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SYMBOLIC INTEGRATION OF DIFFERENTIAL FORMS: FROM ABEL TO ZEILBERGER

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Abstract. This paper focuses on symbolic integration of differential forms, with a particular emphasis on historical and modern developments, from Abel's addition theorems for Abelian integrals to Zeilberger's creative telescoping for parameterized integrals. It explores closed rational p-forms and provides algorithmic approaches for their integration, extending classical results like Hermite reduction and Liouville's theorem. The integration of closed differential forms with parameters is further examined through telescopers, offering a unified framework for handling both algebraic and transcendental cases.

1. Introduction

Differential forms play a central role in mathematics in both theoretical and computational aspects. They provide an elegant language to unify different kinds of integrals in multivariable calculus and are used in differential geometry [48], algebraic topology [9], and algebraic geometry [31] whenever integrals are involved. Symbolic integration of univariate functions has been well-developed starting with the celebrated work by Risch [46] and summarized comprehensively in Bronstein's book [10] and the foundational references collected in [45]. A long-standing challenging project in symbolic computation is developing symbolic algorithms for computing integrals of mutltivariate functions. In the multivariate setting, differential forms provide a natural language for studying integration problems. Symbolic computation of differential forms has been studied in computational algebraic geometry [11, 12] with applications to computing periods in [37]. Packages in computer algebra systems such as Maple and Mathematica rely on a variety of heuristic techniques to integrate differential forms. The goal of this paper is to enrich the algorithmic methods available for symbolic integration of differential forms, motivated by Abel's classical addition theorem on integrals of algebraic functions.

In 1826, Abel submitted a remarkable memoir [1] to the Paris Academy, who did not publish it until 1841, long after his death. His main focus is on what we now call *Abelian integrals*, which are integrals of the form

$$\int f(x,y)\,dx,$$

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where x and y are related by a polynomial equation $\chi(x, y) = 0$. Abel considers an auxiliary equation $\theta(x, y) = 0$ that depends on parameters $a, a', a'', \ldots$, so that the solutions of the simultaneous equations

$$\chi(x, y) = \theta(x, y) = 0$$

are algebraic functions of the parameters. Abel focuses on the nonconstant solutions and writes them as (x_i, y_i), $i = 1, \ldots, \mu$. In Section 2 of [1], Abel states the following theorem:

> I say that if we denote by $f(x, y)$ some rational function of x and y, and if we set
>
> $$dv = f(x_1, y_1)\, dx_1 + f(x_2, y_2)\, dx_2 + \cdots + f(x_\mu, y_\mu)\, dx_\mu,$$
>
> the differential dv will be a rational function of the quantities a, a', a'', etc.

This is one of several results in [1] that go under the name "Abel's Theorem" (see [35]). Abel's first proof of the rationality of dv is very terse. A more detailed argument is given in his last paper [2], and modern explanations can be found in [25], [29, Ch. 9], and [33]. In Section 4 of [1], Abel notes that this method is "in general very long, and for functions that are a little complicated, almost impractical." He then gives a direct proof of the rationality of dv by expressing it as an explicit rational differntial of the parameters. His astonishing formula is described in [25].

Later in Section 2 of [1], Abel notes that:

> If now dv is a rational function of the quantities $a, a', a'' \ldots$ its integral or the quantity v will be an algebraic and logarithmic function of $a, a', a'' \ldots$[1]

For us, this is where things get interesting. In going from dv to v, Abel is clearly referring to the method of partial fractions, which is where the logarithms come from. But this method applies to rational 1-forms in one variable, while here, dv is a rational 1-form in the parameters $a, a', a'', \ldots$. What's missing is the recognition that dv is *closed* (which is easy to prove) and that one needs a result like Theorem 3.1 to guarantee that v is a rational and logarithmic function of the parameters. Abel avoids this difficulty since his explicit formula for dv leads immediately to an explicit formula for v.

In contrast, we show in Theorem 3.1 that this is the case for *any* closed rational 1-form. A natural question is whether Theorem 3.1 still holds for closed rational p-forms. This is answered affirmatively by Theorem 4.3 in Section 4.

Liouville's classical theorem is crucial for determining whether an elementary function has an elementary integral. The first step towards symbolic integration of differential forms with elementary-function coefficients would be establishing Liouville's theorem in this setting. In this direction, the first work was done by Caviness and Rothstein [14] who proved a version of Liouville's theorem for integration in

[1]While Abel says "algebraic and logarithmic" in his result, Theorem 3.1 shows that "algebraic" can be replaced with "rational" (this is also proved in [25]). Abel's preference for "algebraic" is related to a change made in Section 6 of [1]. If α is the number of parameters $a, a', a'', \ldots$ and (x_i, y_i), $i = 1, \ldots, \mu$, are the solutions considered above, then Abel claims that $a, a', a'', \ldots$ are algebraic functions of the first α of $x_1, \ldots, x_\mu$. This makes v an algebraic and logarithmic function of $x_1, \ldots, x_\alpha$.

finite terms of line integrals. Using the techniques introduced in Section 3, we provide a more direct proof in the language of differential 1-forms. It is still challenging to extend this theorem to the case of p-forms.

Symbolic computation of parameterized integrals is an active topic [44] with rich applications to Gauss–Manin connections and Picard–Fuchs equations in algebraic geometry [39, 37] and Feymann diagrams in mathematical physics [40, 4]. A powerful tool for evaluating parameterized integrals is Zeilberger's method of creative telescoping [3, 54]. The formulation of creative telescoping can be generalized from functions to differential forms. Given a differential form ω in $x_1, \ldots, x_m$ with a parameter t, a linear differential operator L with respect to t is called a *telescoper* for ω if $L(\omega)$ is exact. The existence problem of telescopers was first studied by Manin in [39] for the case of m-forms with algebraic functions as coefficients and recently was investigated in [17, 18] for the case of differential forms with D-finite functions as coefficients. We will present an algorithm that uses Hermite reduction to compute minimal telescopers of rational differential 1-forms with one parameter.

The outline of the paper is as follows. After establishing some notation and preliminary lemmas in Section 2, we generalize the 1-forms dv studied by Abel to the case of closed rational 1-forms and develop the corresponding Hermite reduction in Section 3. The case of closed rational p-forms is considered in Section 4. In Section 5, we recall Picard's problem about the exactness of differential 3-forms and Griffiths-Dwork reduction for solving the smooth case together with an example and a conjecture related to Fermat hypersurfaces. In Section 6, we generalize Liouville's theorem to differential 1-forms. We conclude the paper by studying Zeilberger's creative telescoping for closed rational 1-forms in Section 7.

2. Preliminaries

In this section, we first recall some basic facts about differential fields and their extensions [10]. We also discuss differential forms.

Definition 2.1. Let R be a ring (resp. field). A *derivation* on R is a map $D : R \to R$ such that for any $a, b \in R$, we have $D(a + b) = D(a) + D(b)$ and $D(ab) = aD(b) + bD(a)$. The pair (R, D) is called a *differential ring* (resp. *differential field*). The set $\mathrm{Const}_D(R) := \{r \in R \mid D(r) = 0\}$ is a subring (resp. subfield) of R. A pair (R^*, D^*) is called a *differential extension* of (R, D) if $R \subseteq R^*$ and $D^*|_R = D$.

The following fact is used frequently in symbolic integration [10].

Lemma 2.2. *For a field F of characteristic zero, let $F(y)$ be the field of rational functions in y over F. Let D_y denote the derivative d/dy on $F(y)$ satisfying $D_y(y) = 1$ and $D_y(c) = 0$ for all $c \in F$. Given pairwise coprime polynomials $u_1, \ldots, u_n \in F[y] \setminus F$, constants $c_1, \ldots, c_n \in F$, and a rational function $v \in F(y)$, suppose that*

$$\sum_{i=1}^{n} c_i \frac{D_y(u_i)}{u_i} + D_y(v) = 0.$$

Then $c_1 = \cdots = c_n = 0$ and $v \in F$.

Proof. Fix i and let β be a root of u_i of multiplicity ℓ in some differential extension of F. By hypothesis, $u_j(\beta) \neq 0$ for $j \neq i$. A standard calculation shows that

$$\mathrm{res}_{y=\beta}\Big(\frac{D_y(u_i)}{u_i}\Big) = \ell \neq 0.$$

The residue of the derivative of a rational function always vanishes, so taking the residue at $y = \beta$ of each side of the equation in the lemma implies $c_i\ell = 0$, and $c_1 = \cdots = c_n = 0$ follows. Then $D_y(v) = 0$, which implies $v \in F$. $\square$

For a field k of characteristic zero, let $K := k(x_1, \ldots, x_m)$ be the field of rational functions in $x_1, \ldots, x_m$ over k. Let ∂_i denote the usual partial derivative $\partial/\partial x_i$ that satisfies $\partial_i\partial_j(f) = \partial_j\partial_i(f)$ for all $f \in K$ and $\partial_i(c) = 0$ for all $c \in k(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)$. Let $\mathcal{U}$ be the universal differential extension of K in which any system of algebraic differential equations has a solution in $\mathcal{U}$ if it has a solution in some extension of K (see [36, Chapter III, Section 7] for the existence of such universal fields). Note that the ∂_i's also commute on $\mathcal{U}$.

We now recall some basic notions about exterior algebras over $\mathcal{U}$ from Lang's book [38, Chapter XIX]. Given a vector space $\mathcal{M}$ over $\mathcal{U}$ of dimension m with basis $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$, let

$$\bigwedge \mathcal{M} = \bigoplus_{p=0}^{m} \bigwedge^p \mathcal{M}$$

be the exterior algebra of $\mathcal{M}$, where $\bigwedge^p \mathcal{M}$ denotes the vector space over $\mathcal{U}$ with basis $\{\mathbf{a}_{i_1} \wedge \cdots \wedge \mathbf{a}_{i_p} \mid 1 \le i_1 < \cdots < i_p \le m\}$. We call an element in $\bigwedge^p \mathcal{M}$ a p-form, and an element $f \in \mathcal{U}$ is called a 0-form. Let $d\colon \mathcal{U} \to \mathcal{M}$ be the map defined by

$$df = \partial_1(f)\,\mathbf{a}_1 + \cdots + \partial_m(f)\,\mathbf{a}_m.$$

Hence $dx_i = \mathbf{a}_i$ by taking $f = x_i$. For the remainder of this paper, we will use $\{dx_1, \ldots, dx_m\}$ instead of $\{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$. We call $\mathcal{M}$ the *space of differential* 1*-forms*.

We recall some properties of operations on differential forms from [52, 13]. The exterior product $\omega \wedge \eta$ is bilinear, anticommutative, and associative as follows:

$$\begin{aligned} f_I\,dx_I \wedge g_J\,dx_J &= f_I g_J\,dx_I \wedge dx_J \\ (f_I\,dx_I + f'_{I'}\,dx_{I'}) \wedge g_J\,dx_J &= f_I g_J\,dx_I dx_J + f'_{I'} g_J\,dx_{I'} dx_J \\ f_I\,dx_I \wedge g_J\,dx_J &= (-1)^{|I|\cdot|J|} g_J\,dx_J \wedge f_I\,dx_I \\ (f_I\,dx_I \wedge g_J\,dx_J) \wedge h_R\,dx_R &= f_I\,dx_I \wedge (g_J\,dx_J \wedge h_R\,dx_R). \end{aligned}$$

Here, dx_I stands for $dx_{i_1} \wedge \cdots \wedge dx_{i_p}$, where $I = \{i_1, \ldots, i_p\}$, often written $dx_I = dx_{i_1} \cdots dx_{i_p}$. If dx_I is any string with some dx_i appearing more than once, then $dx_I = 0$. If $i_1, \ldots, i_k$ are distinct and $\sigma : \{i_1, \ldots, i_k\} \to \{i_1, \ldots, i_k\}$ is a permutation, then

$$dx_{\sigma(i_1)} \cdots dx_{\sigma(i_k)} = \operatorname{sign}(\sigma)\,dx_{i_1} \cdots dx_{i_k},$$

where $\operatorname{sign}(\sigma) = \pm 1$ is the sign of the permutation. The map d can be extended to a derivation on $\bigwedge \mathcal{M}$ defined recursively by

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge d\omega_2,$$

for any $\omega_1 \in \bigwedge^p \mathcal{M}$ and $\omega_2 \in \bigwedge^q \mathcal{M}$. The difference between differential forms in $\mathcal{M}$ and Kähler differentials is well explained in [30].

Definition 2.3. We say that $\omega \in \bigwedge \mathcal{M}$ is *closed* if $d\omega = 0$ and $\omega \in \bigwedge \mathcal{M}$ is *exact* if there exists $\eta \in \bigwedge \mathcal{M}$ such that $d\eta = \omega$.

Since $d(d\eta) = 0$, it follows that every exact differential form is closed.

We now define operators d_s and d^s that will be used in our recursive arguments. For any $f \in E$ and $s \in \{1, 2, \dots, m\}$, define

$$d^s(f dx_I) = \partial_s(f)\, dx_s \wedge dx_I = \partial_s(f)\, dx_s dx_I.$$

The definition of d^s extends linearly to all of $\bigwedge M$. Then define

$$d_s = d^1 + \cdots + d^s,$$

and note that

$$d = d^1 + \cdots + d^m = d_{m-1} + d^m.$$

Given $\omega \in \bigwedge^p M$, we can always decompose ω into

$$\omega = \omega_{m-1} + \omega^m,$$

where all summands $f_{i_1,\dots,i_p}\, dx_{i_1} \cdots dx_{i_p}$ of ω_{m-1} do not involve dx_m and $\omega^m = \mu\, dx_m$ with $\mu \in \bigwedge^{p-1} \mathcal{M}$ and free of dx_m. The following lemma is an easy consequence of the above definitions.

Lemma 2.4. *If $\omega = \omega_{m-1} + \omega^m \in \bigwedge^p \mathcal{M}$ as above, then we have*

(i) $d^m(\omega^m) = 0$.
(ii) $d(\omega\, dx_m) = d_{m-1}(\omega)\, dx_m$.
(iii) $d\omega = 0$ *if and only if* $d_{m-1}(\omega_{m-1}) = 0$ *and* $d^m(\omega_{m-1}) + d_{m-1}(\omega^m) = 0$.
(iv) *If ω is free of dx_m, then $d\omega = 0$ if and only if $d_{m-1}(\omega) = 0$ and $d^m(\omega) = 0$.*
(v) *If ω is free of x_m and dx_m, then $\omega = d\mu$ for some $(p-1)$-form μ if and only if $\omega = d_{m-1}(\tilde{\mu})$ for some $(p-1)$-form $\tilde{\mu}$ also free of x_m and dx_m.*

3. Integration of Closed Rational 1-Forms

We first study the integration of closed 1-forms with coefficients in $K = F(x_m)$, where $F = k(x_1, \dots, x_{m-1})$ and k is algebraically closed. Assume that $\omega = f_1\, dx_1 + \cdots + f_m\, dx_m$, $f_1, \dots, f_m \in K$, is a closed 1-form. Let $\overline{F}$ denote the algebraic closure of F. By the classical integration formula for rational functions in the variable x_m (i.e., partial fractions with respect to x_m), we have

$$f_m = \partial_m(g_m) \quad \text{with} \quad g_m = a_m + \sum_j c_{m,j} \log b_{m,j}, \tag{3.1}$$

where $a_m \in K$, $c_{m,j} \in \overline{F}$, and $b_{m,j} \in F(c_{m,j})[x_m]$ are pairwise coprime polynomials of positive degree. Since ω is closed, we have $\partial_i(f_m) = \partial_m(f_i)$ for $i = 1, \dots, m-1$. Thus

$$\begin{aligned}
\partial_m(f_i) &= \partial_i(f_m) = \partial_i(\partial_m(g_m)) = \partial_m(\partial_i(g_m)) \\
&= \partial_m\left(\partial_i(a_m) + \sum_j \left(c_{m,j}\frac{\partial_i(b_{m,j})}{b_{m,j}} + \partial_i(c_{m,j}) \log b_{m,j}\right)\right) \\
&= \partial_m\left(\partial_i(a_m) + \sum_j c_{m,j}\frac{\partial_i(b_{m,j})}{b_{m,j}}\right) + \sum_j \partial_i(c_{m,j})\frac{\partial_m(b_{m,j})}{b_{m,j}},
\end{aligned} \tag{3.2}$$

where the last line follows because $c_{m,j} \in \overline{F}$ implies that $\partial_m(\partial_i(c_{m,j})) = 0$.

For fixed j, Lemma 2.2 implies that $\partial_i(c_{m,j}) = 0$ for all $i = 1, \dots, m-1$. This means that each $c_{m,j}$ is a constant in k since k is algebraically closed. Let $\widetilde{\omega} = \omega - dg_m$. Then $\widetilde{\omega} = \tilde{f}_1\, dx_1 + \cdots + \tilde{f}_{m-1}\, dx_{m-1}$, where for each $i = 1, \dots, m-1$,

$$\tilde{f}_i = f_i - \partial_i(g_m) = f_i - \partial_i(a_m) - \sum_j c_{m,j}\frac{\partial_i(b_{m,j})}{b_{m,j}} \in K.$$

Since ω is closed, so is $\widetilde{\omega}$, which implies that

$$0 = d\widetilde{\omega} = \sum_{i=1}^{m-1} d\tilde{f}_i \wedge dx_i = \sum_{i=1}^{m-1}\sum_{j=1}^{m-1} \partial_j(\tilde{f}_i)\, dx_j dx_i + \sum_{i=1}^{m-1} \partial_m(\tilde{f}_i)\, dx_m dx_i.$$

Thus $\partial_m(\tilde{f}_i) = 0$ for $i = 1, \ldots, m-1$. So $\widetilde{\omega}$ is a differential form that is free of dx_m and has coefficients in $F = k(x_1, \ldots, x_{m-1})$. Repeating this process recursively, we obtain the following result.

Theorem 3.1. *Let $\omega = f_1\, dx_1 + \cdots + f_m\, dx_m$ be a closed 1-form with $f_i \in K$. Then $\omega = dg$ for some g of the form*

$$g = a + \sum_j c_j \log b_j,$$

where $a \in K$, $c_j \in k$, and $b_j \in F[x_m]$ are pairwise coprime polynomials of positive degree.

Remark 3.2. If the field k is not algebraically closed, then the above constants c_j are in $\bar{k}$ and $b_j \in F(c_j)[x_m]$. Note also that a version of Theorem 3.1 appears as Theorem 8 in [22]. In [33, (1.5)], Griffiths states a version of Theorem 3.1 and provides a proof in the case of two variables.

Ostrogradsky [41] and Hermite [34] introduced a reduction method, now called *Hermite reduction*, that decomposes $f \in k(x)$ as

$$f = \frac{dg}{dx} + \frac{a}{b},$$

where $g \in k(x), a, b \in k[x]$ are such that $\gcd(a, b) = 1$, $\deg(a) < \deg(b)$ and b is squarefree. Moreover, f is integrable in $k(x)$ if and only if $a = 0$. Hermite reduction separates the rational part of the integral of a rational function from the logarithmic part that is transcendental over $k(x)$. This reduction has been extended to algebraic functions [49, 20], hyperexponential functions [6], and D-finite functions [7, 50, 16].

Based on the proof of Theorem 3.1, we now extend Hermite reduction from rational functions to closed rational 1-forms. We begin with a statement of the classical Hermite reduction algorithm.

Algorithm `HermiteRat`: Given a field k and $A, D \in k[x]$ with D nonzero and coprime with A, return $g, h \in k(x)$ such that

$$\frac{A}{D} = \frac{dg}{dx} + h,$$

where h has a squarefree denominator of degree greater than its numerator. Readers can refer to [34] and Section 2.2 in [10] for more details.

We summarize the generalized Hermite reduction algorithm as follows.

Algorithm `HermiteOneForm`: Given a closed 1-form $\omega = f_1\, dx_1 + \cdots + f_m\, dx_m$ with $f_i \in K = k(x_1, \ldots, x_m)$, return $g \in K$ and a closed rational 1-form $\widetilde{\omega}$ such that

$$\omega = dg + \widetilde{\omega},$$

where $\widetilde{\omega} = \widetilde{f}_1\, dx_1 + \cdots + \widetilde{f}_m\, dx_m$ and each $\widetilde{f}_i$ has a squarefree denominator as a multivariate polynomial in $k[x_1, \ldots, x_m]$.

When $m = 1$, just apply `HermiteRat` to f_1 and multiply the result by dx_1. When $m > 1$, first apply `HermiteRat` to f_m with respect to x_m to obtain

$$f_m = \partial_m(a_m) + \bar{f}_m, \quad \text{where } a_m \in K.$$

Note that the above a_m is the same a_m that appears in (3.1). This implies that

$$\bar{f}_m = \sum_j c_{m,j} \frac{\partial_m(b_{m,j})}{b_{m,j}},$$

where $c_{m,j} \in k$ and $b_{m,j} \in F[x_m]$ are as in (3.1). Then define

$$\widetilde{r}_m := \sum_j c_{m,j} \frac{\partial_1(b_{m,j})}{b_{m,j}}\, dx_1 + \cdots + \sum_j c_{m,j} \frac{\partial_m(b_{m,j})}{b_{m,j}}\, dx_m.$$

Here $\sum_j c_{m,j}\partial_i(b_{m,j})/b_{m,j}$ has a squarefree denominator in $k[x_1, \ldots, x_m]$ by normalization, and it is easy to verify that $\widetilde{r}_m$ is a closed 1-form. Thus $\omega - da_m - \widetilde{r}_m$ is a closed 1-form and free of x_m and dx_m. By induction,

$$\omega - da_m - \widetilde{r}_m = dg_{m-1} + \widetilde{\omega}_{m-1},$$

where $g_{m-1} \in K$ and $\widetilde{\omega}_{m-1} = \widetilde{f}_{m-1,1}\, dx_1 + \cdots + \widetilde{f}_{m-1,m-1}\, dx_{m-1}$, and each $\widetilde{f}_{m-1,i}$ has a squarefree denominator. Setting $g := a_m + g_{m-1}$ and $\widetilde{\omega} := \widetilde{r}_m + \widetilde{\omega}_{m-1}$ gives the desired reduction $\omega = dg + \widetilde{\omega}$.

Remark 3.3. Let $\omega = dg + \tilde{\omega} \in \mathcal{M}$ be given from Algorithm `HermiteOneForm`. Then ω is exact if and only if $\widetilde{\omega} = 0$.

Example 3.4. Let

$$\omega = \frac{txyz - 1}{x^2yz}\, dx + \frac{txyz - 1}{xy^2z}\, dy + \frac{t^2xyz + xyz - 1}{xyz^2}\, dz.$$

If we let $g := \frac{1}{xyz}$ and

$$\widetilde{\omega} := \frac{t}{x}\, dx + \frac{t}{y}\, dy + \frac{t^2 + 1}{z}\, dz,$$

then $\omega = dg + \widetilde{\omega}$ is the result of Hermite reduction on ω.

Example 7.3 below will revisit this example from the point of view of creative telescoping.

4. Integration of Closed Rational p-Forms

In this section, we will extend Theorem 3.1 to the case of rational closed p-forms in m variables. The strategy is to reduce the problem from differential forms in m variables to forms in $m - 1$ or fewer variables and then proceed recursively. The result for p-forms differs from the case of 1-forms because the coefficients of the logarithmic terms can be more complicated. We continue to use the notation $K = F(x_m)$ with $F = k(x_1, \ldots, x_{m-1})$, where k is now any field of characteristic 0.

We first give an example to show that the factors in front of the logarithms may fail to be constants when considering rational closed p-forms with $p > 1$.

Example 4.1. Let $K = \mathbb{C}(x, y)$ and $\omega = x^{-1}y^{-1}\, dxdy$. Suppose that we can write $\omega = d(A\, dy - B\, dx)$ with

$$A = a_0 + \sum_{i>0} \lambda_i \log a_i \quad \text{and} \quad B = b_0 + \sum_{i>0} \mu_i \log b_i,$$

where $a_i, b_i \in \mathbb{C}(x,y)$ and λ_i, μ_i are constants in $\mathbb{C}$. Then $x^{-1}y^{-1} = \partial_x(A) + \partial_y(B)$. If we view $x^{-1}y^{-1}$, $\partial_x(A)$ and $\partial_y(B)$ as rational functions in y with coefficients in $\overline{\mathbb{C}(x)}$, then by arguments similar to the proof of Lemma 2.2, taking the residue at $y = 0$ of both sides of $x^{-1}y^{-1} = \partial_x(A) + \partial_y(B)$ leads to

$$\frac{1}{x} = \partial_x(\mathrm{res}_{y=0}(a_0)) + c \quad \text{for some constant } c \in \mathbb{C},$$

which is impossible by Lemma 2.2. However, if we no longer require that the factors in front of the logarithms be constants, then we can write

$$\omega = d\Big(\frac{1}{y}\log(x)\,dy\Big) = d\Big(-\frac{1}{x}\log(y)\,dx\Big).$$

Before beginning our study of general p-forms, we need some preliminary work. Any rational function $f \in \overline{F}(x_m)$, $F = k(x_1, \ldots, x_{m-1})$ can be written as

$$f = c_0 + c_1 x_m + \cdots + c_\ell x_m^\ell + \frac{p}{q}, \tag{4.1}$$

where $c_i \in \overline{F}$, $\ell \geq 0$ and $p, q \in \overline{F}[x_m]$ with $\deg_{x_m}(p) < \deg_{x_m}(q)$. While there are many choices for p and q, the polynomial part $c_0 + c_1 x_m + \cdots + c_\ell x_m^\ell$ is unique, as can be seen from the series expansion of f in descending powers of x_m. We call c_0 the *constant part* of f.

Lemma 4.2. *If $f \in \overline{F}(x_m)$ and $\partial_m(f) \in K = F(x_m)$, then $f = g + c_0$ where $g \in K$ and $c_0 \in \overline{F}$ is the constant part of f. Thus*

$$f \in K \iff c_0 \in F.$$

Proof. Let $E \subseteq \overline{F}(x_m)$ be the minimal Galois extension of K that contains f and set $d := [E : K] < \infty$. If G is the Galois group of E over K, then the trace map $\mathrm{Tr}_{E/K} \colon E \to K$ is defined by $\mathrm{Tr}_{E/K}(a) = \sum_{\sigma \in G} \sigma(a)$ for $a \in E$. According to Theorem 3.2.4 in [10], each $\sigma \in G$ commutes with ∂_i for all $i \in \{1, \ldots, m\}$.

If $\partial_m(f) = h \in K$, then taking the trace yields

$$\partial_m(\mathrm{Tr}_{E/K}(f)) = \mathrm{Tr}_{E/K}\big(\partial_m(f)\big) = \mathrm{Tr}_{E/K}(h) = d \cdot h = d \cdot \partial_m(f),$$

which implies that $\partial_m\big(f - 1/d \cdot \mathrm{Tr}_{E/K}(f)\big) = 0$. Since $\mathrm{Const}_{\partial_m}(E) \subseteq \overline{F}$, f can be written as

$$f = \frac{1}{d}\,\mathrm{Tr}_{E/K}(f) + c, \text{ for some } c \in \overline{F}.$$

Note that $\tilde{g} := 1/d \cdot \mathrm{Tr}_{E/K}(f)$ is in K. Thus $f = \tilde{g} + c$.

To bring c_0 into the picture, note that writing f as in (4.1) gives $\tilde{g} = f - c = c_0 - c + c_1 x_m + \cdots + c_\ell x_m^\ell + \frac{p}{q}$, which shows that $c_0 - c$ is the constant part of $\tilde{g}$. Then $\tilde{g} \in K$ implies $c_0 - c \in F$, so that $g := \tilde{g} + c - c_0 \in K$ satisfies

$$g + c_0 = \tilde{g} + c = f.$$

The final assertion of the lemma follows immediately, and the proof is complete. $\square$

Now let ω be a closed p-form with coefficients in $K = k(x_1, \ldots, x_m) = F(x_m)$. To explore its integration, write $\omega = \omega_{m-1} + \omega^m$ and $d = d_{m-1} + d^m$ as in Section 2. Following the proof of the Poincaré Lemma in [52], we write

$$\omega^m = dx_m(A_1\,dx_{I_1} + \cdots + A_n\,dx_{I_n}),$$

where $A_i \in K$ and the differentials dx_{I_i} are free of dx_m. By the classical integration formula in x_m that we have already used several times, we have

$$A_i = \partial_m(B_i) \quad \text{with } B_i = b_{i,0} + \sum_{j>0} c_{i,j} \log b_{i,j},$$

where $b_{i,0} \in K$, $c_{i,j} \in \overline{F}$, and $b_{i,j} \in F(c_{i,j})[x_m]$ are monic and pairwise coprime polynomials of positive degree when $j > 0$. Then the p-form $\Psi^m := B_1\, dx_{I_1} + \cdots + B_n\, dx_{I_n}$ satisfies $\omega^m = d^m(\Psi^m)$, and we can define

$$\widetilde{\omega} := \omega - d\,\Psi^m = (\omega_{m-1} + \omega^m) - (d_{m-1} + d^m)(\Psi^m) = \omega_{m-1} - d_{m-1}(\Psi^m).$$

Since $d\omega = 0$, we have $d\widetilde{\omega} = 0$. Note that $\widetilde{\omega}$ is free of dx_m. By Lemma 2.4, we have $d_{m-1}(\widetilde{\omega}) = 0$ and $d^m(\widetilde{\omega}) = 0$, which implies that $\widetilde{\omega}$ is free of x_m, i.e., the coefficients of $\widetilde{\omega}$ are constants with respect to x_m.

As defined, $\widetilde{\omega}$ appears to involve logarithms and elements of $\overline{F}$ since these appear in Ψ^m. We claim that the coefficients of $\widetilde{\omega}$ actually lie in $k(x_1, \ldots, x_{m-1})$. This is proved as follows. For $\ell = 1, \ldots, m-1$ and $i = 1, \ldots, n$, we have

$$\begin{aligned}\partial_\ell(B_i) &= \partial_\ell\big(b_{i,0} + \sum_{j>0} c_{i,j} \log b_{i,j}\big)\\ &= \partial_\ell(b_{i,0}) + \sum_{j>0} c_{i,j} \frac{\partial_\ell(b_{i,j})}{b_{i,j}} + \sum_{j>0} \partial_\ell(c_{i,j}) \log b_{i,j}.\end{aligned}$$

Thus

$$\begin{aligned}\widetilde{\omega} &= \omega_{m-1} - d_{m-1}(\Psi^m) = \omega_{m-1} - d_{m-1}\bigg(\sum_{i=1}^{n} B_i\, dx_{I_i}\bigg)\\ &= \omega_{m-1} - \sum_{\ell=1}^{m-1} \sum_{i=1}^{n} \partial_\ell(B_i)\, dx_\ell dx_{I_i}\\ &= \omega_{m-1} - \sum_{\ell=1}^{m-1} \sum_{i=1}^{n} \bigg(\partial_\ell(b_{i,0}) + \sum_{j>0} c_{i,j} \frac{\partial_\ell(b_{i,j})}{b_{i,j}} + \sum_{j>0} \partial_\ell(c_{i,j}) \log b_{i,j}\bigg)\, dx_\ell dx_{I_i}\\ &= \widetilde{\omega}_{m-1} - \sum_{\ell=1}^{m-1} \sum_{i=1}^{n} \bigg(\sum_{j>0} c_{i,j} \frac{\partial_\ell(b_{i,j})}{b_{i,j}} + \sum_{j>0} \partial_\ell(c_{i,j}) \log b_{i,j}\bigg)\, dx_\ell dx_{I_i},\end{aligned}$$

where on the last line, $\widetilde{\omega}_{m-1} := \omega_{m-1} - \sum_{\ell=1}^{m-1} \sum_{i=1}^{n} \partial_\ell(b_{i,0})\, dx_\ell dx_{I_i}$ has coefficients in K. It follows that we can write $\widetilde{\omega}$ as

$$\widetilde{\omega} = f_{\tilde{I}_1}\, dx_{\tilde{I}_1} + f_{\tilde{I}_2}\, dx_{\tilde{I}_2} + \cdots,$$

where each $dx_{\tilde{I}_i}$ is free of dx_m and $\partial_m(f_{\tilde{I}_i}) = 0$ since $d^m(\widetilde{\omega}) = 0$. Observe that each $f_{\tilde{I}_i}$ is an element in K minus the sum

$$\sum_{\substack{i,\ell\\ dx_\ell \wedge dx_{I_i} = \varepsilon_{i,\ell} dx_{\tilde{I}_i}}} \varepsilon_{i,\ell}\Big(\sum_{j>0} c_{i,j} \frac{\partial_\ell(b_{i,j})}{b_{i,j}} + \sum_{j>0} \partial_\ell(c_{i,j}) \log b_{i,j}\Big),$$

where $\varepsilon_{i,\ell} = \pm 1$. By Lemma 2.2, the logarithmic part of $f_{\tilde{I}_j}$ vanishes. Then $\partial_{x_m}(f_{\tilde{I}_i}) = 0$ implies that

$$f := \sum_{\substack{i,\ell \\ dx_\ell \wedge dx_{I_i} = \varepsilon_{i,\ell} dx_{\tilde{I}_i}}} \varepsilon_{i,\ell} \sum_{j>0} c_{i,j} \frac{\partial_\ell(b_{i,j})}{b_{i,j}}$$

satisfies the hypothesis of Lemma 4.2. We claim that this sum lies in K. Note that $\deg_{x_m}(\partial_{x_\ell}(b_{i,j})) < \deg_{x_m}(b_{i,j})$ since b_{ij} is monic of positive degree in x_m and $\ell < m$. This makes it easy to see that the constant part of f is $0 \in F$. By Lemma 4.2, it follows that f and hence $f_{\tilde{I}_j}$ lie in K. Thus $\widetilde{\omega} = \omega - d(\Psi^m)$ is free of x_m and dx_m and has coefficients in $k(x_1, \ldots, x_{m-1})$. Furthermore, since $d = d_m$ in the m-variable case, we see that $\omega = \widetilde{\omega} + d_m(\Psi^m)$.

If we repeat the above process on the closed rational form $\widetilde{\omega}$ in $x_1, \ldots, x_{m-1}$, we obtain $\widetilde{\widetilde{\omega}}$ and Ψ^{m-1}, where $\widetilde{\widetilde{\omega}}$ is a closed rational form in $x_1, \ldots, x_{m-2}$ and Ψ^{m-1} is similar to Ψ^m except that x_m no longer appears. The result is that $\widetilde{\omega} = \widetilde{\widetilde{\omega}} + d_{m-1}(\Psi^{m-1})$, where d_{m-1} is defined in Section 2. Thus

$$\omega = \widetilde{\widetilde{\omega}} + d_{m-1}(\Psi^{m-1}) + d_m(\Psi^m)$$

Repeating this process recursively, we must stop when only p variables remain since there are no p-forms in fewer than p variables. Hence we have proved the following result.

Theorem 4.3. *Let ω be a closed p-form with coefficients in $K = k(x_1, \ldots, x_m)$. Then ω can be written in the form*

$$\omega = d_p(\Psi^p) + \cdots + d_m(\Psi^m) = d(\Psi^p + \cdots + \Psi^m),$$

where for each $i = p, \ldots, m$, the coefficients $f_{i,k}$ of Ψ^i are of the form

$$f_{i,k} = b_{i,k,0} + \sum_{j>0} c_{i,k,j} \log b_{i,k,j}$$

with

$$\begin{aligned} b_{i,k,0} &\in k(x_1, \ldots, x_i) \\ c_{i,k,j} &\in \overline{k(x_1, \ldots, x_{i-1})} \\ b_{i,k,j} &\in k(x_1, \ldots, x_{i-1})(c_{i,k,j})[x_i], \ j > 0. \end{aligned}$$

Remark 4.4. The above theorem says that Theorem 3.1 can be extended to general p-forms provided that the coefficients of the logarithms are allowed to be more general (specifically, algebraic over $k(x_1, \ldots, x_{i-1})$ for some $i \in \{p, \ldots, m\}$).

Example 4.5. Let us apply Theorem 4.3 to the 2-form $\omega = x^{-1}y^{-1}\, dxdy$ from Example 4.1. In the notation of the theorem with x, y in place of x_1, x_2, we have

$$\omega^2 = \omega = x^{-1}y^{-1}\, dxdy = dy(A_1\, dx) \text{ with } A_1 = -x^{-1}y^{-1}.$$

Furthermore,

$$A_1 = \partial_y(B_1) \text{ with } B_1 = -x^{-1}\log(y),$$

and the above procedure gives

$$\omega = d_2(\Psi^2) = d(B_1\, dx) = d\Big(-\frac{1}{x}\log(y)\, dx\Big),$$

exactly as in Example 4.1. Switching x and y gives the other representation of ω in the example.

Example 4.6. Consider the closed 2-form

$$\omega = \big(\frac{1}{z^2-x} + \frac{1}{xy}\big)\, dxdy + \frac{1}{z^2-x}\, dydz + \frac{y-2yz}{(z^2-x)^2}\, dxdz.$$

In the notation of Theorem 4.3 with x, y, z in place of x_1, x_2, x_3, we have $\omega = \big(1/(z^2-x) + 1/(xy)\big)\, dxdy + dz(A_1\, dx + A_2\, dy)$ with

$$A_1 := -\frac{y-2yz}{(z^2-x)^2} \text{ and } A_2 := -\frac{1}{z^2-x}.$$

Furthermore, $A_1 = \partial_z(B_1)$ with

$$B_1 := \frac{\sqrt{x}y}{4x^2} \log \frac{z-\sqrt{x}}{z+\sqrt{x}} + \frac{2yz-4xy}{4x(z^2-x)},$$

and $A_2 = \partial_z(B_2)$ with

$$B_2 := -\frac{\sqrt{x}}{2x} \log \frac{z-\sqrt{x}}{z+\sqrt{x}}.$$

Hence, $\Psi^3 := B_1\, dx + B_2\, dy$ and

$$\widetilde{\omega} := \omega - d(\Psi^3) = \left(\frac{1}{z^2-x} + \frac{1}{xy} + \partial_y(B_1) - \partial_x(B_2)\right) dxdy = \frac{1}{xy}\, dxdy.$$

Example 4.5 shows that $\widetilde{\omega} = d(\Psi^2)$ with $\Psi^2 := -\frac{1}{x}\log(y)\, dx$, so $\omega = \widetilde{\omega} + d(\Psi^3)$ is given by

$$\begin{aligned}\omega &= d(\Psi^2) + d(\Psi^3) = d(\Psi^2 + \Psi^3)\\ &= d\Big(\big(-\frac{1}{x}\log(y) + \frac{\sqrt{x}y}{4x^2} \log \frac{z-\sqrt{x}}{z+\sqrt{x}} + \frac{2yz-4xy}{4x(z^2-x)}\big)\, dx - \frac{\sqrt{x}}{2x} \log \frac{z-\sqrt{x}}{z+\sqrt{x}}\, dy\Big).\end{aligned}$$

Notice that the coefficients of the logs in Ψ^3 involve x and y, while the coefficient of the log in Ψ^2 involves only x, exactly as described in Theorem 4.3.

5. Picard's Problem and Griffiths-Dwork Reduction

Let k be a field of characteristic zero and $k(x, y, z)$ be the field of rational functions in x, y and z over k. Let $\partial_x, \partial_y, \partial_z$ denote the usual derivations on $k(x, y, z)$ with respect to x, y, z, respectively. In his books [43, Vol. II, pp. 475–479] and [42, pp. 209–215], Picard posed the following problem.

Picard's Problem. *Given $f \in k(x, y, z)$, decide whether there exist $u, v, w \in k(x, y, z)$ such that*

$$f = \partial_x(u) + \partial_y(v) + \partial_z(w). \tag{5.1}$$

In this section, we will study an explicit example and reformulate Picard's problem in terms of differential forms. In the case of m variables, we will then explain how the Griffiths-Dwork order of pole reduction relates to Picard's problem. Note that the discrete analogue of Picard's problem has been recently solved in [15].

Set $K = k(x, y)$ and let $\overline{K}$ be the algebraic closure of K. Any rational function $f = P/Q \in K(z)$ can be decomposed into

$$f = p + \sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{\alpha_{i,j}}{(z-\beta_i)^j},$$

where $p \in K[z]$, the β_i's are the roots in $\overline{K}$ of the denominator Q, and $\alpha_{i,j} \in \overline{K}$. We call the element $\alpha_{i,1}$ the *z-residue* of f at β_i.

A rational function $f \in k(x,y,z)$ is said to be *rationally integrable* with respect to x,y,z if $f = \partial_x(u) + \partial_y(v) + \partial_z(w)$ for some $u,v,w \in k(x,y,z)$. An algebraic function $\alpha \in \overline{k(x,y)}$ is said to be *algebraically integrable* with respect to x,y if $\alpha = \partial_x(\beta) + \partial_y(\gamma)$ for some $\beta,\gamma \in \overline{k(x,y)}$. We recall a lemma from [21] on the relation between those two notions of being integrable.

Lemma 5.1. *Let $f \in k(x,y,z)$. Then f is rationally integrable with respect to x,y,z if and only if all z-residues of f are algebraically integrable with respect to x,y.*

We now show that some rational functions are not rationally integrable.

Example 5.2. For a fixed $n \in \mathbb{N}$, consider the rational function

$$f = \frac{1}{x^n + y^n + z^n}.$$

We will prove that f is rationally integrable with respect to x,y,z if and only if $n \neq 3$. It is straightforward to check that when $n \neq 3$ we have

$$f = \partial_x\left(\frac{(3-n)^{-1}x}{x^n+y^n+z^n}\right) + \partial_y\left(\frac{(3-n)^{-1}y}{x^n+y^n+z^n}\right) + \partial_z\left(\frac{(3-n)^{-1}z}{x^n+y^n+z^n}\right).$$

So it remains to show that f is not rationally integrable with respect to x,y,z when $n=3$. A root (in $\overline{k(x,y)}$) of the denominator of $f = 1/(x^3+y^3+z^3)$ is $\lambda(x^3+y^3)^{1/3}$ with $\lambda^3 = -1$. Then the residue of f at this root is

$$\frac{1}{3\lambda^2(x^3+y^3)^{2/3}}.$$

By Lemma 5.1, it suffices to show that the algebraic function

$$\alpha(x,y) = \frac{1}{(x^3+y^3)^{2/3}}$$

is not algebraically integrable. Consider the birational transformation

$$\bar{x} = x \quad \text{and} \quad \bar{y} = y/x.$$

Then one computes that

$$\alpha(x,y)\,dxdy = \bar{\alpha}(\bar{x},\bar{y})\,d\bar{x}d\bar{y}, \quad \text{where } \bar{\alpha}(\bar{x},\bar{y}) = \frac{1}{\bar{x}(1+\bar{y}^3)^{2/3}}. \tag{5.2}$$

Note that $\bar{\alpha}$ is a rational function in $\bar{x}$. By a similar result to Lemma 5.1, $\bar{\alpha}$ is algebraically integrable with respect to $\bar{x},\bar{y}$ if and only if the residue of $\bar{\alpha}$ at $\bar{x}=0$ is algebraically integrable with respect to $\bar{y}$. The residue of f at $\bar{x}=0$ is

$$\beta(\bar{y}) = \frac{1}{(1+\bar{y}^3)^{2/3}}.$$

This is a simple radical and one can show that it is not algebraically integrable with respect to $\bar{y}$. In fact, if $\beta(\bar{y})$ is algebraically integrable, then

$$\beta(\bar{y}) = D_{\bar{y}}(\gamma(\bar{y})),$$

where $\gamma(\bar{y})$ is algebraic over $k(\bar{y})$ and $D_{\bar{y}} = d/d\bar{y}$, By [10, Lemma 5.1.1], it follows that $D_{\bar{y}}(\gamma(\bar{y}))$ is the derivative of an element of $k(\bar{y},\beta(\bar{y}))$. Using $D_{\bar{y}}(\beta) = \frac{-2\bar{y}^2}{1+\bar{y}^3}\beta$

and the basis $\{1, \beta(\bar{y}), \beta(\bar{y})^2\}$ of $k(\bar{y}, \beta(\bar{y}))$ over $k(\bar{y})$, one can show that $\gamma(\bar{y}) = r\beta(\bar{y}) + c$, where $r \in k(\bar{y})$ and $c \in k$. Thus

$$\beta(\bar{y}) = D_{\bar{y}}\left(\frac{r}{(1+\bar{y}^3)^{2/3}}\right),$$

where $r \in k(\bar{y})$. It follows that

$$D_{\bar{y}}(r) - \frac{2\bar{y}^2}{1+\bar{y}^3} \cdot r = 1. \tag{5.3}$$

A straightforward application of the Risch Algorithm from [10, Ch. 6] to (5.3) shows that the rational solution r is in fact a polynomial of degree at most 2 (see Def. 6.11, Thm. 6.1.2, and Cor. 6.3.1 of [10]). An easy computation now leads to a contradiction, so that (5.3) has no rational solution. As we have seen, this implies that $\bar{\alpha}$ is not algebraically integrable.

We claim that the same is true for α. Here is a classical argument using Picard and Simart [43] (see Remark 5.3 below for a modern proof). Suppose that $\alpha = \partial_x(\beta) + \partial_y(\gamma)$ for $\beta, \gamma \in \overline{k(x,y)}$. In terms of integrals, this means that

$$\iint \alpha \, dxdy = \iint \partial_x(\beta) + \partial_y(\gamma) \, dxdy.$$

By [43, Vol. II, pp. 160–161], integrals such as the one on the right are preserved under birational transformations when β and γ are rational functions. The proof adapts easily to the case of algebraic functions, so that for the birational transformation given above, we have

$$\iint \partial_x(\beta) + \partial_y(\gamma) \, dxdy = \iint \partial_{\bar{x}}(\bar{\beta}) + \partial_{\bar{y}}(\bar{\gamma}) \, d\bar{x}d\bar{y}$$

for algebraic functions $\bar{\beta}$ and $\bar{\gamma}$. Combining this with (5.2), we conclude that $\bar{\alpha}$ is algebraically integrable, which contradicts what we have proved above.

It follows that α is not algebraically integrable. As noted above, Lemma 5.1 then implies that $f = \frac{1}{x^3+y^3+z^3}$ is not rationally integrable.

The 2-form $(\partial_x(\beta) + \partial_y(\gamma))\, dxdy$ appearing in Example 5.2 suggests that the 3-form $(\partial_x(u) + \partial_y(v) + \partial_z(w))\, dxdydz$ may be relevant to Picard's problem. In fact,

$$d(u\, dydz - v\, dxdz + w\, dxdy) = (\partial_x(u) + \partial_y(v) + \partial_z(w))\, dxdydz,$$

which implies that Picard's problem (5.1) is equivalent to saying that the 3-form $f(x,y,z)\, dxdydz$ is exact, i.e.,

$$f\, dxdydz = d(u\, dydz - v\, dxdz + w\, dxdy), \quad \text{where } u, v, w \in k(x,y,z).$$

Remark 5.3. Similar formulas also apply to algebraic functions in two variables, so that $\alpha(x,y)$ from Example 5.2 is algebraically integrable if and only if the 2-form $\alpha(x,y)\, dxdy$ is the exterior derivative of an algebraic 1-form. Since the exterior derivative is compatible with pull-backs, we get an immediate proof that in (5.2), $\alpha(x,y)$ is not algebraically integrable if and only if the same is true for $\bar{\alpha}(\bar{x}, \bar{y})$.

We can also relate Example 5.2 to Theorem 4.3 as follows.

Example 5.4. By the previous example, $f = 1/(x^3 + y^3 + z^3)$ is not rationally integrable with respect to x, y, z, so that $f\, dxdydz$ is not the exterior derivative of a rational 2-form. However, Theorem 4.3 shows how to write $f\,dxdydz$ as the exterior derivative of a rational and logarithmic 2-form as follows.

Let $\alpha = -(x^3+y^3)^{1/3}$. Then the three roots in $\overline{\mathbb{Q}(x,y)}$ of the equation $z^3+x^3+y^3=0$ are $z=\alpha,\alpha\omega,\alpha\omega^2$, where ω is a primitive cube root of unity. Then

$$f = \frac{1/(3\alpha^2)}{z-\alpha} + \frac{\omega/(3\alpha^2)}{z-\alpha\omega} + \frac{\omega^2/(3\alpha^2)}{z-\alpha\omega},$$

which implies that $f=\partial_z(g)$ with

$$g = \frac{1}{3\alpha^2}\log(z-\alpha) + \frac{\omega}{3\alpha^2}\log(z-\alpha\omega) + \frac{\omega^2}{3\alpha^2}\log(z-\alpha\omega^2).$$

If follows easily that $f\,dxdydz = d(g\,dxdy)$.

Conjecture 5.5. For a fixed $n\in\mathbb{N}$, the rational function

$$f = \frac{1}{x_1^n+\cdots+x_m^n}$$

is rationally integrable with respect to $x_1,\ldots,x_m$ with $m\geq 4$ if and only if $n\neq m$.

The general case of Picard's problem asks when $f\in K = k(x_1,\ldots,x_m)$ can be written in the form

$$f = \partial_{x_1}(u_1)+\cdots+\partial_{x_m}(u_m), \quad \text{where } u_1,\ldots,u_m\in K.$$

Since

$$d\Big(\sum_{i=0}^m(-1)^{i-1}u_i\,dx_1\cdots\widehat{dx_i}\cdots dx_m\Big) = \Big(\sum_{i=1}^m\partial_i(u_i)\Big)\,dx_1\cdots dx_m,$$

Picard's problem is thus equivalent to deciding when the m-form

$$\omega = f\,dx_1\cdots dx_m, \quad \text{where } f\in K,$$

is rationally integrable, i.e., is the exterior derivative of a rational $(m-1)$-form.

Under a regularity assumption, the Griffiths-Dwork method [27, §3], [28, §8], [32, §4] solves Picard's problem for a general m-variable case. Here we sketch a solution based on the modern presentation [8]. One difference is that we make explicit use of differential forms, which are suppressed in [8].

The regularity assumption is that the polar locus of ω is a smooth hypersurface $V\subseteq\mathbb{P}^m$. Let ℓ be the order of the pole of ω along V, and write the defining equation of V as $Q=0$, where Q is irreducible in the homogeneous coordinates $\xi_0,\ldots,\xi_m$ of $\mathbb{P}^m$. Here $x_i=\xi_i/\xi_0$ for $i=1,\ldots,m$. By [32, Cor. 2.11], this implies that in homogeneous coordinates, ω can be written as

$$\omega = \frac{P\Omega}{Q^\ell}, \tag{5.4}$$

where Q and ℓ are as above, P is homogeneous with $\ell\deg(Q)=\deg(P)+m+1$, and Ω is the m-form

$$\Omega := \sum_{i=0}^m(-1)^i\xi_i(d\xi_0\cdots\widehat{d\xi_i}\cdots d\xi_m).$$

In [32], Griffiths studies ω using *order of pole reduction*. A key tool is the *Jacobian ideal*

$$J(Q) := \langle\partial_0(Q),\ldots,\partial_m(Q)\rangle\subseteq k[\xi_0,\ldots,\xi_m],$$

where $\partial_i = \partial_{\xi_i}$ for $i = 0, \dots, m$. Our regularity assumption implies that the quotient

$$A(Q) := k[\xi_0, \dots, \xi_m]/J(Q)$$

is a finite-dimensional graded algebra over k.

Now suppose that we have $\omega = \frac{P\Omega}{Q^\ell}$ as in (5.4) with $\ell \deg(Q) = \deg(P) + m + 1$. If $\ell > 1$, consider the $(m-1)$-form

$$\varphi_1 = \frac{1}{\ell - 1} \sum_{0 \le i < j \le m} (\xi_i A_j - \xi_j A_i)\, d\xi_0 \cdots \widehat{d\xi_i} \cdots \widehat{d\xi_j} \cdots d\xi_m$$

for homogeneous polynomials $A_0, \dots, A_m$ with $\deg(A_i) + m = (\ell - 1)\deg(Q)$. Griffiths [32, §2 and §4] shows that φ_1 represents an $(m-1)$ form on $\mathbb{P}^m$ with exterior derivative

$$d\varphi_1 = \frac{\big(\sum_{i=0}^m A_i \partial_i(Q)\big)\Omega}{Q^\ell} - \frac{\frac{1}{\ell-1}\big(\sum_{i=0}^m \partial_i(A_i)\big)\Omega}{Q^{\ell-1}}.$$

Note that $\sum_{i=0}^m A_i \partial_i(Q) \in J(Q)$. The rough idea is that subtracting $d\varphi_1$ from $\omega = \frac{P\Omega}{Q^\ell}$ for carefully chosen A_i can reduce the order of the pole in part of ω. To make this precise, let G be a Gröbner basis of $J(Q)$ and divide P by G to obtain

$$P = \sum_{i=0}^m A_i \partial_i(Q) + r_1,$$

where r_1 is the remainder (which can be identified with an element of $A(Q)$). Combining this with the above formula for $d\varphi_1$ gives

$$\omega = \frac{P\Omega}{Q^\ell} = \frac{\big(\sum_{i=0}^m A_i \partial_i(Q)\big)\Omega}{Q^\ell} + \frac{r_1\Omega}{Q^\ell} = d\varphi_1 + \underbrace{\frac{\big(\overbrace{\tfrac{1}{\ell-1}\textstyle\sum_{i=0}^m \partial_i(A_i)}^{P_1}\big)\Omega}{Q^{\ell-1}}}_{\omega_1} + \frac{r_1\Omega}{Q^\ell}.$$

Applying this recursively to $\omega_1 = \frac{P_1\Omega}{Q^{\ell-1}}$ gives $\omega_1 = d\varphi_2 + \omega_2 + \frac{r_2\Omega}{Q^{\ell-1}}$ with $\omega_2 = \frac{P_2\Omega}{Q^{\ell-2}}$. We can continue until we get $\omega_{\ell-1} = \frac{P_{\ell-1}\Omega}{Q}$ with $\deg(Q) = \deg(P_{\ell-1}) + m + 1$. Then $\deg(P_{\ell-1}) < \deg(Q) - 1 = \deg(\partial_i(Q))$. It follows that $P_{\ell-1}$ is already a remainder r_ℓ modulo $J(Q)$, so that $\omega_{\ell-1} = \frac{r_\ell\Omega}{Q}$ and $d\varphi_\ell = \omega_\ell = 0$. Thus we obtain

$$\omega = d\varphi_1 + \cdots + d\varphi_{\ell-1} + [\omega], \quad \text{where } [\omega] = \frac{r_1\Omega}{Q^\ell} + \frac{r_2\Omega}{Q^{\ell-1}} + \cdots + \frac{r_\ell\Omega}{Q}.$$

We call $[\omega]$ the *Griffiths-Dwork reduction* of ω.

This reduction algorithm is Algorithm 1 from [8] recast in the language of differential forms. The results of [32, §4] lead to Theorem 1 of [8], which we restate as follows.

Proposition 5.6. *Assume that $\omega = \frac{P\Omega}{Q^\ell}$ satisfies the regularity condition that $Q = 0$ defines a smooth hypersurface in $\mathbb{P}^m$. Then ω is rationally integrable if and only if its Griffiths-Dwork reduction $[\omega]$ vanishes.*

Since the summands in $[\omega] = \frac{r_1\Omega}{Q^\ell} + \frac{r_2\Omega}{Q^{\ell-1}} + \cdots + \frac{r_\ell\Omega}{Q}$ are easily seen to be linearly independent, it follows that ω is not rationally integrable as soon as one of the remainders $r_1, r_2, \dots$ is nonzero. So if one is only interested in knowing whether ω is rationally integrable, it is often not necessary to compute the full Griffiths-Dwork reduction.

Example 5.7. Consider the 3-form

$$\omega = \frac{dxdydz}{x^3 + y^3 + z^3}$$

from Example 5.4. Setting $(x, y, z) = (\xi_1/\xi_0, \xi_2/\xi_0, \xi_3/\xi_0)$, one computes that $dxdydz = \frac{\Omega}{\xi_0^4}$, so that

$$\omega = \frac{\Omega}{\xi_0(\xi_1^3 + \xi_2^3 + \xi_3^3)}.$$

The polar locus of this form is not smooth and irreducible, so the regularity assumption is violated. Hence we cannot use Proposition 5.6 to decide if ω is exact.

In [37], Lairez uses results of Dimca [26] to solve Picard's problem without assuming regularity. In this setup, we instead assume that Q is squarefree. Then any m-form with polar locus V defined by $Q = 0$ can be written

$$\omega = \frac{P\Omega}{Q^\ell}$$

for some ℓ. Lairez [37, Sec. 4.2] defines a series of reductions $[\omega]_r$ for $r \geq 1$, where $[\omega]_1$ closely related to the Griffiths-Dwork reduction defined above (see [37, Rem. 17]). The challenges involved in computing $[\omega]_r$ are discussed in [37, Sec. 4.2 and 4.3].

Using these reductions, we can solve Picard's problem as follows.

Proposition 5.8.

(i) *ω is rationally integrable if and only if $[\omega]_r = 0$ for some $r \geq 1$.*
(ii) *There is an integer r_Q depending only on Q such that ω is rationally integrable if and only if $[\omega]_{r_Q} = 0$.*

Proof. Part (i) follows from [37, Cor. 16] after unwinding a lot of notation, and part (ii) then follows from [37, Cor. 24]. □

In order to use Proposition 5.8, we need to find an effective bound for r_Q. One known bound comes from the proof of Theorem B of [26] and is computed using an embedded resolution of singularities of the polar locus V of ω. A conjecture made in [26] would imply that $r_Q \leq m + 1$ (see [37, Conj. 27]), but this is still an open question.

The upshot is that although the methods of [37] apply to examples like those in Example 5.7, a substantial amount of work would be involved. The Griffiths–Dwork reduction provides a partial resolution to the question of the rational integrability of m-forms. It also offers a novel approach to handling multiple integrals through creative telescoping, see [8]. Nevertheless, the problem of determining the rational integrability of general p-forms remains open.

6. Liouville's Theorem for Differential 1-forms

We will extend Liouville's classical theorem to integrals of differential 1-forms in $\mathcal{M}$. Let $K \subseteq \mathcal{U}$ be the universal differential extension from Section 2.

Definition 6.1. Let $E \subseteq \mathcal{U}$ be a differential extension of K and $t \in E$.

(1) t is a *logarithm* over K if there exists nonzero $b \in K$ such that $\partial_i t = \partial_i b/b$ for each $i \in \{1, \ldots, m\}$.

(2) t is an *exponential* over K if there exists $b \in K$ such that $\partial_i t/t = \partial_i b$ for each $i \in \{1, \ldots, m\}$.
(3) E is an *elementary extension* of K if there exist $t_1, \ldots, t_n \in E$ such that $E = K(t_1, \ldots, t_n)$ and for each $j \in \{1, \ldots, n\}$, t_j is either algebraic or a logarithm or an exponential over $K(t_1, \ldots, t_{j-1})$.

The following theorem is Risch's strong version of the classical Liouville's theorem (see [47, 46] and Section 5.5 in [10] for its proof).

Theorem 6.2 (Liouville's theorem)**.** *Let (K, D) be a differential field and $f \in K$. If there exists an elementary extension E of K and $g \in E$ such that $D(g) = f$, then there are $v \in K, u_1, \ldots, u_n \in K^*$ and $c_1, \ldots, c_n \in \overline{\mathrm{Const}_D(K)}$ such that*

$$f = D(v) + \sum_{i=1}^{n} c_i \frac{D(u_i)}{u_i}.$$

The following theorem was first proved by Caviness and Rothstein in [14]. We now present a more direct proof inspired by the recursive argument used in the proof of Theorem 3.1.

Theorem 6.3. *Let $F \subseteq \mathcal{U}$ be a differential extension K and $C := \mathrm{Const}_{\partial_1,\ldots,\partial_m}(F)$. Let $\omega = f_1\, dx_1 + \cdots + f_m\, dx_m \in \mathcal{M}$ be a closed 1-form with $f_i \in F$. If there exist an elementary extension E of F and $g \in E$ such that $\omega = dg$, then there are $a \in F$, $c_1, \ldots, c_n \in \overline{C}$, and $b_1, \ldots, b_n \in F(c_1, \ldots, c_n)^*$ such that*

$$g = a + \sum_j c_j \log b_j.$$

Proof. Denote $C_m := \mathrm{Const}_{\partial_m}(F)$. Since f_m has an elementary integral over F with respect to ∂_m, Theorem 6.2 implies that there exist $a_m \in F$, $c_{m,1}, \ldots, c_{m,n} \in \overline{C_m}$, $b_{m,1}, \ldots, b_{m,n} \in F(c_{m,1}, \ldots, c_{m,n})$ such that

$$f_m = \partial_m(g_m) \quad \text{with} \quad g_m = a_m + \sum_{j=1}^{n} c_{m,j} \log b_{m,j}$$

where $\partial_m(b_{m,j}) \neq 0$ for each j. Note that $F' := \{\partial_m(f) \mid f \in F\}$ is a linear subspace of F over C_m. By Corollary 3.3.1 from [10], we have $\mathrm{Const}_{\partial_m}(\overline{F}) = \overline{C_m}$, so $\overline{F}' := \{\partial_m(f) \mid f \in \overline{F}\}$ is a linear subspace of $\overline{F}$ over $\overline{C_m}$. For every $f \in \overline{F}$, let $\widehat{f}$ be the image of f in the quotient space $\overline{F}/\overline{F}'$. We can always select maximal many elements of the $\widehat{\frac{\partial_m(b_{m,j})}{b_{m,j}}}$'s so that they are linearly independent over $\overline{C_m}$ and then rearrange the logarithmic part of g_m. After this process, we can always assume the $\widehat{\frac{\partial_m(b_{m,j})}{b_{m,j}}}$'s appearing in g_m are linearly independent over $\overline{C_m}$, though a_m may now be in $\overline{F}$. Since ω is closed, we have $\partial_m(f_i) = \partial_i(f_m)$ for all $1 \leq i \leq m$. Then (3.1) and (3.2) imply that

$$\partial_m(f_i) = \partial_m(\partial_i(a_m)) + \sum_{j=1}^{n} \partial_i(c_{m,j}) \frac{\partial_m(b_{m,j})}{b_{m,j}} + \partial_m\Big(\sum_{j=1}^{n} c_{m,j} \frac{\partial_i(b_{m,j})}{b_{m,j}}\Big).$$

Taking the image of both sides in the quotient space $\overline{F}/\overline{F}'$ gives

$$\sum_{j=1}^{n} \partial_i(c_{m,j}) \widehat{\frac{\partial_m(b_{m,j})}{b_{m,j}}} = 0,$$

where $\partial_i(c_{m,j}) \in \overline{C_m}$ because $c_{m,j} \in \overline{C_m}$. Since the $\widehat{\frac{\partial_m(b_{m,j})}{b_{m,j}}}$'s are linearly independent over $\overline{C_m}$, $\partial_i(c_{m,j}) = 0$ for each j, which implies that $c_{m,j} \in \overline{C_i}$. Since this holds for all $i \in \{1, \dots, m-1\}$, we see that $c_{m,j} \in \overline{C}$.

Let $\widetilde{\omega} := \omega - dg_m$. Then $\widetilde{\omega} = \widetilde{f}_1\, dx_1 + \cdots + \widetilde{f}_{m-1}\, dx_{m-1}$, where for each i,

$$\tilde{f}_i := f_i - \partial_i(g_m) = f_i - \partial_i(a_m) - \sum_{j=1}^{n} c_{m,j} \frac{\partial_i(b_{m,j})}{b_{m,j}} \in \overline{F}.$$

This shows that the coefficient field is extended only by algebraically. And the constant field is also extended algebraically, i.e., $\mathrm{Const}_{\partial_1,\dots,\partial_m}(\overline{F}) = \overline{C} \cap \overline{F} = \overline{C}$. Since ω is closed, so is $\widetilde{\omega}$, which implies that $\partial_m(\tilde{f}_i) = 0$ for $i = 1, \dots, m-1$. So $\widetilde{\omega}$ is a differential form that is free of dx_m and has coefficients in $C_m(c_{m,1}, \dots, c_{m,n})$.

Repeating this process recursively shows that there are $a \in \overline{F}$, $c_j \in \overline{C}$ and $b_j \in \overline{C}F^*$ such that

$$f_i = \partial_i(a) + \sum_j c_j \frac{\partial_i(b_j)}{b_j}$$

for all $i \in \{1, \dots, m\}$. This almost has the required form. The problem is that a need not lie in F and the b_j's need not lie in $F(\{c_j\})$. We have more work to do.

Let A be the minimal Galois extension of F generated by a, the c_j's, and the b_j's with $d := [A : F] < \infty$. As in the proof of Lemma 4.2, we have the trace map $\mathrm{Tr}_{A/F}(a) = \sum_{\sigma \in G} \sigma(a)$ for $a \in A$, where G be the Galois group of A over F. Using the trace map. we obtain

$$d \cdot f_i = \mathrm{Tr}_{A/F}(f_i) = \partial_i\big(\mathrm{Tr}_{A/F}(a)\big) + \sum_j \sum_{\sigma \in G} \sigma(c_j) \frac{\partial_i(\sigma(b_j))}{\sigma(b_j)}.$$

Thus

$$f_i = \partial_i\Big(\frac{\mathrm{Tr}_{A/F}(a)}{d}\Big) + \sum_j \sum_{\sigma \in G} \frac{\sigma(c_j)}{d} \frac{\partial_i(\sigma(b_j))}{\sigma(b_j)},$$

where

$$\frac{\mathrm{Tr}_{A/F}(a)}{d} \in F, \quad \frac{\sigma(c_j)}{d} \in \overline{C} \quad \text{and} \quad \sigma(b_j) \in \overline{C}F.$$

This is closer to what we want, but the $\sigma(b_j)$'s need not lie in $F(\{\sigma(c_j)\}_{j,\sigma})$. So there is still more work to do.

Now let L be the field extension of F generated by $\sigma(c_j)$'s. Note that $F \subseteq L \subseteq A$. For the extension $L \subseteq A$ of degree $\ell := [A : L]$, let $\mathrm{Tr}_{A/L} \colon A \to L$ and $N_{A/L} \colon A \to L$ be the trace and norm maps. Recall that for any $b \in A^*$, we have the well-known identity

$$\mathrm{Tr}_{A/L}\Big(\frac{\partial_i(b)}{b}\Big) = \frac{\partial_i(N_{A/L}(b))}{N_{A/L}(b)}.$$

Then applying $\mathrm{Tr}_{A/L}$ to the above expression for f_i yields

$$\begin{aligned}\ell \cdot f_i = \mathrm{Tr}_{A/L}(f_i) &= \ell \cdot \partial_i\Big(\frac{\mathrm{Tr}_{A/F}(a)}{d}\Big) + \sum_j \sum_{\sigma \in G} \frac{\sigma(c_j)}{d} \mathrm{Tr}_{A/L}\Big(\frac{\partial_i(\sigma(b_i))}{\sigma(b_i)}\Big) \\ &= \ell \cdot \partial_i\Big(\frac{\mathrm{Tr}_{A/F}(a)}{d}\Big) + \sum_j \sum_{\sigma \in G} \frac{\sigma(c_j)}{d} \frac{\partial_i(N_{A/L}(\sigma(b_i)))}{N_{A/L}(\sigma(b_i))},\end{aligned}$$

where the second line uses the above identity. Hence,

$$f_i = \partial_i\Big(\frac{\mathrm{Tr}_{A/F}(a)}{d}\Big) + \sum_j \sum_{\sigma\in G} \frac{\sigma(c_j)}{d\cdot\ell}\frac{\partial_i(N_{A/L}(\sigma(b_i)))}{N_{A/L}(\sigma(b_i))},$$

where $\mathrm{Tr}_{A/F}(a)/d \in F$, $\dfrac{\sigma(c_j)}{d\cdot\ell} \in \overline{C}$, and $N_{A/L}(\sigma(b_j)) \in L = F(\{\sigma(c_j)\}_{j,\sigma})$. Since this holds for $i \in \{1,\dots,m\}$, it follows that $\omega = f_1\,dx_1 + \cdots + f_m\,dx_m = dg$, where

$$g = \frac{\mathrm{Tr}_{A/F}(a)}{d} + \sum_j \sum_{\sigma\in G} \frac{\sigma(c_j)}{d\cdot\ell}\log N_{A/L}(\sigma(b_i))$$

has the required form. This completes the proof. □

The following example indicates the difficulty of extending Theorem 6.3 to p-forms for $p > 1$.

Example 6.4. Let $\omega = \big(\exp(x+y^2)+\exp(y+x^2)\big)\,dxdy$. It is easy to check that $\omega = d(\eta)$ with $\eta := \exp(x+y^2)\,dy - \exp(y+x^2)\,dx$.

The recursive strategy used to prove Theorem 3.1 for 1-forms adapts nicely to prove Theorem 4.3 for p-forms. However, this strategy fails when we try to adapt the proof of Theorem 6.3 to the above 2-form, which we write as $\omega = f\,dxdy$ for $f := \exp(x+y^2)+\exp(y+x^2)$. To see why, we follow the strategy of Example 4.5. The first step is to write $\omega = dy(-f\,dx)$ and then find g such that $-f = \partial_y(g)$. But no elementary function has this property because f contains $\exp(x+y^2)$. Also, switching x and y does not help, since here $\omega = dx(f\,dy)$, but there is no elementary function g satisfying $f = \partial_x(g)$ because f contains $\exp(y+x^2)$.

7. Zeilberger's Creative Telescoping for Differential Forms with One Parameter

The method of *creative telescoping* first appeared in van der Poorten's report [51] on Apéry's proof of the irrationality of $\zeta(3)$. Specifically, the method was used to show that the sum

$$b_n := \sum_\ell b_{n,\ell} := \sum_\ell \binom{n}{\ell}^2\binom{n+\ell}{\ell}^2$$

satisfies the recurrence

$$n^3 b_n - (34n^3 - 51n^2 + 27n - 5)b_{n-1} + (n-1)^3 b_{n-2} = 0, \text{ for all } n \geq 2.$$

The essential idea is to "cleverly construct" [51] the product

$$B_{n,\ell} = 4(2n+1)\big(\ell(2\ell+1)-(2n+1)^2\big)\cdot b_{n,\ell}$$

and observe that if Δ_ℓ is the usual difference operator, then the identity

$$\begin{aligned} n^3 b_{n,\ell} + (n-1)^3 b_{n-2,\ell} - (34n^3 - 51n^2 + 27n - 5)b_{n-1,\ell} &= B_{n,\ell} - B_{n,\ell-1} \\ &= \Delta_\ell(B_{n,\ell}). \end{aligned}$$

follows from a straightforward algebraic computation since $b_{n,\ell} = \binom{n}{\ell}^2\binom{n+\ell}{\ell}^2$. Note that both $b_{n,\ell}$ and $B_{n,\ell}$ vanish automatically if $\ell < 0$ or $\ell > n$. If we fix $n \geq 2$ and sum over all integers ℓ, the right-hand side turns to zero, which proves the recurrence.

In general, for a given sequence $b_{n,\ell}$, the process of creative telescoping finds a nontrivial linear recurrence $L \in k(n)\langle S_n \rangle$ and another operator $Q \in k(n,\ell)\langle S_n, S_\ell \rangle$ such that

$$L(n, S_n)(b_{n,\ell}) = \Delta_\ell(Q(b_{n,\ell})).$$

Here, n is the operator of multiplication by n and S_n is the shift operator that maps n to $n+1$. The notation $k(n)\langle S_n \rangle$ reflects the fact that n and S_n do not commute since $S_n n = (n+1)S_n$. In the above identity, the operator L is called a *telescoper* for $b_{n,\ell}$ and $Q(b_{n,\ell})$ is the *certificate* of L for $b_{n,\ell}$. Zeilberger's fast algorithm [53, 54] for creative telescoping was developed to solve this problem when restricted to so-called *hypergeometric terms* $b_{n,\ell}$, meaning that both $S_n(b_{n,\ell})/b_{n,\ell}$ and $S_\ell(b_{n,\ell})/b_{n,\ell}$ are rational functions of n and ℓ.

In the differential analogue of creative telescoping [3], sums and difference operators are replaced with integrals and differential operators, respectively. Thus, given $F(x,y)$ in place of $b_{n,\ell}$, we seek a linear differential equation satisfied by the integral

$$f(x) = \int_{-\infty}^{+\infty} F(x,y)\, dy,$$

where $F(x,y)$ is *hyperexponential* with respect to x, y, i.e., $\partial_x(F(x,y))/F(x,y)$ and $\partial_y(F(x,y))/F(x,y)$ are both rational functions of x and y. In this case, there always exists a nonzero telescoper $L \in k(x)\langle \partial_x \rangle$ for $F(x,y)$ with respect to y with certificate $G(x,y)$ such that

$$L(F(x,y)) = \partial_y(G(x,y)).$$

(As above, the notation $k(x)\langle \partial_x \rangle$ reflects the fact that x and ∂_x do not commute since $\partial_x x = x\partial_x + 1$.) Then integrating with respect to y leads to

$$L(f(x)) = 0.$$

In recent years, reduction-based creative telescoping method [5, 6, 8, 19, 20, 7, 24] is a new trend because of its efficiency in computing telescopers without certificates. The existence of telescopers for differential forms with D-finite function coefficients is studied in [18].

In this section, we assume that all integrals involve one parameter t and the derivation with respect to t is denoted by ∂_t. We will prove that telescopers exist for closed algebraic 1-forms and use `HermiteOneForm` from Section 3 to give a reduction-based creative telescoping algorithm to compute the telescoper for closed rational 1-forms.

Definition 7.1. Given a 1-form $\omega = f_1\, dx_1 + \cdots + f_m\, dx_m$ with f_i algebraic over $k(t, x_1, \ldots, x_m)$, a linear differential operator L in $k(t)\langle \partial_t \rangle$ is called a *telescoper* for ω if $L(\omega) := L(f_1)\, dx_1 + \cdots + L(f_m)\, dx_m = dg$ for some algebraic function g over $k(t, x_1, \ldots, x_m)$. The function g is called a *certificate* of L. A nonzero telescoper for ω of minimal degree in ∂_t is called a *minimal telescoper*.

Observe that $L(\omega) = dg$ is equivalent to $L(f_i) = \partial_i(g)$ for each $i \in \{1, \ldots, m\}$. In the terminology of [17], L is called a *parallel telescoper*.

Theorem 7.2. *Let $\omega = f_1\, dx_1 + \cdots + f_m\, dx_m$ be a 1-form with coefficients algebraic over $k(t, x_1, \ldots, x_m)$. If ω is closed, then it always has a nonzero telescoper.*

Proof. We proceed by induction on m, the number of variables $x_1, \ldots, x_m$. For $m = 1$, one can construct a nonzero telescoper for f_1 with respect to x_1 by the method in [21, 20]. Suppose now that $m > 1$ and that the theorem holds for the case when the number of variables is less than m. For $f_m \in \overline{k(t, x_1, \ldots, x_m)}$, there exists a telescoper $L_m \in k(t, x_1, \ldots, x_{m-1}) \langle \partial_t \rangle$ with respect to x_m and its certificate g_m such that

$$L_m(f_m) = \partial_m(g_m).$$

We claim that L_m can be chosen to be free of $x_1, \ldots, x_{m-1}$. *W.L.O.G.*, assume that all coefficients of L_m lie in the polynomial ring $k[t, x_1, \ldots, x_{m-1}]$, otherwise we can multiply both sides of the above equation by the least common multiple of the denominators. Take L_m with $\deg_{x_1}(L_m)$ minimal, denoted by n, and write

$$L_m = x_1^n L_{m,n} + \cdots + x_1 L_{m,1} + L_{m,0},$$

where $L_{m,j} \in k[t, x_2, \ldots, x_{m-1}] \langle \partial_t \rangle$ for each $j \in \{1, \ldots, n\}$. To make the following computation clearer, we use "$\circ$" to represent the composition of operators. Observe that

$$\partial_1 \circ x_1^j = x_1^j \circ \partial_1 + j x_1^{j-1}.$$

Thus,

$$\partial_1 \circ L_m = x_1^n \circ \partial_1 \circ L_{m,n} + \cdots + x_1 \circ \partial_1 \circ L_{m,1} + \partial_1 \circ L_{m,0} + \sum_{j=1}^{n} j x_1^{j-1} \circ L_{m,j}.$$

Since ω is closed, we have $\partial_1(f_m) = \partial_m(f_1)$. For each $j \in \{1, \ldots, n\}$,

$$x_1^j \circ \partial_1 \circ L_{m,j}(f_m) = x_1^j \circ L_{m,j} \circ \partial_m(f_1) = \partial_m(x_1^j L_{m,j}(f_1)).$$

Now define $\tilde{L}_m := \partial_1 \circ L_{m,0} + \sum_{j=1}^{n} j x_1^{j-1} \circ L_{m,j}$. Putting this all together and using $\partial_1(L_m(f_m)) = \partial_1(\partial_m(g_m)) = \partial_m(\partial_1(g_m))$, we obtain

$$\tilde{L}_m(f_m) = \partial_m\Big(\partial_1(g_m) - \sum_{j=1}^{n} x_1^j L_{m,j}(f_1)\Big),$$

which implies that $\tilde{L}_m$ is a telescoper for f_m with respect to x_m with the degree in x_1 at most $n-1$, leading to a contradiction. So L_m is free of x_1. The same argument holds for $x_2, \ldots, x_{m-1}$. Hence our claim is true, i.e., $L_m \in k(t) \langle \partial_t \rangle$.

Since $\partial_i \circ L_m = L_m \circ \partial_i$ for each $i \in \{1, \ldots, m\}$, we see that $L_m(\omega)$ is closed with coefficients algebraic over $k(t, x_1, \ldots, x_m)$. Applying L_m to ω leads to

$$L_m(\omega) = L_m(f_1)\, dx_1 + \cdots + L_m(f_{m-1})\, dx_{m-1} + \partial_m(g_m).$$

By subtracting dg_m, we see that $L_m(\omega) - dg_m$ is a closed algebraic differential form free of x_m. By induction, there exists a nonzero telescoper $\tilde{L}$ with certificate $\tilde{g}$ such that

$$\tilde{L}(L_m(\omega) - dg_m) = d\tilde{g},$$

which implies that

$$\tilde{L} \circ L_m(\omega) = d\big(\tilde{L}(g_m) + \tilde{g}\big).$$

Thus $L := \tilde{L} \circ L_m$ is a nonzero telescoper for ω with certificate $g = \tilde{L}(g_m) + \tilde{g}$. $\square$

Next we present a reduction-based algorithm to construct a minimal telescoper for any closed rational 1-form $\omega = f_1\,dx_1 + \dots f_m\,dx_m$ with $f_i \in K(t)$. By Section 3, write

$$\omega = dg + \widetilde{\omega},$$

where $g \in K(t)$ and $\widetilde{\omega} = \widetilde{f}_1\,dx_1 + \cdots + \widetilde{f}_m\,dx_m$ and each $\widetilde{f}_i$ is a rational function of x_i with squarefree denominator. According to [23] and the analysis above, we can construct a minimal telescoper $L_m \in k(t)\,\langle \partial_t \rangle$ for $\widetilde{f}_m$ with respect to x_m such that

$$L_m(\widetilde{f}_m) = \partial_m(g_m),$$

for some $g_m \in K(t)$. Using these ingredients, we give a recursive algorithm of computing the minimal telescoper for a closed rational 1-form.

Algorithm `CTOneForm`: Given a closed rational 1-form $\omega = f_1\,dx_1 + \cdots + f_m\,dx_m$, where $f_i \in K$, compute a minimal telescoper $L \in K(t)\,\langle \partial_t \rangle$ and a certificate g of L.

(1) Decompose ω as $\omega = d\widetilde{g} + \widetilde{\omega}$ by `HermiteOneForm` from Section 3. Set

$$\widetilde{\omega} := \widetilde{f}_1\,dx_1 + \cdots + \widetilde{f}_m\,dx_m.$$

(2) Compute a minimal telescoper $L_m \in k(t)\,\langle \partial_t \rangle$ for $\widetilde{f}_m$ with respect to x_m and its certificate g_m by algorithms in [3, 5].

(3) If $m = 1$, then return $(L_m, L_m(\widetilde{g}) + g_m)$, otherwise, set $\bar{\omega} := L_m(\widetilde{\omega}) - dg_m$ and write

$$\bar{\omega} = \bar{f}_1\,dx_1 + \cdots + \bar{f}_{m-1}\,dx_{m-1}, \text{ where } \bar{f}_i := L_m(\widetilde{f}_i) - \partial_i(g_m) \text{ for } i = 1, \dots, m-1.$$

(4) Decompose $\bar{\omega}$ as $\bar{\omega} = d_{m-1}\widetilde{\bar{g}} + \widetilde{\bar{\omega}}$ by `HermiteOneForm` from Section 3.

(5) Run `CTOneForm` for $\widetilde{\bar{\omega}}$ to get $(\bar{L}, \bar{g})$, where $\bar{L}$ is of minimal order and $\bar{g}$ is its certificate.

(6) Return $L := \bar{L} \circ L_m$, and $g := \bar{g} + \bar{L}(\widetilde{\bar{g}}) + \bar{L}(g_m) + \bar{L}\big(L_m(\widetilde{g})\big)$.

The minimality of L follows from Proposition 13 of [17].

Example 7.3. Let ω be as the same 1-form as in Example 3.4. Then Hermite reduction gives $\widetilde{g} = 1/(xyz)$ and

$$\widetilde{\omega} := \frac{t}{x}\,dx + \frac{t}{y}\,dy + \frac{t^2+1}{z}\,dz.$$

Then we compute the minimal telescoper for $(t^2+1)/z$ with respect to z:

$$L_3 := (t^2+1)\partial_t - 2t,$$

and its certificate is $g_3 = 0$. Update ω as

$$\bar{\omega} := L_3(\widetilde{\omega}) - dg_3 = \frac{1-t^2}{x}\,dx + \frac{1-t^2}{y}\,dy.$$

Note that each coefficient of $\bar{\omega}$ is already squarefree, so Hermite reduction for $\bar{\omega}$ is not needed. The minimal telescoper for $(1-t^2)/y$ with respect to y is

$$\bar{L} := (t^2-1)\partial_t - 2t,$$

and its certificate is $\bar{g} = 0$. By computation, we obtain

$$\bar{L}(\bar{\omega}) = 0.$$

Thus

$$\bar{L}L_3(\omega) = \bar{L}L_3(d\widetilde{g} + \widetilde{\omega}) = d\bar{L}L_3(\widetilde{g}) + \bar{L}(L_3(\widetilde{\omega})) = d\bar{L}L_3(\widetilde{g}) + \bar{L}(\bar{\omega}) = d\bar{L}L_3(\widetilde{g}).$$

It follows that the minimal telescoper for ω is

$$L := \bar{L}L_3 = (t^4 - 1)\partial_t^2 - (2t^3 + 2t)\partial_t + (2t^2 + 2),$$

and its certificate is $g = \bar{L}L_3(\widetilde{g}) = (2t^2 + 2)/xyz$.

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