of length one, and since the $\alpha_{i,j}$ are in L, we see that the claim holds for each product of length 2. Now suppose that the claim holds for all products of length less than m with $m \geq 3$ and consider a product $u_{i_1} \cdots u_{i_m}$ of length m. Then by the induction hypothesis, there exist $\gamma_1, \ldots, \gamma_d \in T$ such that

$$u_{i_2} \cdots u_{i_m} - \sum_{j=1}^{d} \gamma_j u_j \in L.$$

Since L is a left ideal, we may left multiply by u_{i_1} and we see that

$$u_{i_1} \cdots u_{i_m} - \sum_{j=1}^{d} \gamma_i u_{i_1} u_j \in L.$$

Since each $\alpha_{i_1,j} \in L$, we can express $u_{i_1}u_j$ as a T-linear combination of $u_1, \ldots, u_d$, which completes the proof of the claim.

It follows that if $f \in I$ then $f \equiv \sum t_i u_i \pmod{L}$ for some $t_1, \ldots, t_d \in T$. But since $L \subseteq I$, $t_1u_1 + \cdots + t_du_d \in I$ and so by construction, $t_1u_1 + \cdots + t_du_d$ is a T-linear combination of the β_k and hence it is in L. It then follows that $f \in L$, giving us that $I \subseteq L$ and showing that $I = L$ and so I is finitely generated as a left ideal. □

Corollary 2.4. *Let S be a finitely generated monoid acting on a set Z, let A and B be abelian groups, and let $u \in A^Z$ and $v \in B^Z$ be sequences satisfying S-linear recurrences. Then $(u, v) = (u_z, v_z)_{z \in Z} \in (A \oplus B)^Z$ also satisfies an S-linear recurrence.*

Proof. Let I and J be respectively the annihilators of u and v. Then $\mathbb{Z}[S]/I$ and $\mathbb{Z}[S]/J$ are finitely generated $\mathbb{Z}$-modules. Observe that $\mathbb{Z}[S]/(I \cap J)$ embeds in $\mathbb{Z}[S]/I \times \mathbb{Z}[S]/J$ via the map

$$z + I \cap J \mapsto (z + I, z + J) \text{ for } z \in \mathbb{Z}[S]$$

and so $\mathbb{Z}[S]/I \cap J$ is a finitely generated $\mathbb{Z}$-module as it is a submodule of a finitely generated $\mathbb{Z}$-module. Since $I \cap J$ annihilates both u and v, it also annihilates (u, v) and so the annihilator of (u, v) contains $I \cap J$. Therefore if L denotes the annihilator of (u, v), we have $\mathbb{Z}[S]/L$ is a finitely generated $\mathbb{Z}$-module, since it is a quotient of $\mathbb{Z}[S]/I \cap J$. Thus (u, v) satisfies an S-linear recurrence. □

We now prove a result we will use repeatedly throughout the remainder of the paper; this result generalizes the classical Fatou's lemma on rational power series in $\mathbb{Z}[[x]]$.

Proposition 2.5. *Let A be a finitely generated abelian group, let S be a finitely generated monoid acting on a set Z, and let $u \in A^Z$. If $u = (u_z)_{z \in Z}$ satisfies an S-quasilinear recurrence then u satisfies an S-linear recurrence.*

In order to prove this result, we require a technical lemma concerning quasilinear sequences taking values in cyclic groups.

Lemma 2.6. *Let P be a prime ideal of $\mathbb{Z}$, let S be a finitely generated monoid acting on a set Z, let u be a sequence in $(\mathbb{Z}/P)^Z$, and let I be the annihilator ideal of u in $\mathbb{Z}[S]$. If $u = (u_z)_{z \in Z}$ satisfies an S-quasilinear recurrence then*

$$R := \mathbb{Z}[S]/I \otimes_{\mathbb{Z}/P} \mathrm{Frac}(\mathbb{Z}/P)$$

is a finite-dimensional $\mathrm{Frac}(\mathbb{Z}/P)$-vector space, where $\mathrm{Frac}(\mathbb{Z}/P)$ is the field of fractions of $\mathbb{Z}/P$.

Proof. Since $u \in (\mathbb{Z}/P)^Z$, we see that $P \subseteq I$ and so we may regard $\mathbb{Z}[S]/I$ as a $\mathbb{Z}/P$-algebra. Since u satisfies an S-quasilinear recurrence, there is a finite generating set $T = \{t_1, \ldots, t_d\}$ of S and a positive integer M such that whenever $(i_1, \ldots, i_M) \in \{1, \ldots, d\}^M$, there exist integers $c_1, \ldots, c_M \in \mathbb{Z}$ with

$\gcd(c_1, \ldots, c_M) = 1$ such that

$$\sum_{j=1}^{M} c_j t_{i_j} \cdots t_{i_M} \in I.$$

We claim that R is spanned as a $\mathrm{Frac}(\mathbb{Z}/P)$-vector space by the finite set $\mathcal{X}$, consisting of elements of the form $(t + I) \otimes 1$ where t is a product of length at most M of elements from T. To see this, suppose towards a contradiction that this is not the case. Since R is spanned as a vector space by elements of the form $(s + I) \otimes 1$ with $s \in S$ and since T generates S, there is then an element

$$s = t_{j_1} \cdots t_{j_L} \in S,$$

with $1 \leq j_1, \ldots, j_L \leq d$, such that $(s + I) \otimes 1$ is not in the span of $\mathcal{X}$. Among all such s, we pick one with L minimal. Then $L > M$, since otherwise $(s + I) \otimes 1 \in \mathcal{X}$. By quasilinearity, there exist integers $c_1, \ldots, c_M \in \mathbb{Z}$ with $\gcd(c_1, \ldots, c_M) = 1$ such that

$$\sum_{k=1}^{M} c_k t_{j_{L-M+k}} \cdots t_{j_L} \in I.$$

Since $\gcd(c_1, \ldots, c_M) = 1$, there is some smallest $k_0 \in \{1, \ldots, M\}$ such that $c_{k_0} \notin P$. In particular, c_{k_0} is a unit in $\mathrm{Frac}(\mathbb{Z}/P)$ and $c_j \in I$ for $j < k_0$. Thus

$$(t_{j_{L-M+k_0}} \cdots t_{j_L} + I) \otimes 1 = \sum_{k=k_0+1}^{M} (t_{j_{L-M+k}} \cdots t_{j_L} + I) \otimes (-c_k c_{k_0}^{-1})$$

in R. Left multiplying both sides by the image of $t_{j_1} \cdots t_{j_{L-M+k_0-1}} \otimes 1$ in R, we have

$$(s + I) \otimes 1 = \sum_{k=k_0+1}^{M} (t_{j_1} \cdots t_{j_{L-M+k_0-1}} t_{j_{L-M+k}} \cdots t_{j_L} + I) \otimes (-c_k c_{k_0}^{-1}).$$

In particular, s is a $\mathrm{Frac}(\mathbb{Z}/P)$-linear combination of words of the form

$$(s' + I) \otimes 1$$

with each s' a product of elements of T of length strictly less than L. But by minimality of L, we then see that these elements are in the span of $\mathcal{X}$ and hence so is $(s + I) \otimes 1$, a contradiction. The result now follows. $\square$

Proof of Proposition 2.5. We let I denote the annihilator of u in $\mathbb{Z}[S]$. Since every finitely generated abelian group is a direct sum of cyclic groups, by Corollary 2.4, it suffices to prove this in the case when A is a non-trivial cyclic group, and so we divide the proof into two cases.

Case I. $A = \mathbb{Z}$.

Since u satisfies an S-quasilinear recurrence, by Lemma 2.6, taking the prime $P = (0)$, $\mathbb{Z}[S]/I \otimes_{\mathbb{Z}} \mathbb{Q}$ is finite-dimensional as a $\mathbb{Q}$-vector space. We pick $t_1, \ldots, t_d \in S$ with the property that the set $\{(t_i + I) \otimes 1 \colon i = 1, \ldots, d\}$

spans $\mathbb{Z}[S]/I \otimes_{\mathbb{Z}} \mathbb{Q}$ as a vector space and we let $R = \mathbb{Z}[S]/I$. Consider the $\mathbb{Z}$-submodule W of $\mathbb{Z}^d = A^d$ spanned by elements of the form

$$v_z := (u(t_1 \cdot z), \ldots, u(t_d \cdot z))$$

with $z \in Z$. Then W is finitely generated as a $\mathbb{Z}$-module and hence there exist $z_1, \ldots, z_m \in Z$ such that W is generated by $v_{z_1}, \ldots, v_{z_m}$.

We define a homomorphism of additive abelian groups $\Psi : \mathbb{Z}[S] \to A^m$ given by

$$s \mapsto (u(s \cdot z_1), \ldots, u(s \cdot z_m)).$$

We claim that $f \in \mathbb{Z}[S]$ is in the kernel of Ψ if and only if f annihilates u. It is clear that if f annihilates u then it is in the kernel of Ψ. Conversely, suppose that f is in the kernel of Ψ. Then since the images of $t_1 \otimes 1, \ldots, t_d \otimes 1$ span $R \otimes_{\mathbb{Z}} \mathbb{Q}$, there is some positive integer m and some integers $c_1, \ldots, c_d$ such that

$$mf - c_1 t_1 - \cdots - c_d t_d \in I.$$

Then for $z \in Z$,

$$mf \cdot u_z = (c_1 t_1 + \cdots + c_d t_d) \cdot u_z = \sum_{i=1}^{d} c_i u(t_i \cdot z). \tag{1}$$

Observe that the right-hand side of Equation (1) is zero if and only if

$$\sum_{i=1}^{d} c_i u(t_i z_j) = 0$$

for $j = 1, \ldots, m$, since the vectors v_z with $z \in Z$ are all in the $\mathbb{Z}$-span of $v_{z_1}, \ldots, v_{z_m}$. Since $f \in \ker(\Psi)$, $mf \in \ker(\Psi)$, we have $mf \cdot u_{z_j} = 0$ for $j = 1, \ldots, m$. It follows from Equation (1) and the preceding remarks that $m \cdot f$ annihilates u and so the positive integer m annihilates $f \cdot u$. But since A is torsion-free, we necessarily have f is in I, giving us the claim. It follows that Ψ induces an injective map from R into A^m. Since A^m is a finitely generated abelian group, we then have R is a finitely generated abelian group. Thus $\mathbb{Z}[S]/I$ is a finitely generated $\mathbb{Z}$-module and so u satisfies an S-linear recurrence, which completes the proof in this case.

Case II. $A = \mathbb{Z}/n\mathbb{Z}$ with $n > 1$.

In this case, we assume towards a contradiction that there exists a sequence $u \in A^Z$ that satisfies an S-quasilinear recurrence but does not satisfy an S-linear recurrence. We may also assume that $n > 1$ is minimal among all positive integers for which there exists such a sequence in $(\mathbb{Z}/n\mathbb{Z})^Z$.

Observe that if n is prime then by Lemma 2.6, $\mathbb{Z}[S]/I$ is a finite-dimensional $\mathbb{Z}/n\mathbb{Z}$-vector space and hence $\mathbb{Z}[S]/I$ is a finitely generated $\mathbb{Z}$-module, which contradicts the fact that u does not satisfy an S-linear recurrence. Thus n is composite, and so there is a prime number p such that $n = pn_0$ with $n_0 > 1$. Now let

$$A_0 = \{x \in A \colon px = 0\}.$$

Then

$$\bar{u} := (u_z + A_0)_{z\in Z}$$

satisfies an S-quasilinear recurrence and since A/A_0 is cyclic of order $n_0 < n$, we have that $\bar{u}$ satisfies an S-linear recurrence by minimality of n.

Hence if J denotes the annihilator of $\bar{u}$ then $\mathbb{Z}[S]/J$ is a finitely generated $\mathbb{Z}$-module. Then for $f \in J$, we have $f \cdot u \in A_0^Z$ and $f \cdot u$ satisfies an S-quasilinear recurrence, since u does. Since $|A_0| = p$, $f \cdot u$ satisfies an S-linear recurrence by the above remarks. In particular, for each $f \in J$, if we let J_f denote the annihilator of $f \cdot u$, then $\mathbb{Z}[S]/J_f$ is a finitely generated $\mathbb{Z}$-module, since $f \cdot u$ satisfies an S-linear recurrence.

Since $\mathbb{Z}[S]/J$ is a finitely generated $\mathbb{Z}$-module and S is a finitely generated monoid, we have that J is finitely generated as a left ideal by Lemma 2.3. We let $f_1, \ldots, f_q$ denote a set of generators of J as a left ideal. Then by construction the left ideal $J' := J_{f_1}f_1 + \cdots + J_{f_q}f_q$ annihilates u and hence $J' \subseteq I$. To finish the proof, observe that each $\mathbb{Z}[S]/J_{f_i}$ is a finitely generated $\mathbb{Z}$-module, and so by an induction using Lemma 2.3, $\mathbb{Z}[S]/L$ is a finitely generated $\mathbb{Z}$-module, where

$$L := J_{f_1} \cdots J_{f_q}.$$

By construction

$$I \supseteq J' \supseteq Lf_1 + \cdots + Lf_q = LJ,$$

and since $\mathbb{Z}[S]/L$ and $\mathbb{Z}[S]/J$ are both finitely generated $\mathbb{Z}$-modules, so is $\mathbb{Z}[S]/LJ$ by Lemma 2.3. Hence $\mathbb{Z}[S]/I$ is finitely generated as a $\mathbb{Z}$-module as it is a homomorphic image of $\mathbb{Z}[S]/LJ$. It now follows that u satisfies an S-linear recurrence. $\square$

We require a few more basic facts about recurrences.

Lemma 2.7. *Let A be an abelian group, let S be a finitely generated monoid, and let $u = (u_s)_{s\in S}$ be a sequence in A^S. Suppose there is a surjective semigroup homomorphism $\Psi : S \to G$ where G is a finite group and let T be the semigroup $\Psi^{-1}(1)$. Then T acts on the set $Z_g := \Psi^{-1}(g)$ for each $g \in G$. Furthermore, if T is finitely generated as a monoid and if for each $g \in G$ $u_g := (u_z)_{z\in Z_g}$ satisfies a T-linear recurrence, then (u_s) satisfies an S-linear recurrence.*

Proof. Notice that if $t \in T$ and $z \in Z_g$ then

$$\Psi(t \cdot z) = \Psi(t)\Psi(z) = 1 \cdot g = g,$$

and so $t \cdot z \in Z_g$. Hence T acts on Z_g.

For $g \in G$, we let $I_g \subseteq \mathbb{Z}[T]$ denote the annihilator of u_g. Then by assumption $\mathbb{Z}[T]/I_g$ is a finitely generated $\mathbb{Z}$-module and since there are only finitely many ideals $(I_g)_{g\in G}$, inductively applying Lemma 2.3 gives that $\mathbb{Z}[T]/J$ is also a finitely generated $\mathbb{Z}$-module, where

$$J := \prod_{g\in G} I_g.$$

Since each I_g is contained in J, if $f \in J$ then f annihilates each u_g. Observe that after reindexing S, we have $u = (u_g)_{g\in G}$, and since T acts on Z_g, we have

$$f \cdot u = (f \cdot u_g)_{g\in G} = 0,$$

and so f annihilates u.

It follows that the ideal $I := \mathbb{Z}[S]J\mathbb{Z}[S] \subseteq \mathbb{Z}[S]$ is contained in the annihilator of u. To finish the proof, it suffices to show that $\mathbb{Z}[S]/I$ is a finitely generated $\mathbb{Z}$-module.

We claim that there is a finite subset U of S such that every element of S can be expressed in the form

$$u_1t_1u_2t_2 \cdots u_{m-1}t_{m-1}u_m,$$

with $m \le |G|$, $u_1, \ldots, u_{m-1} \in U$, and $t_1, \ldots, t_{m-1} \in T$. To see this, we pick a set of generators $s_1, \ldots, s_d$ of S and let U denote the set of elements of S that can be expressed as a product of elements in $s_1, \ldots, s_d$ of length at most $|G|$. Then it is immediate that if s is an element of S, then s has an expression of the form $u_1t_1u_2 \cdots u_{p-1}t_{p-1}u_p$ for some p. For this element s, we pick such an expression with p minimal. If $p \le |G|$, there is nothing to prove, so we may assume that $p > |G|$. Then

$$\Psi(u_1), \Psi(u_1u_2), \ldots, \Psi(u_1 \cdots u_p)$$

are p elements of G and since $p > |G|$, two of them must be the same. Thus there exist i, j with $1 \le i < j \le p$ such that

$$\Psi(u_1 \cdots u_i) = \Psi(u_1 \cdots u_j),$$

and so $\Psi(u_{i+1} \cdots u_j) = 1$. In particular,

$$\Psi(u_{i+1}t_{i+1} \cdots u_{j-1}t_{j-1}u_j) = 1$$

and so $t := u_{i+1}t_{i+1} \cdots u_{j-1}t_{j-1}u_j \in T$. Thus we can rewrite s as

$$u_1t_1 \cdots u_i(t_itt_j)u_{j+1} \cdots t_{p-1}u_p,$$

which contradicts the minimality of p in our expression for s. The claim now follows.

Since $\mathbb{Z}[T]/J$ is a finitely generated $\mathbb{Z}$-module, there exists a finite subset V of T such that $\mathbb{Z}[T]/J$ is spanned by images of elements of V. It follows that $\mathbb{Z}[S]/I$ is spanned as a $\mathbb{Z}$-module by images of elements of the form $u_1t_1u_2t_2 \cdots u_{m-1}t_{m-1}u_m$ with $u_i \in U$ and $t_i \in V$ and $m \le |G|$. Thus $\mathbb{Z}[S]/I$ is a finitely generated $\mathbb{Z}$-module and so (u_s) satisfies an S-linear recurrence, as required. □

The following result is connected to the classical Skolem-Mahler-Lech theorem [Lec53].

Proposition 2.8. *Let A be a finitely generated abelian group, let $B \le A$ be a subgroup, and let $(u_n)_{n\in\mathbb{N}_0}$ be an A-valued sequence that satisfies an $\mathbb{N}_0$-linear recurrence. Then $\{n \colon u_n \in B\}$ is eventually periodic.*

Proof. We may replace (u_n) by the sequence $(u_n + B)$ taking values in the quotient group A/B and thus we may assume without loss of generality that $B = (0)$.

Then A/B is a finitely generated abelian group and hence is a direct sum of cyclic groups. We write

$$A/B = C_1 \oplus \cdots \oplus C_r,$$

with each C_i cyclic. Then

$$u_n = (u_{1,n}, \ldots, u_{r,n})$$

with $(u_{i,n})$ a C_i-valued sequence satisfying an $\mathbb{N}_0$-linear recurrence for $i = 1, \ldots, r$. Then

$$\{n \colon u_n = 0\} = \bigcap_{i=1}^{r} \{n \colon u_{i,n} = 0\}.$$

Since a finite intersection of eventually periodic subsets of $\mathbb{N}_0$ is again eventually periodic, we see that it suffices to consider the case when A is cyclic and $B = (0)$. There are now two cases. First, when $A = \mathbb{Z}$, the result follows from the Skolem-Mahler-Lech theorem [Lec53], since we have an integer-valued sequence u_n that satisfies a linear recurrence and the Skolem-Mahler-Lech theorem asserts that the set of n for which $u_n = 0$ is eventually periodic. The other case is when A is a finite cyclic group. In this case, u_n satisfies a recurrence of the form

$$u_n + c_1 u_{n-1} + \cdots + c_d u_{n-d} = 0$$

for some $d \geq 1$, which holds for $n \geq d$. Since A is finite, there exist indices p and q with $d \leq p < q$ such that $(u_{p-1}, \ldots, u_{p-d}) = (u_{q-1}, \ldots, u_{q-d})$. The recurrence then gives that $u_{n+p} = u_{n+q}$ for $n \geq 0$, and so u_n is eventually periodic. In particular the set of n for which $u_n = 0$ is eventually periodic. The result follows. □

We now turn our attention to multiplicative recurrences in K^* with K a field.

Proposition 2.9. *Let K be a finitely generated extension of $\mathbb{Q}$, and let $(u_n) \in (K^*)^{\mathbb{N}_0}$ be a sequence satisfying a multiplicative $\mathbb{N}_0$-quasilinear recurrence. Then in fact (u_n) satisfies a (multiplicative) linear recurrence and if H is a finitely generated subgroup of K^* then $\{n \colon u_n \in H\}$ is eventually periodic.*

This is not true without the hypothesis that K is finitely generated as an extension of $\mathbb{Q}$. For example if $K = \mathbb{C}$ and $u_n = \exp(2\pi i/2^n)$, then $u_n^2 = u_{n-1}$ and so (u_n) satisfies a quasilinear recurrence. But since u_n is never in the subfield generated by $u_1, \ldots, u_{n-1}$, we see that u_n does not satisfy a linear recurrence.

Proof of Proposition 2.9. The assumption that (u_n) is a quasilinear recurrence means that there is some $d \geq 0$ and integers $i_0, \ldots, i_d$ with $\gcd(i_0, \ldots, i_d) = 1$ so that the following relation holds for all $n \geq 0$:

$$u_n^{i_0} \cdots u_{n+d}^{i_d} = 1.$$

Then if G is the subgroup generated by $u_0, \dots, u_d$, it follows that u_n lies in the *radical* of G for all $n \geq 0$:

$$\sqrt{G} := \{g \in K^* : g^m \in G \text{ for some } m \geq 1\}.$$

We will show that $\sqrt{G}$ is finitely generated, from which the desired conclusion follows from Proposition 2.5.

To see that $\sqrt{G}$ is finitely generated: since $K/\mathbb{Q}$ is finitely generated, K is a finite extension of a function field $L = \mathbb{Q}(t_1, \dots, t_m)$. Now let $R = \mathbb{Z}[G, t_1, \dots, t_d]$ be the subring of K generated by $t_1, \dots, t_d$ and G, and let $F := \operatorname{Frac}(R)$ so that K/F is finite. Finally let $\overline{R}$ denote the integral closure of R in K. Then R is a finitely generated $\mathbb{Z}$-algebra, so the same is true for $\overline{R}$ [Eis95, Corollary 13.13]. It follows that the group of units $\overline{R}^*$ is finitely generated by Roquette's Theorem [Roq57]. But $\sqrt{G} \leq \overline{R}^*$ since every element of $\sqrt{G}$ is integral over R. Thus $\sqrt{G}$ is finitely generated.

Now let $H_0 := H \cap \sqrt{G}$. Then by Proposition 2.8 we have $\{n \colon u_n \in H_0\}$ is eventually periodic, and since $u_n \in H$ if and only if $u_n \in H_0$ we obtain the result. $\square$

3. Multiplicative dependence and S-unit equations

The goal of this section is to establish a key lemma which converts the statement of Theorem 1.2 into a problem about linear recurrences (in the sense of §2). Thus we may apply the results of §2 to obtain the main theorem.

Definition 3.1. Let K be a field and let k be a subfield of K. Elements $a_1, \dots, a_n \in K^*$ are *multiplicatively dependent modulo* k^* if there are integers $i_1, \dots, i_n \in \mathbb{Z}$, not all zero, such that $a_1^{i_1} \cdots a_n^{i_n} \in k^*$. If $a_1, \dots, a_n \in K^*$ are not multiplicatively dependent modulo k^* then they are *multiplicatively independent modulo* k^*.

Observe that if k is algebraically closed in K, the integers $i_0, \dots, i_d$ in Definition 3.1 can be chosen to satisfy $\gcd(i_0, \dots, i_d) = 1$. Indeed, if $m = \gcd(i_0, \dots, i_d)$, then $(i_0, \dots, i_d) = (mj_0, \dots, mj_d)$ for some $j_0, \dots, j_d$, and set $g := f_0^{j_0} \cdots f_d^{j_d}$. Then g^m is in k^*. But then $g \in k^*$ as k is algebraically closed. Since $\gcd(j_0, \dots, j_d) = 1$, this is the required multiplicative dependence modulo k^*.

Let K be an algebraically closed field and let X be an irreducible quasiprojective variety over K. For a group $G \leq K^*$ and rational functions $f_1, \dots, f_n \in K(X)$, we set

$$X_G(f_1, \dots, f_n) := \{x \in X : f_1(x), \dots, f_n(x) \in G\} = \bigcap_{i=1}^{n} f_i^{-1}(G). \tag{2}$$

Notice that if $X = \mathbb{A}^n$ and $f_i(x_1, \dots, x_n) := x_i$ is a coordinate function, then $X_G(f_1, \dots, f_n)$ is the set $X(G)$ of affine points with coordinates in G. The set X_G has been studied in [BOSS] in the case $X = \mathbb{P}^1$ and $f_0 = f_1 = f : \mathbb{P}^1 \to \mathbb{P}^1$ is a rational function; they determine exactly the form of such f so that X_G

is infinite. Similarly, multiplicative dependence of values of rational functions has been studied in [OSSZ19].

Now we state our key lemma.

Lemma 3.2. *Let K be an algebraically closed field of characteristic zero, let G be a finitely generated multiplicative subgroup of K^*, let X be an irreducible quasiprojective variety over K of dimension d, and let $f_0, \dots, f_d \in K(X)$ be $d+1$ rational functions on X. If $X_G := X_G(f_0, \dots, f_d)$ is Zariski dense in X, then $f_0, \dots, f_d$ are multiplicatively dependent modulo K^*.*

Proof. Since the field extension $K(X)/K$ has transcendence degree d, the functions $f_0, \dots, f_d$ must be algebraically dependent over K. Thus there is a polynomial relation

$$\sum_{i_0,\dots,i_d} c_{i_0\cdots i_d} f_0^{i_0} \cdots f_d^{i_d} = 0$$

where $c_{i_0\cdots i_d} \in K$ and the sum is over a finite set of indices in $\mathbb{N}_0^{d+1}$; this holds on some open subset of X. To simplify notation, let I be the (finite) set of those indices $\alpha = (i_0, \dots, i_d) \in \mathbb{N}_0^{d+1}$ where $c_{i_0\cdots i_d}$ is nonzero. For $\gamma = (i_0, \dots, i_d) \in \mathbb{Z}^{d+1}$, we set

$$f_\gamma := f_0^{i_0} \cdots f_d^{i_d} \quad \text{and} \quad c_\gamma := c_{i_0\cdots i_d}.$$

Then for every $y \in X_G$, the I-tuple $(c_\alpha f_\alpha(y))_{\alpha\in I}$ is a solution to the S-unit equation

$$\sum_{\alpha\in I} X_\alpha = 0$$

in the group $\overline{G}$ generated by $G \cup \{c_\alpha : \alpha \in I\}$. This tuple may be *degenerate* in the sense that some subsum vanishes, so we partition it into nondegenerate subtuples. Thus, for each partition $\pi \vdash I$, say $\pi = \{I_1, \dots, I_m\}$, we let $X_{G,\pi}$ be the set of points $y \in X_G$ such that, for each $s = 1, \dots, m$, the I_s-tuple $(c_\alpha f_\alpha(y))_{\alpha\in I_s}$ is nondegenerate (*i.e.* its sum vanishes, but no subsum vanishes). Note that there is a decomposition $X_G = \bigcup_{\pi\vdash I} X_{G,\pi}$ and hence there is some partition π of I such that $X_{G,\pi}$ is Zariski dense in X. Notice $X_{G,\pi}$ is empty if π has some part of size 1 since $c_\alpha f_\alpha(y) \neq 0$ for $\alpha \in I$ and $y \in X_G$, and hence if $\pi = (I_1, \dots, I_m)$ then each I_k has size at least two. Thus there exist α, β, two distinct indices in the same component I_s of π.

By the Main Theorem on S-unit equations [ESS02], an S-unit equation in characteristic zero has only finitely many nondegenerate solutions up to scalar multiplication [ESS02]. Let $(t_{1,\mu})_{\mu\in I_s}, \dots, (t_{n,\mu})_{\mu\in I_s}$ be all solutions to the equation

$$\sum_{\mu\in I_s} t_\mu = 0$$

up to scaling. Then for each $y \in X_{G,\pi}$, we know that $(c_\mu f_\mu(y))_{\mu\in I_s}$ is a multiple of some $(t_{j,\mu})_{\mu\in I_s}$, so there is some $g \in \overline{G}$ such that

$$c_\mu f_\mu(y) = g t_{j,\mu} \qquad \text{for all } \mu \in I_s.$$

Here g, j may depend on y. But then for $\alpha, \beta \in I_s$, we can take quotients to get

$$f_{\alpha-\beta}(y) = \frac{c_\beta t_{j,\alpha}}{c_\alpha t_{j,\beta}}$$

and there are only finitely many possible values for the right-hand side of this equation, independent of y. Taking $\gamma = \alpha - \beta$, we have $f_\gamma(y)$ takes only finitely many values for $y \in X_{G,\pi}$. Since X is irreducible and f_γ is constant on $X_{G,\pi}$, which is Zariski dense in X, we have $f_\gamma \in K^*$, which completes the proof. $\square$

3.1. **Interpolation of G-valued orbits as recurrences.** Now we enter the setup of our semigroup-version of Theorem 1.2. We find it convenient to fix the following assumptions and notation for the remainder of this section.

Notation 3.3. *We introduce the following notation:*

1. *we let K be an algebraically closed field of characteristic zero;*
2. *we let G be a finitely generated subgroup of K^*;*
3. *we let X be an irreducible quasiprojective variety over K;*
4. *we let $\varphi_1, \ldots, \varphi_m$ be dominant rational self-maps of X and we let S denote the monoid generated by these maps under composition;*
5. *we let S^{op} denote the* opposite semigroup, *which is, as a set, just S but with multiplication $\star$ given by $\mu_1 \star \mu_2 = \mu_2 \circ \mu_1$;*
6. *we let $f : X \to \mathbb{P}^1$ be a non-constant rational map;*
7. *we assume that $x_0 \in X$ has the property that its forward orbit under S is Zariski dense and each point avoids the indeterminacy loci of the maps $\varphi_1, \ldots, \varphi_m$ and f.*

With these data fixed, we may thus define a sequence u in K^S by

$$u_\varphi := f(\varphi(x_0)) \quad \text{for } \varphi \in S.$$

Notice that the semigroup algebra $\mathbb{Z}[S^{\mathrm{op}}]$ acts on K^S via the rule

$$\varphi \cdot (v_\mu)_{\mu \in S} = (v_{\mu\circ\varphi})_{\mu\in S} = (v_{\varphi\star\mu})_{\mu\in S} \quad \text{for } \varphi \in S.$$

In this section, we analyze the case when $u_\varphi \in G$ for every $\varphi \in S$.

Proposition 3.4. *Adopt the assumptions and notation of Notation 3.3. If $u_\varphi = f(\varphi(x_0)) \in G$ for every $\varphi \in S$ then $(f(\varphi(x_0)))_{\varphi\in S}$ satisfies a multiplicative S^{op}-linear recurrence.*

Proof. We let $\mathbf{C} \subseteq (K^*)^S$ denote the set of constant sequences and let v denote the image of u in $K^S/\mathbf{C}$. We first show that v satisfies an S^{op}-quasilinear recurrence. Let d denote the dimension of X and let $\varphi_{i_1} \circ \cdots \circ \varphi_{i_{d+1}}$ be a $(d+1)$-fold composition of $\varphi_1, \ldots, \varphi_m$. For $j = 1, 2, \ldots, d+1$, we let

$$\mu_j := \varphi_{i_j} \circ \cdots \circ \varphi_{i_{d+1}}$$

and let $f_j = f \circ \mu_j$. Then we take $X_G := X_G(f_1, \ldots, f_{d+1})$, as in Equation (2). The assumption that $u_\varphi \in G$ for $\varphi \in S$ implies that X_G contains the orbit of x_0 under S, which is dense. Thus $\overline{X_G} = X$ and it follows from Lemma 3.2 that $f_1, \ldots, f_{d+1}$ are multiplicatively dependent modulo K^*. Hence

$$f_1^{p_1} \cdots f_{d+1}^{p_{d+1}} \equiv c$$

where $c \in K^*$ is a constant and $p_1, \ldots, p_{d+1} \in \mathbb{Z}$ with $\gcd(p_1, \ldots, p_{d+1}) = 1$. Now evaluating this at $\varphi(x_0)$ gives

$$u_{\varphi\star\mu_1}^{p_1} \cdots u_{\varphi\star\mu_{d+1}}^{p_{d+1}} = c \quad \text{for all } \varphi \in S.$$

In particular, $p_1\mu_1 + \cdots + p_{d+1}\mu_{d+1} \in \mathbb{Z}[S^{\mathrm{op}}]$ annihilates v. It follows that v satisfies an S^{op}-quasilinear recurrence and it now follows from Proposition 2.5 that it satisfies an S^{op}-linear recurrence. We now claim that u satisfies an S^{op}-linear recurrence. To see this, let $I \subseteq \mathbb{Z}[S^{\mathrm{op}}]$ denote the annihilator of v. Then we have shown that $R := \mathbb{Z}[S^{\mathrm{op}}]/I$ is a finitely generated $\mathbb{Z}$-module. In particular, there exists some M such that R is spanned as a $\mathbb{Z}$-module by compositions of $\varphi_1, \ldots, \varphi_m$ of length at most M. Now let J denote the annihilator of u. We claim that $\mathbb{Z}[S^{\mathrm{op}}]/J$ is spanned as a $\mathbb{Z}$-module by compositions of length at most $M+1$, which will complete the proof that u satisfies an S^{op}-linear recurrence. So to show this, let φ be a composition of length $\ell \geq M+1$. We shall show by induction on ℓ that φ is equivalent mod J to a $\mathbb{Z}$-linear combination of compositions of $\varphi_1, \ldots, \varphi_m$ of length $M+1$, with the base case $\ell = M+1$ being immediate. So suppose that the claim holds whenever $\ell < q$ and consider the case when $\ell = q$. Then we can write $\varphi = \mu \circ \varphi_j$ for some μ that is a composition of length $q-1$ and some $j \in \{1, \ldots, m\}$. Then $\mu \equiv \sum m_j \mu_j \pmod{I}$, where the m_j are integers and the μ_j are all compositions of $\varphi_1, \ldots, \varphi_m$ of length at most M. In particular, $(\mu - \sum m_j\mu_j) \cdot u \in \mathbf{C}$ and so $(\varphi_j - 1) \star (\mu - \sum m_j \mu_j) \cdot u = 0$. Hence

$$\mu \circ \varphi_j \equiv \mu - \sum_j m_j(\mu_j \circ \varphi_j - \mu_j) \pmod{J}.$$

By the induction hypothesis the right-hand side is equivalent mod J to a $\mathbb{Z}$-linear combination of compositions of $\varphi_1, \ldots, \varphi_m$ of length $M+1$, and so we now obtain the result. □

Proposition 3.4 gives a combinatorial description of the sequence $(u_\varphi)_{\varphi \in S}$. We now give a more geometric interpretation of this result.

Corollary 3.5. *Adopt the assumptions and notation of Notation 3.3. If $u_\varphi = f(\varphi(x_0)) \in G$ for every φ in the monoid S generated by the dominant self-maps $\varphi_1, \ldots, \varphi_m$ then there exists a dominant rational map $\Theta : X \dashrightarrow \mathbb{G}_m^d$ with $d \leq \dim(X)$ that is defined at each point in $O_\varphi(x_0) = (\varphi(x_0))_{\varphi \in S}$ and endomorphisms $\Phi_1, \ldots, \Phi_m : \mathbb{G}_m^d \to \mathbb{G}_m^d$ such that the following diagram commutes*

$$\begin{tikzcd} X \arrow[r, dashed, "\varphi_1, \ldots, \varphi_m"] \arrow[d, dashed, "\Theta"'] & X \arrow[d, dashed, "\Theta"] \\ \mathbb{G}_m^d \arrow[r, "\Phi_1, \ldots, \Phi_m"'] & \mathbb{G}_m^d. \end{tikzcd}$$

Moreover, $f = g \circ \Theta$, where $g : \mathbb{G}_m^d \to \mathbb{G}_m$ is a map of the form

$$g(t_1, \ldots, t_d) = Ct_1^{i_1} \cdots t_d^{i_d}$$

for some $i_1, \ldots, i_d \in \mathbb{Z}$ and some $C \in G$.

Proof. We let S^{op} denote the opposite monoid of S. Then by Proposition 3.4 the sequence $u_\varphi := (f(\varphi(x_0)))_{\varphi \in S} \in G^S$ satisfies a multiplicative S^{op}-linear recurrence. It follows that there is some M such that every M-fold composition of $\varphi_1, \ldots, \varphi_m$ is congruent, modulo the annihilator of u, to a $\mathbb{Z}$-linear combination of j-fold compositions of these endomorphisms, as j ranges over numbers $< M$. Let W denote the set of j-fold compositions of $\varphi_1, \ldots, \varphi_m$ with $j < M$. Then we construct a rational map $\Theta : X \dashrightarrow \mathbb{G}_m^L$, where $L = |W|$, given by $\Theta(x) = (f \circ \varphi(x))_{\varphi \in W}$. Now let $i \in \{1, \ldots, m\}$ and consider $\Theta(\varphi_i(x)) = (f \circ \varphi \circ \varphi_i(x))_{\varphi \in W}$. By construction $f = \pi \circ \Theta$, where π is a suitable projection.

Then for $\varphi \in W$ and $i \in \{1, \ldots, m\}$, $\varphi \circ \varphi_i$ either remains in W or it is an M-fold composition of $\varphi_1, \ldots, \varphi_m$, in which case the fact that u satisfies an S^{op}-linear recurrence gives that there exist integers p_μ for each $\mu \in W$ such that

$$f \circ \varphi \circ \varphi_i(x) = \prod_{\mu \in W} (f \circ \mu(x))^{p_\mu}$$

for all x in the S-orbit of x_0. In particular, since the S-orbit of x_0 is Zariski dense in X, $\Theta \circ \varphi_i = \Psi_i \circ \Theta$ for some self-map Ψ_i of $\mathbb{G}_m^L$ of the form

$$(u_1, \ldots, u_L) \mapsto \left(\prod_j u_j^{p_{1,j}}, \ldots, \prod_j u_j^{p_{L,j}} \right).$$

In particular, each Ψ_i is a group endomorphism of the multiplicative torus. Now let Y denote the Zariski closure of the S-orbit of x_0 under Θ. Then by construction Y has a Zariski dense set of points in $G^L \leq \mathbb{G}_m^L$ and is irreducible. Then a theorem of Laurent [Lau84, Théorème 2] gives that Y is a translation of a subtorus of $\mathbb{G}_m^L$. In particular, $Y \cong \mathbb{G}_m^d$ for some $d \leq \dim(X)$ and $\Psi_1, \ldots, \Psi_d$ restrict to endomorphisms of Y. Moreover, since Y is a translation of a subtorus, the restriction of π to Y induces a map $g : \mathbb{G}_m^d \to \mathbb{G}_m$ of the form $g(u_1, \ldots, u_d) \mapsto Cu_1^{q_1} \cdots u_d^{q_d}$. The result now follows. $\square$

We point out that Theorem 1.2 follows from Corollary 3.5 by taking $m = 1$.

Remark 3.6. In fact, it can be observed that $\Theta(x_0) \in G^d$ and that the Ψ_i induce maps of $\mathbb{G}_m^d$ of the form

$$(x_1, \ldots, x_d) \mapsto (\lambda_1 x_1^{p_{1,1}} \cdots x_d^{p_{1,d}}, \ldots, \lambda_d x_1^{p_{d,1}} \cdots x_d^{p_{d,d}})$$

with $\lambda_1, \ldots, \lambda_d \in G$; finally, $g(x_1, \ldots, x_d) \mapsto C x_1^{q_1} \cdots x_d^{q_d}$ with $C \in G$.

The following example shows that the conclusion to Corollary 3.5 does not necessarily hold if K has positive characteristic.

Example 3.7. Let $K = \bar{\mathbb{F}}_p(u)$ and let $X = \mathbb{P}^1_K$. Then we have a map $\varphi : X \to X$ given by $t \mapsto t^p + 1$ and let $f : X \to \mathbb{P}^1$ be the map $f(t) = t$. Notice that if we take $x_0 = u$ then $f \circ \varphi^n(u) = u^{p^n} + n = u^{p^n}(1 + n/u)^{p^n}$ and hence $\varphi^n(u)$ lies in the finitely generated subgroup G of K^* generated by u and $1 + n/u$ for $n = 1, 2, \ldots, p-1$. Then if the conclusion to Corollary 3.5 held, we would necessarily have $d = 1$ since Θ is dominant and $f \circ \varphi^n(u)$ has infinite orbit. Thus the function fields of $\mathbb{P}^1$ and $\mathbb{G}_m^d$ are both isomorphic to $K(t)$ and the commutative diagram given in the statement of Corollary 3.5 would give rise to a corresponding diagram at the level of functions fields:

$$\begin{tikzcd} K(t) \arrow[r, dashed, "\Phi^*"] \arrow[d, dashed, "\Theta^*"'] & K(t) \arrow[d, dashed, "\Theta^*"] \\ K(t) \arrow[r, dashed, "\varphi^*"'] & K(t) \end{tikzcd}$$

with $\varphi^*(t) = t^p + 1$ and $\Phi^*(t) = Ct^a$ for some integer a and some $C \in K^*$. Moreover, $f^* = \Theta^* \circ g^*$ and since f^* is the identity map of $K(t)$, Θ^* and g^* are automorphisms of $K(t)$; since $g^*(t) = C' t^b$ for some integer a and some $C' \in K^*$, we have $b = \pm 1$, and so $\Theta^*(t) = C'^{-b} t^b$. But now $\Theta^* \circ \Phi^*(t) = (C')^{-ab} C t^{ab}$ and $\varphi^* \circ \Theta^*(t) = (C')^{-b}(t^p + 1)^{-b}$, and so the two sides do not agree.

Proof of Corollary 1.3. For each $n \geq 1$, we let $X_{\geq n}$ denote the Zariski closure of $\{\varphi^m(x_0) \colon m \geq n\}$. Since the $X_{\geq i}$ form a descending chain of closed sets and since X endowed with the Zariski topology is a noetherian topological space, there is some m such that $X_{\geq m} = X_{\geq m+1} = \cdots$. We let $Y = X_{\geq m}$ and we let $Z_1, \ldots, Z_r$ denote the irreducible components of Y. By our choice of m, φ induces a dominant rational self-map of Y and in particular there is some permutation σ of $\{1, \ldots, r\}$ such that $\varphi(Z_i)$ is Zariski dense in $Z_{\sigma(i)}$. It follows that there is some L such that φ^L maps each Z_i into itself. Let $j \in \{m, \ldots, m+L-1\}$. Then $\varphi^j(x_0) \in Z_i$ for some i. Then by the above, we have $\{\varphi^{Ln+j}(x_0) \colon n \geq 0\}$ is Zariski dense in Z_i. Moreover, $f(\varphi^{Ln+j}(x_0)) \in G$ for every $n \geq 0$ and so Theorem 1.2 implies that there is some $e \geq 0$ and some endomorphism $\Psi : \mathbb{G}_m^e \to \mathbb{G}_m^e$ and a map $g : \mathbb{G}_m^e \to \mathbb{G}_m$ such that $f(\varphi^{Ln+j}(x_0)) = g \circ \Psi^n(z_0)$ for some $z_0 \in \mathbb{G}_m^e$ whose coordinates lie in G. Let $h_1, \ldots, h_m$ be a set of generators for G. Then

$$\Psi^n(z_0) = (h_1^{a_{1,1}(n)} \cdots h_m^{a_{1,m}(n)}, \ldots, h_1^{a_{e,1}(n)} \cdots h_m^{a_{e,m}(n)})$$

for some integer-valued sequence $a_{i,j}(n)$. (There may be several choices for the sequences $a_{i,j}(n)$ if the h_i are not multiplicatively independent.) Since

$$\Psi(x_1, \ldots, x_e) = (h_1^{p_{1,1}} \cdots h_m^{p_{1,m}} x_1^{q_{1,1}} \cdots x_e^{q_{1,e}}, \ldots, h_1^{p_{e,1}} \cdots h_m^{p_{e,m}} x_1^{q_{e,1}} \cdots x_e^{q_{e,e}}),$$

there is a choice of the sequences $a_{i,j}(n)$ such that there is an integer matrix A and an integer vector $\mathbf{p}$ such that

$$\mathbf{v}(n+1) = A\mathbf{v}(n) + \mathbf{p}$$

for every $n \geq 0$, where $\mathbf{v}(n)$ is the column vector whose entries are $a_{i,j}(n)$ for $i = 1, \ldots, e$, and $j = 1, \ldots, m$ in some fixed ordering of the indices that does not vary with n. In particular, if $Q(x) = q_0 + q_1 x + \cdots + q_r x^r \in \mathbb{Z}[x]$ then

$$Q(A)\mathbf{v}(n) = q_0\mathbf{v}(n) + q_1\mathbf{v}(n+1) + \cdots + q_r\mathbf{v}(n+r) + \mathbf{b}_Q$$

for $n \geq 0$, where $\mathbf{b}_Q$ is an integer vector that depends upon Q but not upon n. In particular, if we take $Q(x)$ to be the characteristic polynomial of A, the Cayley-Hamilton theorem gives that the vectors $\mathbf{v}(n)$ satisfy a non-trivial affine linear recurrence of the form

$$0 = q_0\mathbf{v}(n) + q_1\mathbf{v}(n+1) + \cdots + q_r\mathbf{v}(n+r) + \mathbf{b}_Q$$

for $n \geq 0$. In particular, substituting $n+1$ for n into this equation and subtracting from our original equation gives a recurrence

$$0 = q_0\mathbf{v}(n) + (q_1 - q_0)\mathbf{v}(n+1) + \cdots + (q_r - q_{r-1})\mathbf{v}(n+r) - q_r\mathbf{v}(n+r+1).$$

It follows that each $a_{i,j}(n)$ satisfies a linear recurrence. Then applying the map g and using the fact that a sum of sequences satisfying a linear recurrence also satisfies a linear recurrence now gives the result. $\square$

4. Proof of Theorem 1.1

In this section we prove Theorem 1.1. The setup is as follows: X is a quasiprojective variety defined over a field K of characteristic zero, $\varphi : X \dashrightarrow X$ is a rational map, $x_0 \in X$ is a point whose forward φ-orbit is well-defined, $f : X \dashrightarrow K$ is a rational function defined on this orbit, and G is a finitely generated subgroup of K^*. Finally, we let

$$N := \{n \in \mathbb{N}_0 : f(\varphi^n(x_0)) \in G\}.$$

We first show that if N has a positive upper Banach density then it must contain an infinite arithmetic progression. Once a single arithmetic progression is obtained, we then use noetherian induction to show that N is a union of finitely many arithmetic progressions together with a set of upper Banach density zero.

4.1. **A single arithmetic progression.** With notation as above, in this section we will prove:

Proposition 4.1. *Let X be a quasiprojective variety over an algebraically closed field K of characteristic zero, let $\varphi : X \dashrightarrow X$ be a rational map, let $f : X \dashrightarrow K$ be a rational function, and let $G \leq K^*$ be a finitely generated group. Suppose that $x_0 \in X$ is a point with well-defined forward φ-orbit that also avoids the indeterminacy locus of f. Then if the set*

$$N := \{n \in \mathbb{N}_0 : f(\varphi^n(x_0)) \in G\}$$

has a positive upper Banach density then it contains an infinite arithmetic progression.

To prove this result, we require a lemma.

Lemma 4.2. *Let K be an algebraically closed field of characteristic zero, let X be an irreducible quasiprojective variety over K, let $\varphi : X \to X$ and $f : X \to \mathbb{P}^1$ be rational maps, and let $x_0 \in X$ be a point whose forward orbit under φ is defined and is Zariski dense and avoids the indeterminacy locus of f. If the set of n for which $f \circ \varphi^n(x_0) = 0$ has upper Banach density zero and if $u_n := f \circ \varphi^n(x_0)$ has the property that there exists $C \neq 0$ and integers $i_0, \ldots, i_d$ with $i_0 i_d \neq 0$ and $\gcd(i_0, \ldots, i_d) = 1$ such that $u_n^{i_0} \cdots u_{n+d}^{i_d} = C$ for every $n \geq 0$ then:*

(1) *$u_n \in K^*$ for all n;*
(2) *the set of n for which $u_n \in G$ is an eventually periodic set, whenever G is a finitely generated subgroup of K^*.*

Proof. Since $u_n^{i_0} \cdots u_{n+d}^{i_d} = C$ and since x_0 has a Zariski dense orbit, we have $f^{i_0} = C \prod_{j=1}^{d} (f \circ \varphi^j)^{-i_j}$. In particular, if f has a zero at $\varphi^n(x_0)$ for some n, then there is some $j \in \{1, 2, \ldots, d\}$ for which $f \circ \varphi^j$ has a zero or a pole at $\varphi^n(x_0)$. But since the orbit of x_0 under φ avoids the indeterminacy locus of f, $f(\varphi^{j+n}(x_0)) = 0$ for some $j \in \{1, 2, \ldots, d\}$. Hence if $u_n = 0$ then $u_{n+j} = 0$ for some $j \in \{1, 2, \ldots, d\}$. In particular, $\{n \colon u_n = 0\}$ has a positive upper Banach density, a contradiction. Thus $u_n \in K^*$. In fact, there is a finitely generated extension of $\mathbb{Q}$, $K_0 \subseteq K$, such that $x_0 \in X(K_0)$ and such that φ and f are defined over K_0. It follows that $u_n \in K_0^*$ for all n and using the equation $u_n^{i_0} \cdots u_{n+d}^{i_d} = C$ and substituting $n+1$ for n and taking quotients, we have

$$u_{n+d+1}^{i_d} u_{n+d}^{i_{d-1}-i_d} \cdots u_{n+1}^{i_0-i_1} u_n^{-i_0} = 1.$$

Moreover, it is straightforward to show that $\gcd(i_0, i_0 - i_1, \ldots, i_{d-1} - i_d, i_d) = 1$ and so (u_n) satisfies an $\mathbb{N}_0$-quasilinear recurrence. But that means it satisfies a linear recurrence by Proposition 2.5. In particular, the u_i are all contained in a subfield K_0 of K that is finitely generated over $\mathbb{Q}$ and so the result follows from Proposition 2.9. □

Proof of Proposition 4.1. By [BGT15, Theorem 1.4] there is some positive integer L such that for $j \in \{0, \ldots, L-1\}$ we have $\mathcal{Z}_j := \{n \colon f \circ \varphi^{Ln+j}(x_0) = 0\}$ either is a set of upper Banach density zero or contains all sufficiently large natural numbers. If $\delta(N) > 0$ then there is some j such that $N \cap (L\mathbb{N}_0 + j)$ has a positive upper Banach density and such that $\mathcal{Z}_j$ has upper Banach density zero. Then we can replace φ by φ^L and x_0 by $\varphi^j(x_0)$ and we may assume without loss of generality that the set of n for which $f \circ \varphi^n(x_0) = 0$ has upper Banach density zero.

Let $\mathcal{S}$ denote the collection of Zariski closed subsets Y of X for which there exists a rational self-map $\Psi : Y \dashrightarrow Y$ and $y_0 \in Y$ whose forward orbit under Ψ is well-defined and avoids the indeterminacy locus of f and such that the following hold:

(i) $N(Y, y_0, \Psi, f, G) := \{n \colon f \circ \Psi^n(y_0) \in G\}$ has a positive upper Banach density but does not contain an infinite arithmetic progression;
(ii) $\{n \colon f \circ \Psi^n(y_0) = 0\}$ has upper Banach density zero.

If $\mathcal{S}$ is empty, then we are done. Thus we may assume $\mathcal{S}$ is non-empty and since X is a noetherian topological space, there is some minimal element Y in $\mathcal{S}$. By assumption, there exists a rational self-map $\Psi : Y \dashrightarrow Y$ and $y_0 \in Y$ such that conditions (i) and (ii) hold.

Observe that the orbit of y_0 under Ψ must be Zariski dense in Y, since otherwise, we could replace Y with the Zariski closure of this orbit and construct a smaller counterexample. We also note that Y is necessarily irreducible. To see this, we suppose towards a contradiction that this is not the case and let $Y_1, \ldots, Y_r$ denote the irreducible components of Y, with $r \geq 2$. Then since the orbit of y_0 is Zariski dense, Ψ is dominant and hence it permutes the irreducible components of Y in the sense that there is a permutation σ of $\{1, \ldots, r\}$ with the property that $\Psi(Y_i)$ is Zariski dense in $Y_{\sigma(i)}$. It follows that there is some $M > 1$ such that Ψ^M maps Y_i into Y_i for every i. Now there must be some $j \in \{0, \ldots, M-1\}$ such that $(M\mathbb{N} + j) \cap N(Y, y_0, \Psi, f, G)$ has a positive upper Banach density. Then $\Psi^j(y_0) \in Y_i$ for some i, and so by construction $N(Y_i, \Psi^j(y_0), \Psi^M, f, G)$ has a positive upper Banach density. Since Y_i is a proper closed subset of Y, by minimality of Y, $N(Y_i, \Psi^j(y_0), \Psi^M, f, G)$ contains an infinite arithmetic progression. But $N(Y_i, \Psi^j(y_0), \Psi^M, f, G) \subseteq N(Y, y_0, \Psi, f, G)$ and so $N(Y, y_0, \Psi, f, G)$ contains an infinite arithmetic progression, a contradiction. Thus Y is irreducible.

Let $d := \dim(Y)$. Since the upper Banach density of $N(Y, y_0, \Psi, f, G)$ is positive, a version of Szemerédi's Theorem [Sze75] due to Furstenberg [Fur77, Theorem 1.4] gives that there is a set A of positive upper Banach density and a fixed integer $b \geq 1$ such that $N(Y, y_0, \Psi, f, G)$ contains the finite progression

$$a, a+b, a+2b, \ldots, a+db$$

for every $a \in A$. Setting $f_n := f \circ \Psi^{bn}$ for $n \geq 0$, we have defined $d+1$ rational functions $f_0, \ldots, f_d$, so by Lemma 3.2 either the set

$$Y_G := Y_G(f_0, \ldots, f_d) = \{x \in Y : f_0(x), \ldots, f_d(x) \in G\}$$

is contained in a proper subvariety of Y, or the functions $f_0, \ldots, f_d$ satisfy some multiplicative dependence relation. We proceed by ruling out the first possibility. Suppose that $\overline{Y_G} \subsetneq Y$. Since $\Psi^a(y_0) \in \overline{Y_G}$ for every $a \in A$, the set

$$P := \{n \in \mathbb{N}_0 : \Psi^n(y_0) \in \overline{Y_G}\}$$

has a positive upper Banach density. Hence [BGT15, Theorem 1.4] gives that P is a union of infinite arithmetic progressions $A_1, \ldots, A_r$ and a set of density zero. In particular, since P has a positive upper Banach density, P contains an infinite arithmetic progression. But since $P \subseteq N(Y, y_0, \Psi, f, G)$, we then see $N(Y, y_0, \Psi, f, G)$ contains an infinite arithmetic progression, a contradiction. It follows that Y_G is Zariski dense in Y.

Combining this with Lemma 3.2, we conclude that there is a multiplicative dependence relation

$$\prod_{s=0}^{d} f(\Psi^{sb}(x))^{i_s} = C \in K^*, \tag{3}$$

where $i_0, \ldots, i_d \in \mathbb{Z}$ with $\gcd(i_0, \ldots, i_d) = 1$. Then for $a \in \{0, \ldots, b-1\}$ we let $u_a(n) := f(\Psi^{a+bn}(y_0))$. Evaluating Equation (3) at $x = \Psi^{a+bn}(y_0)$ then gives the relation

$$u_a(n)^{i_0} \cdots u_a(n+d)^{i_d} = C.$$

and so Lemma 4.2 gives that $u_a(n) \in K^*$ for all $n \geq 0$ and that $u_a(n)$ satisfies a multiplicative $\mathbb{N}_0$-linear recurrence and that the set of n for which $u_a(n) \in G$ is eventually periodic. In particular, since there is some a for which the set $\{n\colon u_a(n) \in G\}$ has a positive upper Banach density, for this a, $\{n\colon u_a(n) \in G\}$ contains an infinite arithmetic progression $c + e\mathbb{N}_0$. This then gives that N contains the infinite arithmetic progression $a+b(c+e\mathbb{N}_0) = (a+bc)+be\mathbb{N}_0$, as required. □

4.2. **A union of arithmetic progressions.** We now use Proposition 4.1 to prove Theorem 1.1.

Proof of Theorem 1.1. First, by [BGT15, Theorem 1.4] there is some positive integer L such that for $j \in \{0, \ldots, L-1\}$ we have $\mathcal{Z}_j := \{n\colon f \circ \varphi^{Ln+j}(x_0) = 0\}$ either is a set of upper Banach density zero or contains all sufficiently large natural numbers. Then to prove the result, it suffices to prove that for every natural number $j \in \{0, \ldots, L-1\}$, the set of n for which $f \circ \varphi^{Ln+j}(x_0) \in G$ is a finite union of arithmetic progressions along with a set of upper Banach density zero. In the case that $\mathcal{Z}_j$ contains all sufficiently large natural numbers, this is immediate; hence we may replace φ by φ^L and x_0 by some point in the orbit under φ and assume without loss of generality that the set $\mathcal{Z}$ of n for which $f \circ \varphi^n(x_0) = 0$ has upper Banach density zero. We now let $X_{\geq i}$ denote the Zariski closure of $\{\varphi^n(x_0)\colon n \geq i\}$. Then as in the proof of Corollary 1.3, we have that there is some m such that $X_{\geq m} = X_{\geq m+1} = \cdots$ and without loss of generality we may replace X with $X_{\geq m}$ and x_0 with $\varphi^m(x_0)$ and assume that the orbit of x_0 is Zariski dense in X. Now let $X_1, \ldots, X_r$ denote the irreducible components of X. Then there is some positive integer M such that $\varphi^M(X_i)$ is Zariski dense in X_i for $i = 1, \ldots, r$. Then it suffices to prove that for $j \in \{0, \ldots, M-1\}$ we have $\{n\colon f \circ \varphi^{Mn+j}(x_0) \in G\}$ is a finite union of arithmetic progressions along with a set of upper Banach density zero. Since $\{\varphi^{Mn+j}(x_0)\colon n \geq 0\}$ is Zariski dense in some component X_i, we may replace X by X_i, x_0 by $\varphi^j(x_0)$ and φ with φ^M and we may assume that X is irreducible and that $\{\varphi^n(x_0)\colon n \geq 0\}$ is Zariski dense in X. Now let $N := \{n\colon f(\varphi^n(x_0)) \in G\}$. If N has upper Banach density zero, then there is nothing to prove. Thus we may assume that N has a positive upper Banach density, and hence it contains an infinite arithmetic progression, say $a\mathbb{N}_0 + b$ with $a > 0$.

We point out that the Zariski closure, Y, of the set $\{\varphi^{an+b}(x_0)\colon n \geq 0\}$ must be Zariski dense in X, since the union of the closures Y_i of $\varphi^i(Y)$ for

$i = 0, 1, \dots, a-1$ contains all but finitely many points in the orbit of x_0 and hence is dense in X. Since X is irreducible, we then see that Y_i must be X for some i, which then gives that $Y = X$.

Now for each $i \geq 0$, define a rational function $f_i := f \circ \varphi^{ai}$, and set

$$X_G := \{x \in X : f_0(x), \dots, f_d(x) \in G\},$$

where d is the dimension of X. Then X_G contains $\{\varphi^{an+b}(x_0) : n \geq 0\}$, which is Zariski dense in X and so Lemma 3.2 gives that the functions $f_0, \dots, f_d$ satisfy some multiplicative dependence of the form

$$f_0^{i_0} \cdots f_d^{i_d} = c$$

with c nonzero and $i_0, \dots, i_d$ integers with $\gcd(i_0, \dots, i_d) = 1$. It follows that if we set $u_n = f(\varphi^n(x_0))$ then $u_n^{i_0} u_{n+a}^{i_1} \cdots u_{n+ad}^{i_d} = c$ for every $n \geq 0$. Moreover, by assumption the set of n for which $u_n = 0$ has upper Banach density zero and thus by Lemma 4.2, the set

$$\{n \in \mathbb{N}_0 : u_n \in G\}$$

is eventually periodic. This completes the proof. □

Remark 4.3. We point out that if K is a field of characteristic zero, then a K-valued sequence $g(n)$ that satisfies a linear recurrence can be realized as a sequence of the form $f(\phi^n(x_0))$ with x_0 a point in a suitable affine space $\mathbb{A}^d$, ϕ a linear self-map, and f a projection to $\mathbb{A}^1$. In particular, Theorem 1.1 shows that for a finitely generated subgroup G of K^*, the set of n for which $g(n) \in G$ is a finite union of arithmetic progressions along with a set of upper Banach density zero. In the case when G is finite, one can strengthen the conclusion and replace the density zero set with a finite set; this is the content of the celebrated Skolem-Mahler-Lech theorem [Lec53]. The case when $g(n) \in G \cup \{0\}$ for every n was first dealt with by Pólya [Pól21] and later, more generally, by Bézivin [Béz86].

5. Heights of points in orbits

Corollary 1.3 gives an interesting "gap" about heights of points in the forward orbit of a self-map φ for varieties and maps defined over $\mathbb{Q}$.

In order to state this gap result, we must first recall the definition of the Weil height here. Let K be a number field and let M_K be the set of places of K. For a place v, let $|\cdot|_v$ be the corresponding absolute value, normalized so that $|p|_v = p^{-1}$ when v lies over the p-adic valuation on $\mathbb{Q}$. Let K_v be the completion of K at a place v and let $n_v := [K_v : \mathbb{Q}_v]$. Now define a function $H : \overline{\mathbb{Q}} \to [1, \infty)$ as follows: for $x \in \overline{\mathbb{Q}} \setminus \{0\}$, choose any number field K containing x, and set

$$H(x)^{[K:\mathbb{Q}]} := \prod_{v \in M_K} \max\{|x|_v^{n_v}, 1\}.$$

This is independent of the choice of K and defines a function $H : \overline{\mathbb{Q}} \to [1, \infty)$ called the *absolute Weil height*. We let $h : \overline{\mathbb{Q}} \to [0, \infty)$ be its logarithm; i.e.,

$h(x) := \log H(x)$. For further background on height functions, we refer the reader to [BG06, Chapter 2] and [Sil07, Chapter 3]. We have the following result.

Theorem 5.1. *Let X be an irreducible quasiprojective variety with a dominant self-map $\varphi : X \dashrightarrow X$ and let $f : X \dashrightarrow \mathbb{P}^1$ be a rational map, all defined over $\bar{\mathbb{Q}}$. Suppose that $x_0 \in X$ has the following properties:*

(1) *every point in the orbit of x_0 under φ avoids the indeterminacy loci of φ and f;*
(2) *there is a finitely generated multiplicative subgroup G of $\bar{\mathbb{Q}}^*$ such that $f \circ \varphi^n(x_0) \in G$ for every $n \in \mathbb{N}_0$.*

If $h(f \circ \varphi^n(x_0)) = \mathrm{o}(n^2)$ then the sequence $(f \circ \varphi^n(x_0))_n$ satisfies a linear recurrence. More precisely, there exists an integer $L \geq 1$ such that for each $j \in \{0, \ldots, L-1\}$ there are $\alpha_j, \beta_j \in G$ such that for all n sufficiently large we have

$$f \circ \varphi^{Ln+j}(x_0) = \alpha_j \beta_j^n.$$

Proof. Let d denote the rank of G. Then a result of Schlickewei [Sch97, Theorem 1.1] gives that there exist $g_1, \ldots, g_d \in G$ such that every element of G can be expressed uniquely in the form $\zeta g_1^{n_1} \cdots g_d^{n_d}$ with ζ being a root of unity and $n_1, \ldots, n_d \in \mathbb{Z}$ and such that

$$h(\zeta g_1^{n_1} \cdots g_d^{n_d}) \geq \max_{1 \leq i \leq d} \{|n_i| 4^{-d} h(g_i)\}. \tag{4}$$

Moreover, since G is finitely generated, there exists a positive integer N and an N-th root of unity $g_0 \in G$ such that every element of G is of the form $g_0^{n_0} g_1^{n_1} \cdots g_d^{n_d}$ with $n_0, \ldots, n_d \in \mathbb{Z}$. By Corollary 1.3, there is a positive integer L and integer-valued sequences $b_{i,j}(n)$ for $j = 0, \ldots, L-1$ and $i = 1, \ldots, m$, each of which satisfies a linear recurrence, such that

$$f \circ \varphi^{Ln+j}(x_0) = \prod_{i=0}^{d} g_i^{b_{i,j}(n)}$$

for $n \geq p$. Then multiplication by a root of unity does not affect the height of a number and so by Equation (4),

$$h(f \circ \varphi^{Ln+j}(x_0)) = h\left(\prod_{i=1}^{d} g_i^{b_{i,j}(n)}\right) \geq \max_{1 \leq i \leq d} \{|b_{i,j}(n)| 4^{-d} h(g_i)\}.$$

Thus if $h(f \circ \varphi^{Ln+j}(x_0)) = \mathrm{o}(n^2)$ then $b_{i,j}(n) = \mathrm{o}(n^2)$ for $j = 0, \ldots, L-1$ and $i = 1, \ldots, d$. Since each $b_{i,j}(n)$ is also an integer-valued sequence satisfying a linear recurrence, we have that it is in fact $\mathrm{O}(n)$ and is "piecewise linear", *i.e.,* it has the form $A + Bn$ on progressions of a fixed gap [BNZ, Proposition 3.6]. Formally, this means that there exists a fixed $M \geq 1$ and integers $A_{i,j}, B_{i,j}$ for $j \in \{0, \ldots, M-1\}$ and $i \in \{1, \ldots, d\}$, and integer-valued sequences $c_j(n)$, which satisfy a linear recurrence for $j = 0, \ldots, M-1$, such that for n sufficiently large we have

$$f \circ \varphi^{Mn+j}(x_0) = g_0^{c_j(n)} \prod_{i=1}^{d} g_i^{A_{i,j}+B_{i,j}n}. \tag{5}$$

Then since each $c_j(n)$ is an integer-valued sequence satisfying a linear recurrence, it is eventually periodic modulo N, and since g_0 is an N-th root of unity, we may assume without loss of generality that each sequence $c_j(n)$ is eventually periodic. We set

$$\alpha_j = \prod_{i=1}^{d} g_i^{A_{i,j}} \qquad \text{and} \qquad \beta_j = \prod_{i=1}^{d} g_i^{B_{i,j}}.$$

Then by Equation (5),

$$f \circ \varphi^{Mn+j}(x_0) = \alpha_j \beta_j^n g_0^{c_j(n)},$$

for $j = 0, \ldots, M-1$. Since we are only concerned about what holds for n sufficiently large, it is no loss of generality to assume that each sequence $c_j(n)$ is periodic and we let q be a positive integer that is a common period for each of $c_0, \ldots, c_{M-1}$. Then for $j \in \{0, \ldots, M-1\}$ and $i \in \{0, \ldots, q-1\}$ we have

$$f \circ \varphi^{qMn+Mi+j}(x_0) = \left(\alpha_j g_0^{c_j(i)} \beta_j^i\right) (\beta_j^q)^n.$$

We take $L = qM$. Then for each $s \in \{0, \ldots, L-1\}$ we can find unique $j \in \{0, \ldots, M-1\}$ and $i \in \{0, \ldots, q-1\}$ such that $s = Mi + j$. Then

$$f \circ \varphi^{Ln+s}(x_0) = \left(\alpha_j g_0^{c_j(i)} \beta_j^i\right) (\beta_j^q)^n,$$

which is of the desired form. The result now follows. □

6. Applications to D-finite power series

In this section we apply our results to D-finite power series, showing how Theorem 1.1 generalizes a result of Methfessel [Met00] and Bézivin [Béz89]. We also look at classical results of Pólya [Pól21] and Bézivin [Béz86] through a dynamical lens.

Definition 6.1. Let K be a field. A power series $F(x) \in K[[x]] \subseteq K((x))$ is *D-finite* if the set of formal derivatives $\{F^{(i)}(x) : i \geq 0\}$ is linearly dependent over $K(x) \subseteq K((x))$; equivalently, $F(x)$ satisfies a linear differential equation of the form

$$\sum_{i=0}^{d} p_i(x) F^{(i)}(x) = 0,$$

where $p_0(x), \ldots, p_d(x) \in K[x]$ are polynomials, not all zero.

A sequence $(a_n) \in K^{\mathbb{N}_0}$ is *holonomic* or *P-recursive* over K if it satisfies a recurrence relation

$$a_{n+1} = \sum_{i=0}^{d} r_i(n) a_{n-i},$$

for all $n \geq d$, where $r_0(t), \ldots, r_d(t) \in K(t)$ are rational functions. If each $r_i(t)$ is constant, then (a_n) satisfies a *K-linear recurrence*.

For further background, we refer the reader to the works of Stanley [Sta80, Sta96]. It is well-known that if $F(x) = \sum_{n\geq 0} a_n x^n$ is a formal power series with coefficients in a field K of characteristic zero, then $F(x)$ is D-finite (resp. rational) if and only if (a_n) is P-recursive (resp. K-linearly recursive) [Sta80]. The first application of our main result Theorem 1.1 in this setting is immediate, as follows.

Since $F(x)$ is D-finite, its coefficient sequence is P-recursive: there is a recurrence relation

$$a_{n+1} = \sum_{i=0}^{d} r_i(n) a_{n-i},$$

valid for all sufficiently large n, where the $r_i(x) \in K(x)$ are rational functions [Sta80]. Thus we may define a rational map $\varphi : \mathbb{A}^{d+2} \dashrightarrow \mathbb{A}^{d+2}$ as follows:

$$(t, t_0, t_1, \ldots, t_d) \mapsto \left(t+1, t_1, t_2, \ldots, t_d, \sum_{i=0}^{d} r_i(t) t_i \right).$$

Here $(t, t_0, t_1, \ldots, t_d)$ are coordinates on $\mathbb{A}^{d+2}$. Now there is some $p > 0$ such that none of the $r_i(x)$ have a pole at $x = n$ when $n \geq p$. Now take the initial point to be $x_0 := (p, a_p, \ldots, a_{p+d})$ and the rational function $f(t, t_0, t_1, \ldots, t_d) := t_0$. Then the sequence $(a_n)_{n\geq 0}$ can be recovered as

$$a_{n+p} = f(\varphi^n(x_0)) \text{ for } n \geq 0. \tag{6}$$

Proof of Theorem 1.4. By Equation (6), after taking a suitable shift of the sequence $(a_n)_{n\geq 0}$, it can be recovered as

$$a_n = f(\varphi^n(x_0)).$$

Thus the desired sets N and N_0 are just

$$N = \{n \in \mathbb{N}_0 : f(\varphi^n(x_0)) \in G\} \text{ and } N_0 = \{n \in \mathbb{N}_0 : f(\varphi^n(x_0)) \in G \cup \{0\}\}.$$

Then we obtain the desired decomposition of N from Theorem 1.1; since $N_0 = N \cup Z$, where $Z = \{n \in \mathbb{N}_0 : f(\varphi^n(x_0)) = 0\}$, applying [BGT15, Theorem 1.4] then gives that Z is a finite union of arithmetic progressions along with a set of upper Banach density zero. Then since both N and Z are expressible as a finite union of infinite arithmetic progressions along with a set of upper Banach density zero, so is their union. The result follows. $\square$

6.1. **Theorems of Pólya and Bézivin.** Pólya [Pól21] showed that, given a fixed set of prime numbers S, if $F(x) = \sum a_n x^n \in \mathbb{Z}[[x]]$ is the power series of a rational function and the prime factors of a_n lie inside of S for every n, then there is some natural number L such that for n sufficiently large

$$a_{Ln+j} = \frac{A_j}{B_j} \cdot \beta_j^n,$$

where A_j, B_j, and β_j are integers whose prime factors lie inside of S for $j = 0, \ldots, L-1$ and B_j divides $A_j \beta_j^m$ for some positive integer m. This result was later extended by Bézivin [Béz86], who showed that if K is a field of characteristic zero and $G \leq K^*$ is a finitely generated group then if $F(x) = \sum a_n x^n$ is a D-finite power series such that there is some fixed m such that each a_n is a sum of at most m elements of G, then $F(x)$ is rational; moreover, he gave a precise form of these rational functions. We show how to recover Bézivin's theorem in the case that $m = 1$ from the dynamical results we obtained in the preceding sections. In particular, this recovers Pólya's theorem. We conclude by showing the relationship between these classical theorems and the dynamical results developed in the preceding sections. More precisely, we give a dynamical proof of the following result.

Theorem 6.2. *(Bézivin [Béz86]) Let K be a field of characteristic zero and let $F(x) = \sum a_n x^n \in K[[x]]$ be a D-finite power series such that $a_n \in G \cup \{0\}$ for every n, where G is a finitely generated subgroup of K^*. Then $F(x)$ is rational.*

To do this, we require a basic result on orders of zeros and poles of coefficients in a D-finite series. We recall that if X is a smooth irreducible projective curve over an algebraically closed field k, and if $k(X)$ is the field of rational functions on X, then to each $p \in X$ we have a discrete non-archimedean valuation $\nu_p : k(X)^* \to \mathbb{Z}$ that gives the order of vanishing of a function at p (when the function has a pole at p then this valuation is negative). Then for a function $f \in k(X)^*$ we have a divisor $\operatorname{div}(f) = \sum_{p \in X} \nu_p(f)[p]$, which is a formal $\mathbb{Z}$-linear combination of points of X. The *support* of $\operatorname{div}(f)$ is the (finite) set of points p for which $\nu_p(f) \neq 0$; that is, it is the set of points where f has a zero or a pole. We make use of the fact $\sum_p \nu_p(f) = 0$ [Har77, II, Corollary 6.10].

Lemma 6.3. *Let E be an algebraically closed field of characteristic zero and let K be the field of rational functions of a smooth projective curve $\mathcal{C}$ over E. Suppose that $F(x) = \sum a_n x^n \in K[[x]]$ is D-finite, $a_n \neq 0$ for every n, and that there is a finite subset S of $\mathcal{C}$ such that $\operatorname{div}(a_n)$ is supported on S for every n. Then for each $p \in S$, $\nu_p(a_n) = \mathrm{O}(n)$.*

Proof. We have a polynomial recurrence

$$R_M(n)a_{n+M} + \cdots + R_0(n)a_n = 0$$

for n sufficiently large. Since each $R_i(n) = \sum_{j=0}^{L} r_{i,j} n^j$, we claim there is a fixed number C_i such that $\nu_p(R_i(n)) = C_i$ for sufficiently large n for each nonzero polynomial R_i. To see this, pick a uniformizing parameter u for the local ring $\mathcal{O}_{\mathcal{C},p}$ that consists of all rational functions in K with p being not a pole and suppose that $Q(x) = q_0 + \cdots + q_L x^L$ is a nonzero polynomial in $K[x]$. Then we can rewrite it as $\sum_{i=0}^{N} u^{m_i} q_i' x^L$ where $m_i = \nu_p(q_i)$ and $q_i' \in \mathcal{O}_{X,p}^*$. Let s denote the minimum of $m_0, \ldots, m_N$. Then $Q(x)/u^s = \sum_{i=0}^{N} u^{m_i - s} q_i' x^L$ and by construction

$$\sum_{i=0}^{N} u^{m_i - s}(p) q_i'(p) x^L$$

is a nonzero polynomial in $E[x]$ and hence it is nonzero for n sufficiently large, which shows that $\nu_p(Q(n)) = s$ for all n sufficiently large. Thus in particular if C is the maximum of the C_i as i ranges over the indices for which $R_i(x)$ is nonzero, then for n sufficiently large

$$\begin{aligned}\nu_p(a_{n+M}) &= \nu_p(R_M(n)a_{n+M}) - C\\ &= \nu_p(\sum_{i=0}^{M-1} R_i(n)a_{n+i}) - C\\ &\geq -2C + \min(\nu_p(a_{n+i})\colon i = 0, \ldots, M-1).\end{aligned}$$

It follows that $\nu_p(a_n) \geq -2Cn + B$ for some constant B and all sufficiently large n. It follows that there is a fixed constant C_0 such that $\nu_p(a_n) \geq -C_0 n$ for every $p \in S$, for all n sufficiently large. To get an upper bound, observe that $\sum_{p\in S} \nu_p(a_n) = 0$ [Har77, II, Corollary 6.10] and so for n sufficiently large we have

$$\nu_p(a_n) = \sum_{q\in S\setminus\{p\}} -\nu_q(a_n) \leq (|S| - 1)C_0 n,$$

which now gives $\nu_p(a_n) = \mathrm{O}(n)$. □

We now give a quick overview of how one can recover Theorem 6.2 from the above dynamical framework.

Proof of Theorem 6.2. By Theorem 1.4, $\{n\colon a_n \in G\}$ is a finite union of arithmetic progressions along with a set of upper Banach density zero. Hence there exists some L such that for each $j \in \{0, \ldots, L-1\}$, the set $\{n\colon a_{Ln+j} = 0\}$ either contains all sufficiently large n or is a set of upper Banach density zero. Since $F(x) = \sum a_n x^n$ is D-finite if and only if for each $L \geq 1$ and each $j \in \{0, \ldots, L-1\}$, the series $\sum a_{Ln+j} x^n$ is D-finite, by Lemma 4.2 it suffices to consider the case when $a_n \in G$ for every n. The fact that the coefficients are P-recursive gives that there is a finitely generated field extension K_0 of $\mathbb{Q}$ such that $F(x) \in K_0[[x]]$. We prove the result by induction on $\mathrm{trdeg}_{\mathbb{Q}}(K_0)$. If $[K_0 : \mathbb{Q}] < \infty$ then K_0 is a number field. Then by [BNZ, Theorem 1.6], $h(a_n) = \mathrm{O}(n \log n)$ and by Equation (6) and Theorem 5.1, we then get a_n satisfies a linear recurrence, giving the result when K_0 has transcendence degree zero over $\mathbb{Q}$.

We now suppose that the result holds whenever K_0 has transcendence degree less than m, with $m \geq 1$, and consider the case when $\mathrm{trdeg}_{\mathbb{Q}}(K_0) = m$. Then there is a subfield E of K_0 such that K_0 has transcendence degree 1 over E and such that E is algebraically closed in K_0. Since K_0 has characteristic zero and E is algebraically closed in K_0, K_0 is a regular extension of E, and so $R := K_0 \otimes_E \bar{E}$ is an integral domain. Then the field of fractions of R is the field of regular functions of a smooth projective curve X over $\bar{E}$. Now let $g_1, \ldots, g_d, g_{d+1}, \ldots, g_m$ be generators for G so that $g_1, \ldots, g_d$ generate a free abelian group and $g_{d+1}, \ldots, g_m$ are roots of unity and let $\{p_1, \ldots, p_\ell\} \in X$ denote the collection of points at which some element from $g_1, \ldots, g_d$ has a zero or a pole. Then there are integers $b_{i,j}$ such that

$$\mathrm{div}(g_i) = \sum b_{i,j}[p_j]$$

for $i = 1, \ldots, d$. Now we have $a_n = g_1^{e_1(n)} \cdots g_m^{e_m(n)}$ and so

$$\mathrm{div}(a_n) = \sum_{j=1}^{\ell} \left(\sum_{i=1}^{d} b_{i,j} e_i(n) \right) [p_j].$$

In particular, by Lemma 6.3,

$$\sum_{i=1}^{d} b_{i,j} e_i(n) = \mathrm{O}(n)$$

and since the left-hand side satisfies a linear recurrence, we have that it is piecewise linear in the sense of having the form $A + Bn$ on progressions of a fixed gap [BNZ, Proposition 3.6]. It then follows that there exists some $r \geq 1$ and some fixed $h_0, \ldots, h_{r-1} \in K_0^*$ such that for $j \in \{0, \ldots, r-1\}$, $a_{r(n+1)+j}/a_{rn+j} = C_{j,n} h_j$, where $C_{j,n} \in E^*$. It follows that for $a_{rn+j} = C_{j,n} h_j^n P_j$, where $P_j \in K_0$ is constant. Then since the series $\sum P_j^{-1} h_j^{-n} x^n$ is D-finite and since D-finite series are closed under Hadamard product,

$$G_j(x) := \sum C_{j,n} x^n \in \bar{E}[[x]]$$

is D-finite and takes values in a finitely generated multiplicative group. Thus $G_j(x)$ is rational and then it is straightforward to show that

$$F_j(x) := \sum a_{rn+j} x^n = \sum C_{j,n} P_j h_j^n x^n$$

must also be rational and thus $F(x) = \sum_{j=0}^{r-1} x^j F_j(x^r)$ is also rational, as required. □

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