On the Existence of Telescopers for Rational Functions in Three Variables

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Abstract

Zeilberger's method of creative telescoping is crucial for the computer-generated proofs of combinatorial and special-function identities. Telescopers are linear differential or (q-)recurrence operators computed by algorithms for creative telescoping. Two fundamental problems related to creative telescoping are whether telescopers exist, and how to construct them efficiently when they do. In this paper, we solve the existence problem of telescopers for rational functions in three variables including 18 cases. We reduce the existence problem from the trivariate case to the bivariate case and some related problems. The existence criteria given in this paper enable us to determine the termination of algorithms for creative telescoping with trivariate rational inputs.

Key words: Creative telescoping, Existence criterion, Reduction, Telescoper

1. Introduction

Creative telescoping plays a crucial role in the algorithmic proof theory of combinatorial identities developed by Wilf and Zeilberger in the early 1990s [45, 46, 44]. For a

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given function $f(x, y_1, \ldots, y_n)$, the process of creative telescoping constructs a nonzero linear differential or (q-)recurrence operator L in x such that

$$L(f) = \Theta_{y_1}(g_1) + \cdots + \Theta_{y_n}(g_n),$$

where Θ_{y_i} denotes the derivation or (q-)difference operator in y_i and the g_i's belong to the same class of functions as f. The operator L is then called a *telescoper* for f, and the g_i's are called the *certificates* of L. Two fundamental problems have been studied extensively related to creative telescoping. The first problem is the *existence problem of telescopers*, i.e., deciding the existence of telescopers for a given class of functions. The second one is the *construction problem of telescopers*, i.e., designing efficient algorithms for computing telescopers if they exist. For additional open problems related to creative telescoping, see [19]. In this paper, we will mainly focus on the existence problem of telescopers and will study the construction problem of telescopers in future work.

The existence of telescopers is closely connected to the termination of algorithms for creative telescoping and the hypertranscendence and algebraic dependency of functions defined by indefinite sums or integrals [33, 40]. In [45], Zeilberger first presented a sufficient condition on the existence of telescopers by showing that telescopers always exist for the so-called *holonomic functions* using Bernstein's theory of algebraic D-modules. Soon after this work, Wilf and Zeilberger in [44] proved that telescopers exist for proper hypergeometric terms. However, holonomicity and properness are only sufficient conditions. Abramov and Le [4] gave a necessary and sufficient condition on the existence of telescopers for rational functions in two discrete variables. This work was soon extended to the hypergeometric case by Abramov [3], the q-hypergeometric case in [24], and the mixed rational and hypergeometric case in [22, 13]. All of the above work only focused on the problem for bivariate functions of a special class. The first criterion for the existence of telescopers beyond the bivariate case was given in [18], where a necessary and sufficient condition is presented for the existence problem of telescopers for rational functions in three discrete variables. This paper will continue this project by considering the remaining cases where continuous, discrete and q-discrete variables can appear simultaneously.

The remainder of this paper is organized as follows. We define the existence problem of telescopers precisely in Section 2 and recall different types of reductions that are used in testing the exactness of bivariate rational functions in Section 3. Existence criteria are given for 18 types of telescopers for rational functions in three variables in Section 4.

A preliminary version [14] of this article has appeared in the Proceedings of ISSAC'19. In the present version, the proofs are supplemented with further details, and twelve more cases are treated to cover in addition the q-shift operators.

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2. Preliminaries

Let $\mathbb{K}$ be a field of characteristic zero and $\mathbb{K}(\mathbf{v})$ be the field of rational functions in the variables $\mathbf{v} = \{x, y_1, \ldots, y_n\}$ over $\mathbb{K}$. For each $v \in \mathbf{v}$, the *derivation* δ_v on $\mathbb{K}(\mathbf{v})$ is defined as the usual partial derivation ∂/∂_v with respect to v satisfying that $\delta_v(f+g) = \delta_v(f) + \delta_v(g)$ and $\delta_v(fg) = g\delta_v(f) + f\delta_v(g)$ for all $f, g \in \mathbb{K}(\mathbf{v})$. Moreover, $\delta_v(c) = 0$ if and only if $c \in \mathbb{K}(\mathbf{v} \setminus \{v\})$, i.e., c is free of v. For each $v \in \mathbf{v}$, the *shift operator* σ_v is the $\mathbb{K}$-automorphism of $\mathbb{K}(\mathbf{v})$ defined by $\sigma_v(v) = v + 1$ and $\sigma_v(w) = w$ for all $w \in \mathbf{v} \setminus \{v\}$. Let $q \in \mathbb{K} \setminus \{0\}$ be such that $q^m \neq 1$ for all nonzero $m \in \mathbb{Z}$. For each $v \in \mathbf{v}$, the *q-shift operator* $\tau_{q,v}$ is the $\mathbb{K}$-automorphism defined by $\tau_{q,v}(v) = qv$ and $\tau_{q,v}(w) = w$ for all $w \in \mathbf{v} \setminus \{v\}$. Abusing notation, we let δ_v and θ_v with $\theta_v \in \{\sigma_v, \tau_{q,v}\}$ denote a fixed extension of δ_v and θ_v to the derivation and the $\overline{\mathbb{K}}$-automorphism of $\overline{\mathbb{K}(\mathbf{v})}$, the algebraic closure of $\mathbb{K}(\mathbf{v})$.

For each $v \in \mathbf{v}$, let $\partial_v \in \{D_v, S_v, T_{q,v}\}$, where D_v, S_v and $T_{q,v}$ refer to the differential, shift and q-shift operators, respectively. We consider $\mathcal{D} := \mathbb{K}(\mathbf{v})\langle \partial_x, \partial_{y_1}, \ldots, \partial_{y_n}\rangle$ as a noncommutative ring in $\partial_x, \partial_{y_1}, \ldots, \partial_{y_n}$ subject to the multiplication rules

$$\partial_{v_i}\partial_{v_j} = \partial_{v_j}\partial_{v_i} \quad \text{for all } v_i, v_j \in \mathbf{v}$$

and for any $v \in \mathbf{v}$ and $f \in \mathbb{K}(\mathbf{v})$,

$$\partial_v f = \begin{cases} f\partial_v + \delta_v(f) & \text{if } \partial_v = D_v, \\ \sigma_v(f)\partial_v & \text{if } \partial_v = S_v, \\ \tau_{q,v}(f)\partial_v & \text{if } \partial_v = T_{q,v}. \end{cases} \tag{1}$$

Actually $\mathcal{D}$ is a noncommutative algebra over $\mathbb{K}(\mathbf{v})$, which is also called the ring of linear functional operators or Ore polynomials (for more details, see [10, 26]). Let Δ_v be the difference operator $S_v - 1$ and $\Delta_{q,v}$ be the q-difference operator $T_{q,v} - 1$. For each $v \in \mathbf{v}$, we define

$$\Theta_v := \partial_v - \partial_v(1) = \begin{cases} D_v & \text{if } \partial_v = D_v, \\ \Delta_v & \text{if } \partial_v = S_v, \\ \Delta_{q,v} & \text{if } \partial_v = T_{q,v}. \end{cases} \tag{2}$$

The action of the operator $\partial_v \in \mathcal{D}$ on an element $f \in \mathbb{K}(\mathbf{v})$ is defined as

$$\partial_v(f) = \begin{cases} \delta_v(f) & \text{if } \partial_v = D_v, \\ \sigma_v(f) & \text{if } \partial_v = S_v, \\ \tau_{q,v}(f) & \text{if } \partial_v = T_{q,v}. \end{cases} \tag{3}$$

In general, the action of the operator $L = \sum_{i_0, i_1, \ldots, i_n \geq 0} a_{i_0, i_1, \ldots, i_n} \partial_x^{i_0} \partial_{y_1}^{i_1} \cdots \partial_{y_n}^{i_n} \in \mathcal{D}$ on $f \in \mathbb{K}(\mathbf{v})$ is defined as

$$L(f) = \sum_{i_0, i_1, \ldots, i_n \geq 0} a_{i_0, i_1, \ldots, i_n} \partial_x^{i_0} \partial_{y_1}^{i_1} \cdots \partial_{y_n}^{i_n}(f).$$

Then the field $\mathbb{K}(\mathbf{v})$ becomes a left $\mathcal{D}$-module. In this paper, we will mainly work with rational functions in three variables x, y, z and the operators in $\mathbb{K}(x, y, z)\langle \partial_x, \partial_y, \partial_z\rangle$.

Example 2.1. Let $L = 1 + (x + yz)D_x + 2S_yT_{q,z} \in \mathbb{K}(x, y, z)\langle D_x, S_y, T_{q,z}\rangle$ and $f = 1/(x + yz)$. Then we have

$$L(f) = f + (x + yz)\delta_x(f) + 2\sigma_y(\tau_{q,z}(f)) = \frac{2}{qz + qyz + x}.$$

The functions we consider will be in a certain $\mathcal{D}$-module, such as the field $\mathbb{K}(\mathbf{v})$ or its algebraic closure $\overline{\mathbb{K}(\mathbf{v})}$. The ring $\mathbb{K}(x)\langle\partial_x\rangle$ is a subring of $\mathcal{D}$ that is also a left Euclidean domain. An operator $L \in \mathbb{K}(x)\langle\partial_x\rangle$ is called a *common left multiple* of operators $L_1, \ldots, L_n \in \mathbb{K}(x)\langle\partial_x\rangle$ if there exist $R_1, \ldots, R_n \in \mathbb{K}(x)\langle\partial_x\rangle$ such that

$$L = R_1L_1 = \cdots = R_nL_n.$$

Among all of such multiples, the monic one of minimal degree in ∂_x is called the least common left multiple (LCLM) of operators $L_1, \ldots, L_n$. Efficient algorithms for basic operations in $\mathbb{K}(x)\langle\partial_x\rangle$, such as the LCLM computation, have been developed in [10, 5].

Since field extensions will occur in our studies, let us first recall some basic terminologies from Galois Theory (see [43]). Let $\mathbb{F}$ be a finite algebraic extension of $\mathbb{K}(x)$ with $n = [\mathbb{F} : \mathbb{K}(x)]$. Since $\mathrm{char}(\mathbb{K}) = 0$, $\mathbb{F}$ is also a separable extension and then $\mathbb{F} = \mathbb{K}(x)(\alpha)$ for some $\alpha \in \mathbb{F}$. Let $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_n$ be the conjugates of α and $\tau_1, \ldots, \tau_n$ be the distinct embeddings of $\mathbb{F}$ into the algebraic closure $\overline{\mathbb{K}(x)}$ such that $\tau_i(\alpha) = \alpha_i$ for all $i = 1, \ldots, n$. The derivation D_x on $\mathbb{K}(x)$ can be uniquely extended to $\mathbb{F}$ [9, Theorem 3.2.3] and the extended derivation commutes with all τ_i's [9, Theorem 3.2.4 (i)]. An extension $\mathbb{F}$ of $\mathbb{K}(x)$ is said to be *normal* over $\mathbb{K}(x)$ if $\mathbb{F}$ is a splitting field of some polynomial $p \in \mathbb{K}(x)[y]$ and is a Galois extension of $\mathbb{K}(x)$ if it is both separable and normal over $\mathbb{K}(x)$. Let $\mathrm{Gal}(\mathbb{F}/\mathbb{K}(x))$ be the Galois group of all automorphisms of $\mathbb{F}$ that fix $\mathbb{K}(x)$. For any $\tau \in \mathrm{Gal}(\mathbb{F}/\mathbb{K}(x))$ and $L = \sum_{i=0}^{r} \ell_i D_x^i \in \mathbb{F}\langle D_x\rangle$, we define the action $\tau(L) = \sum_{i=0}^{r} \tau(\ell_i)D_x^i$. Since the derivation D_x commutes with any $\tau \in \mathrm{Gal}(\mathbb{F}/\mathbb{K}(x))$, we have $\tau(L_1L_2) = \tau(L_1)\tau(L_2)$ for all operators $L_1, L_2 \in \mathbb{F}\langle D_x\rangle$. If $\mathbb{F}$ is a Galois extension of $\mathbb{K}(x)$, the Fundamental Theorem of Galois Theory [43, Theorem 2.8.8] implies that $L \in \mathbb{K}(x)\langle D_x\rangle$ if and only if $\tau(L) = L$ for all $\tau \in \mathrm{Gal}(\mathbb{F}/\mathbb{K}(x))$.

Lemma 2.2. For an operator $L = \sum_{i=0}^{\rho} e_iD_x^i \in \overline{\mathbb{K}(x)}\langle D_x\rangle$ with $e_\rho = 1$, we let $\mathbb{F}$ be a finite normal extension of $\mathbb{K}(x)$ containing the coefficients e_i and let G be the Galois group of $\mathbb{F}$ over $\mathbb{K}(x)$. Let T be the LCLM of the operators $\sigma(L) = \sum_{i=0}^{\rho} \sigma(e_i)D_x^i$ for all $\sigma \in G$. Then T belongs to $\mathbb{K}(x)\langle D_x\rangle$.

Proof. It suffices to show that $\tau(T) = T$ for all $\tau \in G$. For each $\sigma \in G$, we have $T = P_\sigma\sigma(L)$ for some $P_\sigma \in \mathbb{F}\langle D_x\rangle$. Since $\tau(L_1L_2) = \tau(L_1)\tau(L_2)$ for all $L_1, L_2 \in \mathbb{F}\langle D_x\rangle$, the operator $\tau(\sigma(L))$ divides $\tau(T)$ for each $\sigma \in G$. When σ runs through all of the elements of G, so does $\tau\sigma$. Hence $\tau(T)$ is also a common left multiple of the operators $\sigma(L)$ for all $\sigma \in G$. Since $\tau(T)$ and T are both monic and of the same degree in D_x, we get $\tau(T) = T$. ∎

Example 2.3. For operator $L = D_x + \sqrt{x}$. The LCLM of L and its conjugate $D_x - \sqrt{x}$ is $D_x^2 - \frac{1}{2x}D_x - x \in \mathbb{K}(x)\langle D_x\rangle$.

Remark 2.4. Note that Lemma 2.2 is not true in the (q-)shift case. For example, take $L = S_x + \sqrt{x}$. The LCLM of L and its conjugate is $S_x^2 - \sqrt{x(x + 1)}$, which is not in $\mathbb{K}(x)\langle S_x\rangle$.

Definition 2.5. For any rational function $f \in \mathbb{K}(x, y_1, \ldots, y_n)$, a nonzero operator $L(x, \partial_x) \in \mathbb{K}(x)\langle \partial_x \rangle$ is called a *telescoper of type* $(\partial_x, \Theta_{y_{i_1}}, \ldots, \Theta_{y_{i_k}})$ for f if there exist rational functions $g_1, \ldots, g_k \in \mathbb{K}(x, y_1, \ldots, y_n)$ such that

$$L(x, \partial_x)(f) = \Theta_{y_{i_1}}(g_1) + \cdots + \Theta_{y_{i_k}}(g_k), \tag{4}$$

where $1 \leq i_\ell \leq n$ for any $1 \leq \ell \leq k$. The rational functions $g_1, \ldots, g_k$ are called the *certificates* of L.

Note that all of the telescopers for a given function together with the zero operator form a left ideal of $\mathbb{K}(x)\langle \partial_x \rangle$ (see [25, Definition 1]). The following lemma summarizes closure properties related to the existence of telescopers.

Lemma 2.6. Let $f, g \in \overline{\mathbb{K}(x, y, z)}$, $a, b \in \mathbb{K}(x)$ and $\alpha, \beta \in \overline{\mathbb{K}(x)}$. Then we have

(i) if both f and g have telescopers in $\mathbb{K}(x)\langle D_x \rangle$ of type $(D_x, \Theta_y, \Theta_z)$, so does $\alpha f + \beta g$;

(ii) if both f and g have telescopers in $\mathbb{K}(x)\langle \partial_x \rangle$ of type $(\partial_x, \Theta_y, \Theta_z)$ with $\partial_x \in \{S_x, T_{q,x}\}$, so does $af + bg$.

Proof. We first show that αf has a telescoper in $\mathbb{K}(x)\langle D_x \rangle$ if f does. When $\alpha = 0$, the conclusion is obvious. Next we assume that $\alpha \neq 0$ and $L = \sum_{i=0}^{\rho} e_i D_x^i \in \overline{\mathbb{K}(x)\langle D_x \rangle}$ is a telescoper for f. Then $L(f) = \Theta_y(u) + \Theta_z(v)$ with $u, v \in \overline{\mathbb{K}(x, y, z)}$. Set $\tilde{L} = L \cdot \frac{1}{\alpha}$, which belongs to $\overline{\mathbb{K}(x)}\langle D_x \rangle$. Then we have $\tilde{L}(\alpha f) = \Theta_y(u) + \Theta_z(v)$, which means $\tilde{L}$ is a telescoper for αf. By Lemma 2.2, there exists $T \in \mathbb{K}(x)\langle D_x \rangle$ such that T is a left multiple of $\tilde{L}$. So T is also a telescoper for αf. When telescopers are in $\mathbb{K}(x)\langle S_x \rangle$ or $\mathbb{K}(x)\langle T_{q,x} \rangle$, the above argument works for af for any $a \in \mathbb{K}(x)$. It remains to show that $f + g$ has a telescoper in $\mathbb{K}(x)\langle \partial_x \rangle$ with $\partial_x \in \{D_x, S_x, T_{q,x}\}$ if both f and g do. Assume that $P, Q \in \mathbb{K}(x)\langle \partial_x \rangle$ are telescopers for f, g, respectively. Then the LCLM of P and Q is a telescoper for $f + g$ by the commutativity between operators in $\mathbb{K}(x)\langle \partial_x \rangle$ and the operators Θ_y and Θ_z. ∎

Let $V = (V_1, \ldots, V_m)$ be any set partition of the variables $\mathbf{v} = \{x, y_1, \ldots, y_n\}$. A rational function $f \in \mathbb{K}(\mathbf{v})$ is said to be *split* with respect to the partition V if $f = f_1 \cdots f_m$ with $f_i \in \mathbb{K}(V_i)$. A polynomial $p \in \mathbb{K}[\mathbf{v}]$ is said to be *integer-linear* in $\mathbb{K}[\mathbf{v}]$ if there exist $r \in \mathbb{K}[z]$ and $a, b_1, \ldots, b_n \in \mathbb{Z}$ such that $p = r(ax + b_1 y_1 + \cdots + b_n y_n)$. A polynomial $p \in \mathbb{K}[\mathbf{v}]$ is said to be *q-integer-linear* in $\mathbb{K}[\mathbf{v}]$ if there exist $r \in \mathbb{K}[z]$ and $a, b_1, \ldots, b_n, s, t_1, \ldots, t_n \in \mathbb{Z}$ such that $p = x^s y_1^{t_1} \cdots y_n^{t_n} r(x^a y_1^{b_1} \cdots y_n^{b_n})$. A rational function $f = P/Q \in \mathbb{K}(\mathbf{v})$ with $P, Q \in \mathbb{K}[\mathbf{v}]$ and $\gcd(P, Q) = 1$ is said to be *(q-)proper* in $\mathbb{K}(\mathbf{v})$ if Q is a product of (q-)integer-linear polynomials over $\mathbb{K}$. Split polynomials and (q-)proper rational functions will be used to state our existence criteria for telescopers in Section 4.

In the subsequent sections, we will study the existence of telescopers for rational functions in three variables. More precisely, we consider the following problem.

Existence Problem for Telescopers. For a rational function $f \in \mathbb{K}(x, y, z)$, decide the existence of telescopers of type $(\partial_x, \Theta_y, \Theta_z)$ for f.

Remark 2.8. In the trivariate case, there are 18 different types of telescopers up to the symmetry among (Θ_y, Θ_z) which are collected into six different classes in Table 2.7 according to different techniques used in the studies.

Classes	Types	Telescoping equations
1.	1.1. (D_x, D_y, D_z)	$L(x, D_x)(f) = D_y(g) + D_z(h)$.
2.	2.1. $(D_x, \Delta_y, \Delta_z)$	$L(x, D_x)(f) = \Delta_y(g) + \Delta_z(h)$
	2.2. $(D_x, \Delta_{q,y}, \Delta_z)$	$L(x, D_x)(f) = \Delta_{q,y}(g) + \Delta_z(h)$
	2.3. $(D_x, \Delta_{q,y}, \Delta_{q,z})$	$L(x, D_x)(f) = \Delta_{q,y}(g) + \Delta_{q,z}(h)$
3.	3.1. (S_x, D_y, D_z)	$L(x, S_x)(f) = D_y(g) + D_z(h)$
	3.2. $(T_{q,x}, D_y, D_z)$	$L(x, T_{q,x})(f) = D_y(g) + D_z(h)$
4.	4.1. (S_x, Δ_y, D_z)	$L(x, S_x)(f) = \Delta_y(g) + D_z(h)$
	4.2. $(S_x, \Delta_{q,y}, D_z)$	$L(x, S_x)(f) = \Delta_{q,y}(g) + D_z(h)$
	4.3. $(T_{q,x}, \Delta_y, D_z)$	$L(x, T_{q,x})(f) = \Delta_y(g) + D_z(h)$
	4.4. $(T_{q,x}, \Delta_{q,y}, D_z)$	$L(x, T_{q,x})(f) = \Delta_{q,y}(g) + D_z(h)$
5.	5.1. $(S_x, \Delta_y, \Delta_z)$	$L(x, S_x)(f) = \Delta_y(g) + \Delta_z(h)$
	5.2. $(S_x, \Delta_{q,y}, \Delta_z)$	$L(x, S_x)(f) = \Delta_{q,y}(g) + \Delta_z(h)$
	5.3. $(S_x, \Delta_{q,y}, \Delta_{q,z})$	$L(x, S_x)(f) = \Delta_{q,y}(g) + \Delta_{q,z}(h)$
	5.4. $(T_{q,x}, \Delta_y, \Delta_z)$	$L(x, T_{q,x})(f) = \Delta_y(g) + \Delta_z(h)$
	5.5. $(T_{q,x}, \Delta_{q,y}, \Delta_z)$	$L(x, T_{q,x})(f) = \Delta_{q,y}(g) + \Delta_z(h)$
	5.6. $(T_{q,x}, \Delta_{q,y}, \Delta_{q,z})$	$L(x, T_{q,x})(f) = \Delta_{q,y}(g) + \Delta_{q,z}(h)$
6.	6.1. (D_x, Δ_y, D_z)	$L(x, D_x)(f) = \Delta_y(g) + D_z(h)$
	6.2. $(D_x, \Delta_{q,y}, D_z)$	$L(x, D_x)(f) = \Delta_{q,y}(g) + D_z(h)$

Table 2.7. Six different classes of existence problems of telescopers

Different types of partial fraction decompositions will be used in solving the existence problems of telescopers. Let $G = \langle \theta_x, \theta_y, \theta_z \rangle$ be the free abelian group generated by the operators $\theta_x, \theta_y, \theta_z$ with $\theta_v \in \{\sigma_v, \tau_{q,v}\}$. Let $f \in \mathbb{K}(x, y, z)$ and H be a subgroup of G. We call the set

$$[f]_H := \{c \cdot \psi(f) \mid \psi \in H \text{ and } c \in \mathbb{K} \setminus \{0\}\}$$

the H-*orbit* at f. Two elements $f, g \in \mathbb{K}(x, y, z)$ are said to be H-*equivalent* if $[f]_H = [g]_H$, denoted by $f \sim_H g$. The relation $\sim_H$ is an equivalence relation in $\mathbb{K}(x, y, z)$. Let $f = P/Q$ and $g = A/B$ with $P, Q, A, B \in \mathbb{K}[x, y, z]$, $\gcd(P, Q) = 1$ and $\gcd(A, B) = 1$. If $f \sim_H g$, then $P \sim_H A$ and $Q \sim_H B$ since any $\psi \in H$ is an automorphism on $\mathbb{K}(x, y, z)$. So detecting the H-equivalence among rational functions can be reduced to that among polynomials. Two irreducible polynomials in distinct H-orbits are clearly coprime. A nonzero rational function $f \in \mathbb{K}(x, y, z)$ is said to be $(\theta_x, \theta_y, \theta_z)$-*invariant* if there exist $m, n, k \in \mathbb{Z}$ not all zero, and $c \in \mathbb{K} \setminus \{0\}$ such that $\theta_x^m \theta_y^n \theta_z^k(f) = c \cdot f$. By comparing the leading coefficients, the constant c in the above relation must be of the form q^s for some $s \in \mathbb{Z}$. Moreover, $c = 1$ if all of θ_x, θ_y, and θ_z are shift operators.

For any subgroup H of G and any polynomial $Q \in \mathbb{K}(x, y)[z]$, one can group all of irreducible factors in z of Q into distinct H-orbits which leads to a factorization

$$Q = c \cdot \prod_{i=1}^{n} \prod_{j=1}^{m_i} \psi_{i,j}(d_i)^{e_{i,j}}, \quad \text{where } c \in \mathbb{K}(x, y), n, m_i, e_{i,j} \in \mathbb{N} \text{ and } \psi_{i,j} \in H$$

and the d_i's are monic irreducible polynomials in distinct H-orbits. With respect to this fixed representation, we have the unique partial fraction decomposition for a rational function $f = P/Q \in \mathbb{K}(x, y, z)$ of the form

$$f = p + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \sum_{\ell=1}^{e_{i,j}} \frac{a_{i,j,\ell}}{\psi_{i,j}(d_i)^{\ell}}, \tag{5}$$

where $p, a_{i,j,\ell} \in \mathbb{K}(x, y)[z]$ satisfy $\deg_z(a_{i,j,\ell}) < \deg_z(d_i)$ for all i, j, ℓ. In the sequel, we will take different H according to different types of existence problems.

Example 2.9. Consider the rational function of the form

$$f = \frac{x}{z^2 + 2x + y} + \frac{y}{z^2 + 2x + y + 1} + \frac{-yz + x}{z^2 + 2qx + y} + \frac{3x^2}{z^2 + 2qx + y + 2z + 2}.$$

If $H = \langle \sigma_y \rangle$, then we have a decomposition

$$f = \frac{x}{d_1} + \frac{y}{\sigma_y(d_1)} + \frac{-yz + x}{d_2} + \frac{3x^2}{d_3},$$

where $d_1 = z^2+2x+y$, $d_2 = z^2+2qx+y$ and $d_3 = z^2+2qx+y+2z+2$. Note that d_1, d_2, d_3 are in distinct $\langle \sigma_y \rangle$-orbits. If $H = \langle \tau_{q,x}, \sigma_y \rangle$, then we have a different decomposition

$$f = \frac{x}{d_1} + \frac{y}{\sigma_y(d_1)} + \frac{-yz + x}{\tau_{q,x}(d_1)} + \frac{3x^2}{d_3},$$

where d_1, d_3 are in distinct $\langle \tau_{q,x}, \sigma_y \rangle$-orbits. If $H = \langle \tau_{q,x}, \sigma_y, \sigma_z \rangle$, then we have another decomposition

$$f = \frac{x}{d_1} + \frac{y}{\sigma_y(d_1)} + \frac{-yz + x}{\tau_{q,x}(d_1)} + \frac{3x^2}{\tau_{q,x}\sigma_y\sigma_z(d_1)}.$$

3. Reductions and Exactness Criteria

In this section, let $\mathbb{F}$ be any field of characteristic zero and later we will specialize $\mathbb{F}$ to the rational function field $\mathbb{K}(x)$ in Section 4. In order to detect the existence of telescopers, we first need to check whether 1 is a telescoper or not. This is equivalent to the so-called exactness problem.

Definition 3.1. Let $\mathbb{E}$ be a differential subfield of $\overline{\mathbb{K}(y_1, \ldots, y_n)}$. A rational function $f \in \mathbb{E}$ is called $(\Theta_{y_{i_1}}, \ldots, \Theta_{y_{i_k}})$-exact in $\mathbb{E}$ if there exist $g_1, \ldots, g_k \in \mathbb{E}$ such that

$$f = \Theta_{y_{i_1}}(g_1) + \cdots + \Theta_{y_{i_k}}(g_k),$$

where $1 \leq i_\ell \leq n$ for any $1 \leq \ell \leq k$.

Exactness Testing Problem. Decide whether a given function in $\mathbb{E}$ is $(\Theta_{y_{i_1}}, \ldots, \Theta_{y_{i_k}})$-exact in $\mathbb{E}$.

Cases	Exactness equations
Continuous case	1.1. $f = D_y(g) + D_z(h)$
Discrete cases	2.1. $f = \Delta_y(g) + \Delta_z(h)$
	2.2. $f = \Delta_{q,y}(g) + \Delta_z(h)$
	2.3. $f = \Delta_{q,y}(g) + \Delta_{q,z}(h)$
Mixed cases	3.1. $f = \Delta_y(g) + D_z(h)$
	3.2. $f = \Delta_{q,y}(g) + D_z(h)$

Table 3.3. Six different cases of exactness testing problems

Remark 3.2. In this paper, we will only focus on the exactness testing problem for bivariate rational functions in $\mathbb{F}(y,z)$. Since there are three choices for each operator in $\{\Theta_y, \Theta_z\}$, by the symmetry between Θ_y and Θ_z, there are 6 different types of exactness testing problems, listed in Table 3.3.

Let $\mathbb{E}$ be a finite separable extension of $\mathbb{F}(y,z)$ with $n = [\mathbb{E} : \mathbb{F}(y,z)]$. Let $\tau_1, \ldots, \tau_n$ be the distinct embeddings of $\mathbb{E}$ into $\overline{\mathbb{F}(y,z)}$. The *trace* map $\mathrm{Tr}_{\mathbb{E}/\mathbb{F}(y,z)} : \mathbb{E} \to \mathbb{F}(y,z)$ is defined by $\mathrm{Tr}_{\mathbb{E}/\mathbb{F}(y,z)}(u) = \sum_{i=1}^{n} \tau_i(u)$ for any $u \in \mathbb{E}$. The following lemma shows that the (non)exactness is preserved even when we are looking for g and h in a larger field.

Lemma 3.4. Let $f \in \mathbb{F}(y,z)$. Then f is (Θ_y, Θ_z)-exact in $\overline{\mathbb{F}(y,z)}$ if and only if it is (Θ_y, Θ_z)-exact in $\mathbb{F}(y,z)$.

Proof. The sufficiency is obvious. For the necessity, we assume that there exist $u, v \in \overline{\mathbb{F}(y,z)}$ such that $f = \Theta_y(u) + \Theta_z(v)$. Let $\mathbb{L}$ be a finite normal extension of $\mathbb{F}(y,z)$ containing u, v and $\Theta_y(u), \Theta_z(v)$ and let $\mathrm{Tr}_{\mathbb{L}/\mathbb{F}(y,z)}$ be the trace from $\mathbb{L}$ to $\mathbb{F}(y,z)$, which commutes with (q-)shift operators by [23, Lemma 3.1] and also with derivations by [9, Theorem 3.2.4 (i)]. Then

$$\mathrm{Tr}_{\mathbb{L}/\mathbb{F}(y,z)}(f) = \mathrm{Tr}_{\mathbb{L}/\mathbb{F}(y,z)}\left(\Theta_y(u) + \Theta_z(v)\right) = \Theta_y(\mathrm{Tr}_{\mathbb{L}/\mathbb{F}(y,z)}(u)) + \Theta_z(\mathrm{Tr}_{\mathbb{L}/\mathbb{F}(y,z)}(v)).$$

Since $f \in \mathbb{F}(y,z)$, we have $\mathrm{Tr}_{\mathbb{L}/\mathbb{F}(y,z)}(f) = mf$ with $m = [\mathbb{L} : \mathbb{F}(y,z)]$. Thus $f = \Theta_y(g) + \Theta_z(h)$ with $g = \frac{1}{m}\mathrm{Tr}_{\mathbb{L}/\mathbb{F}(y,z)}(u)$ and $h = \frac{1}{m}\mathrm{Tr}_{\mathbb{L}/\mathbb{F}(y,z)}(v)$ that are both in $\mathbb{F}(y,z)$. ∎

Let $\mathcal{E}$ denote the set of all (Θ_y, Θ_z)-exact rational functions in $\mathbb{F}(y,z)$. Note that $\mathcal{E}$ forms a subspace of $\mathbb{F}(y,z)$ viewed as an $\mathbb{F}$-vector space. Reduction algorithms have been developed in [21, 23, 35, 11, 41] for simplifying rational functions modulo $\mathcal{E}$ and then reducing the exactness problem from general rational functions to simple fractions. For later use, we summarize these reductions as follows.

3.1. The continuous case

For a rational function $f \in \mathbb{F}(y,z)$, the Ostrogradsky–Hermite reduction [39, 34] with respect to z decomposes f into the form

$$f = D_z(g) + \frac{a}{b}, \tag{6}$$

where $g \in \mathbb{F}(y,z)$ and $a, b \in \mathbb{F}(y)[z]$ with $\gcd(a,b) = 1$, $\deg_z(a) < \deg_z(b)$ and b being squarefree in z over $\mathbb{F}(y)$. Moreover, $f = D_z(u)$ for some $u \in \mathbb{F}(y,z)$ if and only if $a = 0$.

We recall the criterion on the (D_y, D_z)-exactness of bivariate rational functions from [21, Lemma 4].

Lemma 3.5. Let $f \in \mathbb{F}(y,z)$ be of the form (6) and write

$$\frac{a}{b} = \sum_{i=1}^{n} \frac{\alpha_i}{z - \beta_i},$$

where $\alpha_i, \beta_i \in \overline{\mathbb{F}(y)}$ with $\beta_i \neq \beta_j$ for i, j with $1 \leq i, j \leq n$ and $i \neq j$. Then f is (D_y, D_z)-exact in $\mathbb{F}(y,z)$ if and only if for each i with $1 \leq i \leq n$, we have $\alpha_i = D_y(\gamma_i)$ for some $\gamma_i \in \overline{\mathbb{F}(y)}$.

The above lemma reduces the exactness problem in the differential case from bivariate rational functions to univariate algebraic functions. Let $\alpha \in \overline{\mathbb{F}(y)}$ be an algebraic function over $\mathbb{F}(y)$ with $m := [\mathbb{F}(y,\alpha) : \mathbb{F}(y)]$. If $\alpha = D_y(\beta)$ for some $\beta \in \overline{\mathbb{F}(y)}$, then we can find such a $\beta \in \mathbb{F}(y,\alpha)$ by the trace argument as in the proof of Lemma 3.4. Assume that $\beta = b_0 + b_1\alpha + \cdots + b_{m-1}\alpha^{m-1}$ with $b_i \in \mathbb{F}(y)$. Then the equality $\alpha = D_y(\beta)$ leads to a system of linear differential equations in the b_i's, whose rational solutions can be computed by the method in [6]. A generalization of the Ostrogradsky–Hermite reduction to the algebraic case also solves the exactness problem of algebraic functions [20].

3.2. The discrete cases

For any automorphism θ on $\mathbb{F}(y,z)$ and $a, b \in \mathbb{F}(y,z)$, we have the reduction formula

$$\frac{a}{\theta^n(b)} = \theta(g) - g + \frac{\theta^{-n}(a)}{b}, \tag{7}$$

where $g = \sum_{i=0}^{n-1} \frac{\theta^{i-n}(a)}{\theta^i(b)}$ if $n \geq 0$ and $g = -\sum_{i=0}^{-n-1} \frac{\theta^i(a)}{\theta^{n+i}(b)}$ if $n < 0$. By using the above reduction formula with $\theta = \sigma_z$, Abramov's reduction in z [1, 2] decomposes $f \in \mathbb{F}(y,z)$ into the form

$$f = \Delta_z(g) + \frac{a}{b}, \tag{8}$$

where $g \in \mathbb{F}(y,z)$ and $a, b \in \mathbb{F}(y)[z]$ with $\gcd(a,b) = 1$, $\deg_z(a) < \deg_z(b)$ and b being shift-free in z over $\mathbb{F}(y)$, i.e., for any $k \in \mathbb{Z} \setminus \{0\}$ we have $\gcd(b, \sigma_z^k(b)) = 1$. Moreover, $f = \Delta_z(u)$ for some $u \in \mathbb{F}(y,z)$ if and only if $a = 0$. We use the reduction formula (7) with $\theta = \sigma_y$ to further decompose f as

$$f = \Delta_y(u) + \Delta_z(v) + \sum_{i=1}^{I} \sum_{j=1}^{J_i} \frac{a_{i,j}}{d_i^j}, \tag{9}$$

where $u, v \in \mathbb{F}(y,z)$, $a_{i,j} \in \mathbb{F}(y)[z]$, and $d_i \in \mathbb{F}[y,z]$ are such that $\deg_z(a_{i,j}) < \deg_z(d_i)$ and the d_i's are irreducible polynomials in distinct $\langle \sigma_y, \sigma_z \rangle$-orbits. We recall the criterion on the (Δ_y, Δ_z)-exactness of bivariate rational functions by combining Lemma 3.2 and Theorem 3.3 in [35].

Lemma 3.6. Let $f \in \mathbb{F}(y,z)$ be of the form (9). Then f is (Δ_y, Δ_z)-exact in $\mathbb{F}(y,z)$ if and only if for all i, j with $1 \leq i \leq I$, $1 \leq j \leq J_i$, we have $\sigma_y^{m_i}(d_i) = \sigma_z^{n_i}(d_i)$ for some $m_i, n_i \in \mathbb{Z}$ with $m_i > 0$ and $a_{i,j} = \sigma_y^{m_i}\sigma_z^{-n_i}(b_{i,j}) - b_{i,j}$ for some $b_{i,j} \in \mathbb{F}(y)[z]$ with $\deg_z(b_{i,j}) < \deg_z(d_i)$. In particular, if f is (Δ_y, Δ_z)-exact, so is each $a_{i,j}/d_i^j$.

For a rational function $f \in \mathbb{F}(y,z)$, Abramov's reduction in z and its q-analogue in y decompose f into

$$f = \Delta_{q,y}(g) + \Delta_z(h) + \sum_{i=1}^{I} \sum_{j=1}^{J_i} \frac{a_{i,j}}{d_i^j}, \tag{10}$$

where $g, h \in \mathbb{F}(y,z)$, $a_{i,j} \in \mathbb{F}(y)[z]$, $d_i \in \mathbb{F}[y,z]$ satisfy that $\deg_z(a_{i,j}) < \deg_z(d_i)$ and d_i's are irreducible polynomials in distinct $\langle \tau_{q,y}, \sigma_z \rangle$-orbits. We recall the criterion on the $(\Delta_{q,y}, \Delta_z)$-exactness in $\mathbb{F}(y,z)$ from [11, Theorem 3].

Lemma 3.7. Let $f \in \mathbb{F}(y,z)$ be of the form (10). Then f is $(\Delta_{q,y}, \Delta_z)$-exact in $\mathbb{F}(y,z)$ if and only if for each $i \in \{1, \ldots, I\}$, $d_i \in \mathbb{F}[z]$ and for each $j \in \{1, \ldots, J_i\}$, $a_{i,j} = \Delta_{q,y}(b_{i,j})$ for some $b_{i,j} \in \mathbb{F}(y)[z]$. In particular, if f is $(\Delta_{q,y}, \Delta_z)$-exact, so is each $a_{i,j}/d_i^j$.

The q-analogue of Abramov's reduction decomposes $f \in \mathbb{F}(y,z)$ into the form

$$f = \Delta_{q,z}(g) + c + \frac{a}{b}, \tag{11}$$

where $g \in \mathbb{F}(y,z)$, $c \in \mathbb{F}(y)$ and $a, b \in \mathbb{F}(y)[z]$ with $\gcd(a,b) = 1$, $\deg_z(a) < \deg_z(b)$ and b being q-shift-free in z over $\mathbb{F}(y)$, that is $\gcd(b, \tau_{q,z}^k b) = 1$ for any $k \in \mathbb{Z} \setminus \{0\}$. Moreover, $f = \Delta_{q,z}(u)$ for some $u \in \mathbb{F}(y,z)$ if and only if $c = 0$ and $a = 0$.

Applying the reduction formula (7) with $\theta = \tau_{q,y}$, we can further decompose f as

$$f = \Delta_{q,y}(u) + \Delta_{q,z}(v) + c + \sum_{i=1}^{I} \sum_{j=1}^{J_i} \frac{a_{i,j}}{d_i^j}, \tag{12}$$

where $u, v \in \mathbb{F}(y,z)$, $c \in \mathbb{F}(y)$, $a_{i,j} \in \mathbb{F}(y)[z]$, and $d_i \in \mathbb{F}[y,z]$ are such that $\deg_z(a_{i,j}) < \deg_z(d_i)$ and the d_i's are irreducible polynomials in distinct $\langle \tau_{q,y}, \tau_{q,z} \rangle$-orbits. Then the $(\Delta_{q,y}, \Delta_{q,z})$-exactness criterion of f can be given by combining Lemma 3.6 and Theorem 3.8 in [41], which is a q-analogue of Lemma 3.6.

Lemma 3.8. Let $f \in \mathbb{F}(y,z)$ be of the form (12). Then f is $(\Delta_{q,y}, \Delta_{q,z})$-exact in $\mathbb{F}(y,z)$ if and only if $c = \Delta_{q,y}(h)$ for some $h \in \mathbb{F}(y)$ and for all i, j with $1 \le i \le I$, $1 \le j \le J_i$, we have $\sigma_y^{m_i}(d_i) = q^{s_i} \sigma_z^{n_i}(d_i)$ for some $m_i, n_i, s_i \in \mathbb{Z}$ with $m_i > 0$ and for the smallest such positive integer m_i, $a_{i,j} = q^{-js_i} \tau_{q,y}^{m_i} \tau_{q,z}^{-n_i}(b_{i,j}) - b_{i,j}$ for some $b_{i,j} \in \mathbb{F}(y)[z]$ with $\deg_z(b_{i,j}) < \deg_z(d_i)$. In particular, if f is $(\Delta_{q,y}, \Delta_{q,z})$-exact, so is each $a_{i,j}/d_i^j$.

3.3. The mixed cases

For a rational function $f \in \mathbb{F}(y,z)$, applying the Ostrogradsky–Hermite reduction in z and the reduction formula (7) with $\theta = \theta_y \in \{\sigma_y, \tau_{q,y}\}$ to f yields

$$f = \Theta_y(u) + D_z(v) + \sum_{i=1}^{I} \frac{a_i}{d_i}, \tag{13}$$

where $u, v \in \mathbb{F}(y,z), a_i \in \mathbb{F}(y)[z], d_i \in \mathbb{F}[y,z]$ with $\deg_z(a_i) < \deg_z(d_i)$ and the d_i's are irreducible polynomials in distinct $\langle \theta_y \rangle$-orbits. We recall the criterion on the (Θ_y, D_z)-exactness in $\mathbb{F}(y,z)$ from [11, Theorem 2].

Lemma 3.9. Let $\theta_y \in \{\sigma_y, \tau_{q,y}\}$ and $f \in \mathbb{F}(y,z)$ be of the form (13). Then f is (Θ_y, D_z)-exact in $\mathbb{F}(y,z)$ if and only if for each $i \in \{1, \ldots, I\}$, $d_i \in \mathbb{F}[z]$ and $a_i = \Theta_y(b_i)$ for some $b_i \in \mathbb{F}(y)[z]$. In particular, if f is (Θ_y, D_z)-exact, so is each a_i/d_i.

4. Existence Criteria

We will reduce the existence problem of telescopers in the trivariate case to that in the bivariate case and two related problems. To this end, we first recall the existence criteria on telescopers for bivariate rational functions from [4, 38, 3, 22, 13].

Theorem 4.1. Let $f(x, y)$ be a rational function in $\mathbb{K}(x, y)$. Then

(i) Differential case (see [22, Theorem 4.5]): f always has a telescoper of type (D_x, D_y);

(ii) Shift case (see [4, Theorem 1] or [22, Theorem 4.11]): f has a telescoper of type (S_x, Δ_y) if and only if f is of the form $f = \Delta_y(g) + r$ for some $g, r \in \mathbb{K}(x, y)$ and r is proper in $\mathbb{K}(x, y)$.

(iii) q-Shift case (see [38, Theorem 1] or [22, Theorem 4.15]): f has a telescoper of type $(T_{q,x}, \Delta_{q,y})$ if and only if f is of the form $f = \Delta_{q,y}(g) + r$ for some $g, r \in \mathbb{K}(x, y)$ and r is q-proper in $\mathbb{K}(x, y)$.

(iv) Mixed cases (see [22, Theorems 4.6, 4.7, 4.9, 4.12, 4.13, 4.14]): f has a telescoper of type $(\partial_x, \Theta_y) \in \{(S_x, D_y), (T_{q,x}, D_y), (D_x, \Delta_y), (T_{q,x}, \Delta_y), (D_x, \Delta_{q,y}), (S_x, \Delta_{q,y})\}$ if and only if f is of the form $f = \Theta_y(g) + r$ for some $g, r \in \mathbb{K}(x, y)$ and the denominator of r is split with respect to the partition $(\{x\}, \{y\})$.

Example 4.2. Let $f = 1/(x + y)$. It is easy to check that

$$L_1(f) = D_y(f), \quad L_2(f) = \Delta_y(f) \quad \text{and} \quad L_3(f) = \Delta_{q,y}(-\tau_{q,y}^{-1} f),$$

where $L_1 = D_x$, $L_2 = S_x - 1$ and $L_3 = qT_{q,x} - 1$. Then f has a telescoper of type (D_x, D_y), (S_x, Δ_y) and $(T_{q,x}, \Delta_{q,y})$, but f has no telescoper in the mixed cases since $x + y$ is not split.

To verify the existence of a telescoper for a trivariate rational function, we firstly introduce the following (q-)shift equivalence testing problem and separation problem.

Problem 4.3 (Shift Equivalence Testing Problem)**.** Let $\mathbb{F}$ be any computable field of characteristic zero. Given $p \in \mathbb{F}[x_1, ..., x_n]$, decide whether there exist $m_1, \ldots, m_n \in \mathbb{Z}$ with $m_1 > 0$ such that $p(x_1 + m_1, \ldots, x_n + m_n) = p(x_1, \ldots, x_n)$.

This problem is solved by Grigoriev in [31, 32] and more recently by Kauers and Schneider in [36] and Dvir et al. in [28].

Problem 4.4 (q-Shift Equivalence Testing Problem)**.** Let $p \in \mathbb{F}[x_1, ..., x_n]$, decide if there exist $m_0, m_1, \ldots, m_n \in \mathbb{Z}$ with $m_1 > 0$ such that $p(q^{m_1} x_1, \ldots, q^{m_n} x_n) = q^{m_0} p(x_1, \ldots, x_n)$.

This problem is much easier than the shift case, and an algorithm for testing q-shift equivalence has been given in [41].

Problem 4.5 (Separation Problem)**.** Given an algebraic function $\alpha \in \overline{\mathbb{K}(x, y)}$, decide whether there exists a nonzero operator $L \in \mathbb{K}(x)\langle D_x \rangle$ such that $L(\alpha) = 0$. If such an operator exists, we say that α is *separable* in x and y.

As a special case of [15, Proposition 10], a rational function in $\mathbb{K}(x, y)$ is separable if and only if it is of the form $a/(bc)$ with $a \in \mathbb{K}[x, y], b \in \mathbb{K}[x]$ and $c \in \mathbb{K}[y]$. This motivates the nomenclature of Problem 4.5. We will study the separation problem in the forthcoming paper [16], in which an algorithm is presented for constructing such a differential annihilator $L \in \mathbb{K}(x)\langle D_x \rangle$ if it exists.

In the pure differential setting, telescopers always exist for general D-finite functions over $\mathbb{K}(\mathbf{v})$, which was proved by Zeilberger in 1990 using the elimination property of holonomic D-modules [45]. For the sake of completeness, we will give a more direct proof for rational functions in $\mathbb{K}(\mathbf{v})$. We first adapt Wegschaider's "non-commutative trick" in [42, Theorem 3.2] to the differential case.

Lemma 4.6. Let $f \in \mathbb{K}(x, y_1, \ldots, y_n)$ and $A \in \mathbb{K}[x]\langle D_x, D_{y_1}, \ldots, D_{y_n}\rangle$ be a nonzero operator such that $A(f) = 0$. Then there exists a nonzero operator $L \in \mathbb{K}[x]\langle D_x\rangle$ such that $L(f) = D_{y_1}(g_1) + \cdots + D_{y_n}(g_n)$ for some $g_1, \ldots, g_n \in \mathbb{K}(x, y_1, \ldots, y_n)$.

Proof. We will follow the same argument as in the proof of [42, Theorem 3.2]. We claim that for every $\ell \in \{1, \ldots, n+1\}$, there exist $Q_{j,\ell} \in \mathbb{K}[\mathbf{v}]\langle D_x, D_{y_j}, \ldots, D_{y_n}\rangle$ for each $j \in \{1, \ldots, \ell-1\}$ and a nonzero $R_\ell \in \mathbb{K}[x]\langle D_x, D_{y_\ell}, \ldots, D_{y_n}\rangle$ such that f is annihilated by the operator

$$P_\ell = \sum_{j=1}^{\ell-1} D_{y_j} Q_{j,\ell} + R_\ell. \tag{14}$$

The lemma follows from this claim since R_{n+1} is the desired operator $L \in \mathbb{K}[x]\langle D_x\rangle$ with $g_j := -Q_{j,\ell}(f) \in \mathbb{K}(\mathbf{v})$ for $j \in \{1, \ldots, n\}$.

We prove the claim inductively: for $\ell = 1$ take $P_1 = R_1 = A$. Assume that for some $\ell \in \{1, \ldots, n\}$ we have a nonzero operator P_ℓ of the form (14) that annihilates f. We show that by division of R_ℓ by D_ℓ we can construct the operator $P_{\ell+1}$.

Since D_{y_ℓ} commutes with x and $D_x, D_{y_{\ell+1}}, \ldots, D_{y_n}$, we can write $R_\ell = D_{y_\ell}^m(R_{\ell+1} + D_{y_\ell} M)$, where $m \in \mathbb{N}$, $M \in \mathbb{K}[x]\langle D_x, D_{y_\ell}, \ldots, D_{y_n}\rangle$, and $R_{\ell+1}$ is a nonzero operator in $\mathbb{K}[x]\langle D_x, D_{y_{\ell+1}}, \ldots, D_{y_n}\rangle$. For any $w \in \mathbb{K}[y_\ell]$ of degree at most m, we have

$$w D_{y_\ell}^m = D_{y_\ell} \tilde{Q}_\ell + r \tag{15}$$

for some $r \in \mathbb{K}$ and $\tilde{Q}_\ell \in \mathbb{K}[y_\ell]\langle D_{y_\ell}\rangle$. In particular, $r = (-1)^m m! \neq 0$ if we take $w = y_\ell^m$. Using the fact $r D_{y_i} = D_{y_i} r$ for all $i \in \{1, \ldots, n\}$ and (15), we find

$$\begin{aligned}
\frac{y_\ell^m}{(-1)^m m!} P_\ell &= \sum_{j=1}^{\ell-1} D_{y_j} \left(\frac{y_\ell^m}{(-1)^m m!} Q_{j,\ell} \right) + \frac{y_\ell^m}{(-1)^m m!} D_{y_\ell}^m (R_{\ell+1} + D_{y_\ell} M) \\
&= \sum_{j=1}^{\ell-1} D_{y_j} \left(\frac{y_\ell^m}{(-1)^m m!} Q_{j,\ell} \right) + \left(D_{y_\ell} \tilde{Q}_\ell + 1 \right) (R_{\ell+1} + D_{y_\ell} M) \\
&= \sum_{j=1}^{\ell} D_{y_j} \tilde{Q}_{j,\ell} + R_{\ell+1} \triangleq P_{\ell+1} \quad \text{with } \tilde{Q}_{j,\ell} \in \mathbb{K}[\mathbf{v}]\langle D_x, D_{y_j}, \ldots, D_{y_n}\rangle.
\end{aligned}$$

Since $P_\ell(f) = 0$, we have $P_{\ell+1}(f) = 0$. So $P_{\ell+1}$ is the desired operator. ∎

Theorem 4.7. For any rational function $f \in \mathbb{K}(\mathbf{v})$, there exists a nonzero $L \in \mathbb{K}[x]\langle D_x\rangle$ such that $L(f) = D_{y_1}(g_1) + \cdots + D_{y_n}(g_n)$ for some $g_1, \ldots, g_n \in \mathbb{K}(\mathbf{v})$.

Proof. It suffices to show that there exists a nonzero $A \in \mathbb{K}[x]\langle D_x, D_{y_1}, \ldots, D_{y_n}\rangle$ such that $A(f) = 0$ by Lemma 4.6. Write $f = P/Q$ with $P, Q \in \mathbb{K}[\mathbf{v}]$ and $\gcd(P, Q) =$

1. Denote $d_x = \max\{\deg_x(P), \deg_x(Q)\}$ and $d_{y_i} = \max\{\deg_{y_i}(P), \deg_{y_i}(Q)\}$ for $i \in \{1, \ldots, n\}$. Let $\mathbb{W}_N$ be the $\mathbb{K}$-vector space generated by the set

$$\{ x^i D_x^{j_0} D_{y_1}^{j_1} \cdots D_{y_n}^{j_n} \mid 0 \le i + j_0 + \cdots + j_n \le N \}$$

over $\mathbb{K}$. By an easy combinatorial counting, the dimension of $\mathbb{W}_N$ is $\binom{N+n+2}{n+2} = \mathcal{O}(N^{n+2})$ over $\mathbb{K}$. Furthermore, for any $(i, j_0, \ldots, j_n) \in \mathbb{N}^{n+2}$, a direct calculation yields

$$x^i D_x^{j_0} D_{y_1}^{j_1} \cdots D_{y_n}^{j_n}(f) = \frac{P_{i,j_0,\ldots,j_n}}{Q^{i+j_0+\cdots+j_n+1}}, \tag{16}$$

where $P_{i,j_0,\ldots,j_n} \in \mathbb{K}[\mathbf{v}]$ with $\deg_x(P_{i,j_0,\ldots,j_n}) \le (i + j_0 + \cdots + j_n + 1)d_x + i$ and

$$\deg_{y_i}(P_{i,j_0,\ldots,j_n}) \le (i + j_0 + \cdots + j_n + 1)d_{y_i} \quad \text{for } i \in \{1, \ldots, n\}.$$

So the set $\mathbb{W}_N(f)$ is included in the $\mathbb{K}$-vector space $\mathbb{V}_N$ spanned by the set

$$\left\{ \frac{x^{k_0} y_1^{k_1} \cdots y_n^{k_n}}{Q^{N+1}} \middle| 0 \le k_0 \le (N+1)d_x + N,\, 0 \le k_i \le (N+1)d_{y_i} \text{ for } i = 1, \ldots, n \right\},$$

whence the dimension of $\mathbb{V}_N$ is $(N+1)(d_x+1)\prod_{i=1}^n((N+1)d_{y_i}+1) = \mathcal{O}(N^{n+1})$ over $\mathbb{K}$. We now define the linear map $\psi : \mathbb{W}_N \to \mathbb{V}_N$ by $\psi(L) = L(f)$ for any $L \in \mathbb{W}_N$. For sufficiently large N, we have

$$\binom{N+n+2}{n+2} > (N+1)(d_x+1)\prod_{i=1}^n((N+1)d_{y_i}+1),$$

which implies that the kernel of ψ is nontrivial. Therefore, there exists a nonzero operator $A \in \mathbb{W}_N \subseteq \mathbb{K}[x]\langle D_x, D_{y_1}, \ldots, D_{y_n}\rangle$ such that $A(f) = 0$. ∎

Remark 4.8. In the continuous setting, the existence of telescopers for rational functions implies that for algebraic functions by [21, Lemma 4]. Efficient algorithms for computing telescopers have been given in [7, 21, 8, 37].

Example 4.9. Let $f = 1/(2x + y^2 + yz^2) \in \mathbb{K}(x, y, z)$. One can check that

$$L(f) = D_y(-2y \cdot f) + D_z(-z \cdot f),$$

where $L = 4xD_x + 1$. Thus f has a telescoper of type (D_x, D_y, D_z).

4.2. Existence criteria of the second class

We now solve the second class of existence problems where telescopers are linear differential operators in $\mathbb{K}(x)\langle D_x\rangle$ and $(\Theta_y, \Theta_z) \in \{(\Delta_y, \Delta_z), (\Delta_{q,y}, \Delta_z), (\Delta_{q,y}, \Delta_{q,z})\}$.

Problem 4.10. Given $f \in \mathbb{K}(x, y, z)$, determine if there exists a nonzero operator $L \in \mathbb{K}(x)\langle D_x\rangle$ such that $L(f) = \Theta_y(g) + \Theta_z(h)$ for some $g, h \in \mathbb{K}(x, y, z)$.

For $v \in \{y, z\}$, let $\theta_v = \sigma_v$ if $\Theta_v = \Delta_v$ or $\theta_v = \tau_{q,v}$ if $\Theta_v = \Delta_{q,v}$. By partial fraction decomposition w.r.t z, any $f \in \mathbb{K}(x, y, z)$ can be uniquely decomposed into

$$f = \mu + zp + \sum_{i=1}^m \sum_{j=1}^{n_i} \frac{a_{i,j}}{d_i^j},$$

where $\mu \in \mathbb{K}(x, y)$, $m, n_i \in \mathbb{N}$, $p, a_{i,j}, d_i \in \mathbb{K}(x, y)[z]$, $\deg_z(a_{i,j}) < \deg_z(d_i)$, and the d_i's are distinct monic irreducible polynomials. Since $q^n \neq 1$, it is easy to check that zp is Θ_z-exact. Utilizing the transformation (7) first with $\theta = \theta_y$ and subsequently with $\theta = \theta_z$, f can be further decomposed into

$$f = \Theta_y(u) + \Theta_z(v) + \mu + \sum_{i=1}^{I} \sum_{j=1}^{J_i} \frac{a_{i,j}}{d_i^j}, \tag{17}$$

where $u, v \in \mathbb{K}(x, y, z)$, d_i's are irreducible polynomials in distinct $\langle \theta_y, \theta_z \rangle$-orbits and none of the nonzero $a_{i,j}/d_i^j$ is (Θ_y, Θ_z)-exact since otherwise it can be removed by adding into the u and v part.

The following theorem shows that Problem 4.10 can be reduced to the same problem but for simple fractions and bivariate rational functions.

Theorem 4.11. Let $f \in \mathbb{K}(x, y, z)$ be of the form (17). Then f has a telescoper of type $(D_x, \Theta_y, \Theta_z)$ if and only if μ and the fraction $a_{i,j}/d_i^j$ have a telescoper of the same type for all i, j with $1 \leq i \leq I$ and $1 \leq j \leq J_i$.

Proof. The sufficiency follows from Lemma 2.6. For the necessity, when f has a telescoper of type $(D_x, \Theta_y, \Theta_z)$, since D_x does not change the $\langle \theta_y, \theta_z \rangle$-equivalence of the denominators, one can deduce that μ and $r = \sum_{i=1}^{I} \sum_{j=1}^{J_i} \frac{a_{i,j}}{d_i^j}$ both have a telescoper of the same type.

Next we will show each fraction $a_{i,j}/d_i^j$ has a telescoper of the same type when r has a telescoper. To this end, we first show that $D_x(d_i) = 0$, that is $d_i \in \mathbb{K}[y, z]$ for all $1 \leq i \leq I$. Over the field $\overline{\mathbb{K}(x, y)}$, we can decompose r as

$$r = \Theta_y(u^\star) + \Theta_z(v^\star) + r^\star \text{ with } r^\star = \sum_{i=1}^{I'} \sum_{j=1}^{J_i'} \frac{\alpha_{i,j}}{(z - \beta_i)^j},$$

where $u^\star, v^\star \in \overline{\mathbb{K}(x, y)}(z)$, $\alpha_{i,j}, \beta_i \in \overline{\mathbb{K}(x, y)}$ with $\alpha_{i,J_i'} \neq 0$, $z - \beta_i$ and $z - \beta_{i'}$ are not $\langle \theta_y, \theta_z \rangle$-equivalent for all i, i' with $1 \leq i \neq i' \leq I'$. It suffices to show $D_x(\beta_i) = 0$ for all i with $1 \leq i \leq I'$. We will prove this claim by contradiction. Suppose that $D_x(\beta_k) \neq 0$ for some $1 \leq k \leq I'$ and that $L = \sum_{\ell=0}^{\rho} e_\ell D_x^\ell \in \mathbb{K}(x)\langle D_x \rangle$ with $e_\rho \neq 0$ is a telescoper for $r^\star$. Then

$$L(r^\star) = \sum_{i=1}^{I'} \left(\frac{J_i'^{\overline{\rho}} e_\rho \alpha_{i,J_i'} D_x(\beta_i)^\rho}{(z - \beta_i)^{J_i'+\rho}} + \sum_{j=1}^{J_i'+\rho-1} \frac{\tilde{\alpha}_{i,j}}{(z - \beta_i)^j} \right),$$

where $J_i'^{\overline{\rho}} = J_i'(J_i' + 1) \cdots (J_i' + \rho - 1)$ and $\tilde{\alpha}_{i,j} \in \overline{\mathbb{K}(x, y)}$. As $L(r^\star)$ is (Θ_y, Θ_z)-exact and $D_x(\beta_k) \neq 0$, we have

$$\theta_y^{m_k}(z - \beta_k) = q^{s_k} \theta_z^{n_k}(z - \beta_k) \tag{18}$$

for some $m_k, n_k, s_k \in \mathbb{Z}$ with $m_k > 0$ and

$$J_k'^{\overline{\rho}} e_\rho \alpha_{k,J_k'} D_x(\beta_k)^\rho = q^{-(J_k'+\rho)s_k} \theta_y^{m_k}(\gamma_k) - \gamma_k \tag{19}$$

for some $\gamma_k \in \overline{\mathbb{K}(x, y)}$. From Equation (18), we know $\theta_y^{m_k} D_x(\beta_k) = q^{s_k} D_x(\beta_k)$. Dividing Equation (19) by $J_k'^{\overline{\rho}} e_\rho D_x(\beta_k)^\rho$ gives

$$\alpha_{k,J_k'} = q^{-J_k' s_k} \theta_y^{m_k} \left(\frac{\gamma_k}{J_k'^{\overline{\rho}} e_\rho D_x(\beta_k)^\rho} \right) - \frac{\gamma_k}{J_k'^{\overline{\rho}} e_\rho D_x(\beta_k)^\rho}.$$

Thus $\frac{\alpha_{k,J'_k}}{(z-\beta_k)^{J'_k}}$ is (Θ_y, Θ_z)-exact in $\overline{\mathbb{K}(x,y)}(z)$, and hence can be moved into $u^\star$ and $v^\star$. Then by similar discussions as above, one can see $\frac{\alpha_{k,j}}{(z-\beta_k)^j}$ is (Θ_y, Θ_z)-exact for all j with $1 \leq j \leq J_i$. Notice that β_k is a root of d_k for some $1 \leq k \leq I$ and that $D_x(\beta_k) \neq 0$ leads to $D_x(\beta) \neq 0$ for any conjugate root β of d_k. Then all fractions of the form $\frac{\alpha}{(z-\beta)^j}$ in $r^\star$ are also (Θ_y, Θ_z)-exact. Collecting all these fractions together, we get $\frac{a_{k,j}}{d_k^j}$ is (Θ_y, Θ_z)-exact in $\overline{\mathbb{K}(x,y)}(z)$ and hence in $\mathbb{K}(x,y,z)$ by Lemma 3.4, which contradicts the assumption that none of the nonzero $\frac{a_{i,j}}{d_i^j}$ in r is exact. At this stage we have proved $d_i \in \mathbb{K}[y,z]$. Since L is also a telescoper for r, we have

$$L(r) = \sum_{i=1}^{I} \sum_{j=1}^{J_i} \frac{L(a_{i,j})}{d_i^j} = \Theta_y(g) + \Theta_z(h)$$

for some $g, h \in \mathbb{K}(x,y,z)$. Since the d_i's are in distinct $\langle \theta_y, \theta_z \rangle$-orbits and also free of x, we have for each i, j,

$$L\left(\frac{a_{i,j}}{d_i^j}\right) = \frac{L(a_{i,j})}{d_i^j} = \Theta_y(g_{i,j}) + \Theta_z(h_{i,j})$$

for some $g_{i,j}, h_{i,j} \in \mathbb{K}(x,y,z)$. So L is a telescoper for all $a_{i,j}/d_i^j$ with $1 \leq i \leq I$ and $1 \leq j \leq J_i$. ∎

Notice that for $\mu \in \mathbb{K}(x,y)$, having telescopers of type $(D_x, \Theta_y, \Theta_z)$ or (D_x, Θ_y) are equivalent. As the existence problem of telescopers for bivariate rational functions has been settled by Theorem 4.1, we only need to decide when

$$f = \frac{a}{d^j},$$

where $a \in \mathbb{K}(x,y)[z], d \in \mathbb{K}[x,y,z]$ with $\deg_z(a) < \deg_z(d)$ and d being irreducible, has telescopers of type $(D_x, \Theta_y, \Theta_z)$. The same argument as in the proof of Theorem 4.11 implies that if f is not (Θ_y, Θ_z)-exact but has a telescoper of type $(D_x, \Theta_y, \Theta_z)$, then d is free of x. Assume $d \in \mathbb{K}[y,z]$ and $L \in \mathbb{K}(x)\langle D_x \rangle$ is a telescoper of f. Then $L(f) = \frac{L(a)}{d^j}$ is (Θ_y, Θ_z)-exact. We will proceed by checking whether the two conditions for the exactness in Lemmas 3.6–3.8 are satisfied. In order to carry out this case distinction explicitly, we will refer to the corresponding (q)-shift equivalence testing problems.

If $\theta_y^m(d) \neq q^t \theta_z^n(d)$ whenever $m, n, t \in \mathbb{Z}$ and $m > 0$, then we have $L(a) = 0$ which can be reduced to solving the separation problem for bivariate rational functions and settled via GCD computations.

If $\theta_y^m(d) = q^t \theta_z^n(d)$ for some $m, n, t \in \mathbb{Z}$ with m being the smallest such positive integer, then $L(a)$ satisfies an equation. Next we will show how to solve the equation for different (θ_y, θ_z) separately.

(1) When $(\theta_y, \theta_z) = (\sigma_y, \sigma_z)$, Lemma 3.6 shows that $L(x, D_x)(a) = \sigma_y^m \sigma_z^{-n}(b) - b$ for some $b \in \mathbb{K}(x,y)[z]$ with $\deg_z(b) < \deg_z(d)$. Taking $\bar{y} = y/m$ and $\bar{z} = ny + mz$ shows $L(x, D_x)(a) = \sigma_y^m \sigma_z^{-n}(b) - b$ is equivalent to the existence problem of telescopers of type (D_x, Δ_y) for bivariate rational functions, which has been solved by Theorem 4.1.
(2) When $(\theta_y, \theta_z) = (\tau_{q,y}, \sigma_z)$, Lemma 3.7 leads to $L(a) = \Delta_{q,y}(b)$, which is the existence problem of telescopers of type $(D_x, \Delta_{q,y})$ solved by Theorem 4.1.

(3) When $(\theta_y, \theta_z) = (\tau_{q,y}, \tau_{q,z})$, by Lemma 3.8 we know $L(x, D_x)(a) = q^{-jt}\tau_{q,y}^m \tau_{q,z}^{-n}(b) - b$ for some $b \in \mathbb{K}(x, y)[z]$ with $\deg_z(b) < \deg_z(d)$. Define an $\mathbb{F}$-homomorphism φ of $\mathbb{F}(y, z)$ by $y \mapsto y^m, z \mapsto y^{-n}z$. Then the q-difference equation can also be simplified.

Proposition 4.12. Let $f \in \mathbb{F}(y, z)$ be a rational function and $m, n, s \in \mathbb{N}$ integers with $m > 0$. Then $f = q^s \tau_{q,y}^m \tau_{q,z}^{-n}(g) - g$ for some $g \in \mathbb{F}(y, z)$ if and only if $\varphi(f) = q^s \tau_{q,y}(h) - h$ for some $h \in \mathbb{F}(y, z)$.

Proof. Let $\tau = \tau_{q,y}^m \tau_{q,z}^{-n}$. The necessity part follows from the fact that $\varphi \circ \tau = \tau_{q,y} \circ \varphi$. For the sufficiency, define $\psi : \mathbb{F}(y, z) \to \overline{\mathbb{F}(y, z)}$ by $y \mapsto y^{1/m}, z \mapsto y^{n/m}z$, where $\overline{\mathbb{F}(y, z)}$ is the algebraic closure of $\mathbb{F}(y, z)$. It is easy to see $\psi \circ \varphi = id_{\mathbb{F}(y,z)}$ and $\psi \circ \tau_{q,y} = \tau \circ \psi$, where $\tau_{q,y}$ and $\tau_{q,z}$ are extended to $\overline{\mathbb{F}(y, z)}$. Thus $\varphi(f) = q^s \tau_{q,y}(h) - h$ implies $f = q^s \tau_{q,y}^m \tau_{q,z}^{-n}(\tilde{g}) - \tilde{g}$ with $\tilde{g} = \psi(h) \in \overline{\mathbb{F}(y, z)}$. By similar trace arguments as those used in Lemma 3.4, one can see $f = q^s \tau_{q,y}^m \tau_{q,z}^{-n}(\tilde{g}) - \tilde{g}$ if and only if $f = q^s \tau_{q,y}^m \tau_{q,z}^{-n}(g) - g$ for some $g \in \mathbb{F}(y, z)$. ∎

At this stage, by letting $\bar{y} = y^{1/m}$ and $\bar{z} = y^{n/m}z$, we only need to decide whether $L(\bar{a}) = q^{-jt}\tau_{q,y}(\bar{b}) - \bar{b}$ for some $\bar{b} \in \mathbb{K}(x, y, z)$, which can be determined by a similar discussion process as the existence problem of telescopers of type $(D_x, \Delta_{q,y})$.

Example 4.13. Let $f = \frac{x+1}{y+z^2}$. Notice that $d = y + z^2$ is free of x and $\theta_y^m(d) \neq q^t\theta_z^n(d)$ whenever $m, n, t \in \mathbb{Z}$ and $m > 0$. It is easy to check that $L(f) = 0$ for $L = (x + 1)D_x - 1$, which means that f has a telescoper of type $(D_x, \Theta_y, \Theta_z)$. If we add a x to the denominator of f, the obtained function $\frac{x+1}{x+y+z^2}$ does not have a telescoper of type $(D_x, \Theta_y, \Theta_z)$ since $x + y + z^2$ is not free of x.

4.3. Existence criteria of the third class

We now consider the third class of the existence problems of telescopers for rational functions in three variables.

Problem 4.14. Given $f \in \mathbb{K}(x, y, z)$, decide whether there exists a nonzero operator L in $\mathbb{K}(x)\langle\partial_x\rangle$ with $\partial_x \in \{S_x, T_{q,x}\}$ such that $L(f) = D_y(g)+D_z(h)$ for some $g, h \in \mathbb{K}(x, y, z)$.

Let $f \in \mathbb{F}(y, z)$ be of the form (6) with $\mathbb{F} = \mathbb{K}(x)$. If f is (D_y, D_z)-exact in $\mathbb{K}(x, y, z)$, then 1 is a telescoper for f. From now on, we assume that f is not (D_y, D_z)-exact. Let $(\partial_x, \theta_x) \in \{(S_x, \sigma_x), (T_{q,x}, \tau_{q,x})\}$. By dividing the roots of b in $\overline{\mathbb{K}(x, y)}$ into different $\langle\theta_x\rangle$-orbits, we can write f as $f = D_z(u) + r$ with $u \in \mathbb{F}(y, z)$ and

$$r = \sum_{i=1}^{I}\sum_{j=0}^{J_i} \frac{\alpha_{i,j}}{z - \theta_x^j(\beta_i)}, \tag{20}$$

where $\alpha_{i,j}, \beta_i \in \overline{\mathbb{K}(x, y)}$ and the β_i's are in distinct $\langle\theta_x\rangle$-orbits. Note that f has a telescoper of type (∂_x, D_y, D_z) if and only if r has a telescoper of the same type.

Lemma 4.15. Let $r = \sum_{j=0}^{J} \alpha_j/(z - \theta_x^j(\beta))$ with $\alpha_j, \beta \in \overline{\mathbb{K}(x, y)}$ and $\theta_x^m(\beta) \neq \beta$ for any $m \in \mathbb{Z} \setminus \{0\}$. Then r is (D_y, D_z)-exact if it has a telescoper of type (∂_x, D_y, D_z).

Proof. Assume that $L = \sum_{\ell=0}^{\rho} e_\ell \partial_x^\ell \in \mathbb{K}(x)\langle\partial_x\rangle$ with $e_0 \neq 0$ is a telescoper for r of type (∂_x, D_y, D_z). Then

$$L(r) = \sum_{j=0}^{J+\rho} \frac{\tilde{\alpha}_j}{z - \theta_x^j(\beta)} = D_y(u) + D_z(v),$$

where $u, v \in \overline{\mathbb{K}(x,y)}(z)$ and $\tilde{\alpha}_j = \sum_{k=0}^{j} e_k \theta_x^k(\alpha_{j-k})$ with $e_k = 0$ for $k > \rho$ and $\alpha_j = 0$ for $j > J$. Since $\theta_x^m(\beta) \neq \beta$ whenever $m \in \mathbb{Z}\setminus\{0\}$, for each $1 \leq j \leq J+\rho$ we have $\tilde{\alpha}_j = D_y(\tilde{\gamma}_j)$ for some $\tilde{\gamma}_j \in \overline{\mathbb{K}(x,y)}$ by Lemma 3.5. We now prove inductively that for each j with $0 \leq j \leq J$, $\alpha_j = D_y(\gamma_j)$ for some $\gamma_j \in \overline{\mathbb{K}(x,y)}$. Since $\tilde{\alpha}_0 = e_0\alpha_0$ and $e_0 \in \mathbb{K}(x) \setminus \{0\}$, we have $\alpha_0 = D_y(\gamma_0)$ with $\gamma_0 = \tilde{\gamma}_0/e_0$. Suppose that we have shown that $\alpha_j = D_y(\gamma_j)$ for $j = 0, \ldots, k-1$ with $k \leq J$. Note that $\tilde{\alpha}_k = e_0\alpha_k + e_1\theta_x(\alpha_{k-1}) + \cdots + e_k\theta_x^k(\alpha_0) = D_y(\tilde{\gamma}_k)$. Then $\alpha_k = D_y(\gamma_k)$ with $\gamma_k = \frac{1}{e_0}(\tilde{\gamma}_k - \sum_{j=1}^{k} e_j \theta_x^j(\gamma_{k-j}))$. So r is (D_y, D_z)-exact by Lemma 3.5. ∎

Theorem 4.16. Let $r \in \mathbb{K}(x,y,z)$ be of the form (20). Then r has a telescoper of type (∂_x, D_y, D_z) if and only if for each i with $1 \leq i \leq I$, either $\alpha_{i,j}/(z - \theta_x^j(\beta_i))$ is (D_y, D_z)-exact or $\beta_i \in \overline{\mathbb{K}(y)}$ and there exists a nonzero $L_{i,j} \in \mathbb{K}(x)\langle\partial_x\rangle$ such that $L_{i,j}(\alpha_{i,j}) = D_y(\gamma_{i,j})$ for some $\gamma_{i,j} \in \mathbb{K}(x,y)(\beta_i)$.

Proof. The sufficiency part follows from Lemma 2.6 since each fraction $\alpha_{i,j}/(z - \theta_x^j(\beta_i))$ is either (D_y, D_z)-exact or has a telescoper of type (∂_x, D_y, D_z). To show the necessity part, we assume that $L = \sum_{\ell=0}^{\rho} e_\ell \partial_x^\ell \in \mathbb{K}(x)\langle\partial_x\rangle$ with $e_0 \neq 0$ is a telescoper for r of type (∂_x, D_y, D_z). Then we have

$$L(r) = \sum_{i=1}^{I} \sum_{j=0}^{J_i+\rho} \frac{\tilde{\alpha}_{i,j}}{z - \theta_x^j(\beta_i)} = D_y(u) + D_z(v),$$

where $u, v \in \mathbb{K}(x,y,z)$ and $\tilde{\alpha}_{i,j} = \sum_{k=0}^{j} e_k \theta_x^k(\alpha_{i,j-k})$ with $e_k = 0$ for $k > \rho$ and $\alpha_{i,j} = 0$ for $j > J_i$. By Lemma 3.5, we have $r_i = \sum_{j=0}^{J_i+\rho} \frac{\tilde{\alpha}_{i,j}}{z - \theta_x^j(\beta_i)}$ is (D_y, D_z)-exact for each i with $1 \leq i \leq I$ since the β_i's are in distinct $\langle\theta_x\rangle$-orbits. If there exists a nonzero $m_i \in \mathbb{N}$ such that $\theta_x^{m_i}(\beta_i) = \beta_i$, then $\beta_i \in \overline{\mathbb{K}(y)}$ by [22, Lemma 3.4 (i)]. So $J_i = 0$ and $L(\alpha_{i,0}/(z - \beta_i)) = L(\alpha_{i,0})/(z - \beta_i)$ is (D_y, D_z)-exact, which implies that $L(\alpha_{i,0}) = D_y(\gamma_{i,0})$ for some $\gamma_{i,0} \in \overline{\mathbb{K}(x,y)}$. Since $\alpha_{i,0} \in \mathbb{K}(x,y)(\beta_i)$, we can choose $\gamma_{i,0} \in \mathbb{K}(x,y)(\beta_i)$ by the trace argument. If there is no nonzero $m_i \in \mathbb{N}$ such that $\theta_x^{m_i}(\beta_i) = \beta_i$, then the theorem follows from Lemma 4.15. ∎

Problem 4.14 now has been reduced to the exactness testing problem and the following existence problem.

Problem 4.17. Given $\alpha \in \mathbb{K}(x,y)(\beta)$ with β algebraic over $\mathbb{K}(y)$, decide whether α has a telescoper of type (∂_x, D_y) with $\partial_x \in \{S_x, T_{q,x}\}$, i.e., there exists a nonzero $L \in \mathbb{K}(x)\langle\partial_x\rangle$ such that $L(\alpha) = D_y(\gamma)$ for some $\gamma \in \mathbb{K}(x,y)(\beta)$.

In order to solve the above problem, we first present a vector version of the Hermite-like reduction in [29]. Let $\vec{a} = \frac{1}{d}(a_1, \ldots, a_n) \in \mathbb{K}(x,y)^n$ with $a_i, d \in \mathbb{K}[x,y]$ satisfying that $\gcd(d, a_1, \ldots, a_n) = 1$ and $\mathbf{B} = \frac{1}{e}(b_{i,j}) \in \mathbb{K}(x,y)^{n\times n}$ with $e, b_{i,j} \in \mathbb{K}[x,y]$ such that $\gcd(e, b_{1,1}, \ldots, b_{1,n}, \ldots, b_{n,n}) = 1$. Let $p \in \mathbb{K}[x,y]$ be any irreducible factor of d

that is coprime with e. Then $d = p^m d_1$ with $d_1 \in \mathbb{K}[x, y]$ and $\gcd(p, d_1) = 1$. Since $\gcd(p, D_y(p)) = 1$, we have $\gcd(p, D_y(p)d_1) = 1$ and then the Bézout relation

$$a_i = s_i p + t_i D_y(p) d_1,$$

where $s_i, t_i \in \mathbb{K}(x)[y]$. Using integration by parts, we get

$$\frac{a_i}{p^m d_1} = \frac{s_i p + t_i D_y(p) d_1}{p^m d_1} = D_y\left(\frac{u_i}{p^{m-1}}\right) + \frac{v_i}{p^{m-1} d_1},$$

where $u_i = t_i(1-m)^{-1}$ and $v_i = s_i - (1-m)^{-1} D_y(t_i) d_1$. Let $\vec{u} = (u_1, \ldots, u_n)$ and $\vec{v} = (v_1, \ldots, v_n)$. Then we have

$$\vec{a} = D_y\left(\frac{\vec{u}}{p^{m-1}}\right) + \frac{\vec{v}}{p^{m-1} d_1} = D_y\left(\frac{\vec{u}}{p^{m-1}}\right) + \frac{\vec{u}}{p^{m-1}} \cdot \mathbf{B} + \frac{\vec{w}}{p^{m-1} d_1 e},$$

where $\vec{w} \in \mathbb{K}(x)[y]^n$. This process of multiplicity reduction yields

$$\vec{a} = D_y\left(\frac{\vec{g}}{p^{m-1}}\right) + \frac{\vec{g}}{p^{m-1}} \cdot \mathbf{B} + \frac{\vec{h}}{p d_1 e},$$

where $\vec{g}, \vec{h} \in \mathbb{K}(x)[y]^n$. By reducing the multiplicity of each irreducible factor of d that is coprime with e in the above way, we obtain the additive decomposition

$$\vec{a} = D_y(\vec{b}) + \vec{b} \cdot \mathbf{B} + \vec{r}, \tag{21}$$

where $\vec{b} \in \mathbb{K}(x, y)^n$ and $\vec{r} = \frac{1}{pc}(r_1, \ldots, r_n)$ with $r_i \in \mathbb{K}(x)[y]$ and $p, c \in \mathbb{K}[x, y] \setminus \{0\}$ be such that p is a squarefree polynomial and $\gcd(p, e) = 1$ and each irreducible factor of c divides e. We call the above process a *vector Hermite reduction* of $\vec{a}$ with respect to $\mathbf{B}$.

Let $\beta \in \overline{\mathbb{K}(y)}$ and $n = [\mathbb{K}(y, \beta) : \mathbb{K}(y)]$. Assume that $\{\beta_1, \ldots, \beta_n\}$ is a basis for $\mathbb{K}(y, \beta)$ as a linear space over $\mathbb{K}(y)$. Since $D_y(\beta_i) \in \mathbb{K}(y, \beta)$, we have $D_y(\beta_i) = \frac{1}{e}\sum_{j=1}^n b_{j,i}\beta_j$ with $e, b_{j,i} \in \mathbb{K}[y]$. Set $\mathbf{B} = \frac{1}{e}(b_{i,j}) \in \mathbb{K}(y)^{n\times n}$. Then $D_y(\vec{\beta}) = \vec{\beta} \cdot \mathbf{B}$ with $\vec{\beta} = (\beta_1, \ldots, \beta_n)$. Since $\alpha \in \mathbb{K}(x, y)(\beta)$, we can write $\alpha = \vec{a} \cdot \vec{\beta}^T$ for some $\vec{a} = \frac{1}{d}(a_1, \ldots, a_n) \in \mathbb{K}(x, y)^n$ with $d, a_i \in \mathbb{K}[x, y]$. Applying the vector Hermite reduction to $\vec{a}$ with respect to $\mathbf{B}$ yields the additive decomposition (21), which is equivalent to

$$\alpha = D_y(\vec{b} \cdot \vec{\beta}^T) + \tilde{\alpha} \text{ with } \tilde{\alpha} = \frac{1}{pc}\sum_{i=1}^n r_i \beta_i, \tag{22}$$

where $r_i, p, c \in \mathbb{K}[x, y]$ with p being squarefree and $\gcd(p, e) = 1$ and each irreducible factor of c divides $e \in \mathbb{K}[y]$.

Theorem 4.18. Let $\alpha \in \mathbb{K}(x, y)(\beta)$ be of the form (22). Then α has a telescoper of type (∂_x, D_y) if and only if the polynomial p in (22) is split in x and y.

Proof. Assume that p is split in x and y, i.e., $p = p_1 p_2$ for some $p_1 \in \mathbb{K}[x]$ and $p_2 \in \mathbb{K}[y]$. Then $\tilde{\alpha}$ can be written as $\tilde{\alpha} = \sum_{j=1}^m f_j \cdot g_j$ with $f_j \in \mathbb{K}(x)$ and $g_j \in \mathbb{K}(y)(\beta)$ since $\beta_i \in \mathbb{K}(y)(\beta)$ and $c \in \mathbb{K}[y]$. Let $L_j = f_j(x)\partial_x - \theta_x(f_j) \in \mathbb{K}(x)\langle\partial_x\rangle$ for each $1 \le j \le m$. Then $L_j(f_j \cdot g_j) = 0$. So the LCLM of the L_j's annihilates $\tilde{\alpha}$, which then is a telescoper for α of type (∂_x, D_y). To show the necessity, we assume that $L = \sum_{\ell=0}^{\rho} e_\ell \partial_x^\ell \in \mathbb{K}(x)\langle\partial_x\rangle$

with $e_0 e_\rho \neq 0$ is a telescoper for α of type (∂_x, D_y). Then $L(\tilde{\alpha}) = D_y(\tilde{\gamma})$ for some $\tilde{\gamma} \in \mathbb{K}(x,y)(\beta)$. Write $\tilde{\gamma} = \vec{s} \cdot \vec{\beta}^T$ with $\vec{s} \in \mathbb{K}(x,y)^n$ and $\vec{r} = (r_1, \ldots, r_n)$. Then we have

$$L\left(\frac{1}{pc}\vec{r}\right) = \sum_{\ell=0}^{\rho} \frac{e_\ell}{\theta_x^\ell(p)c}\theta_x^\ell(\vec{r}) = D_y(\vec{s}) + \vec{s} \cdot \mathbf{B}.$$

Suppose that p is not split in x and y. Then there exists a non-split irreducible factor p_0 of p such that $\theta_x(p_0) \nmid p$. Then $\theta_x^\rho(p_0)$ is also a non-split irreducible polynomial and only divides the denominator $\theta_x^\rho(p)c$. Since p is squarefree, the valuation of the left-hand side of the above equality at $\theta_x^\rho(p_0)$ is -1. However, the valuation of the right-hand side is either ≥ 0 or < -1 since $\mathbf{B} \in \mathbb{K}(y)^{n \times n}$. This leads to a contradiction. So p is split in x and y. ∎

Example 4.19. Let $f = x/(z^2 - y)$. Then

$$f = \frac{\alpha}{z - \beta} + \frac{-\alpha}{z + \beta},$$

where $\alpha = x/(2\sqrt{y})$ and $\beta = \sqrt{y}$. By Theorem 4.16, f has a telescoper of type (∂_x, D_y, D_z) since $\beta \in \overline{\mathbb{K}(y)}$ and $L(\alpha) = 0$ for $L = x\partial_x - \theta_x(x)$. Hence,

$$L(f) = \frac{L(\alpha)}{z - \beta} + \frac{L(-\alpha)}{z + \beta} = D_y(0) + D_z(0).$$

Example 4.20. Let $f = x/((x+y)(z^2 - y))$. Then

$$f = \frac{\alpha}{z - \beta} + \frac{-\alpha}{z + \beta},$$

where $\alpha = x/(2\sqrt{y}(x+y))$ and $\beta = \sqrt{y}$. Since $x + y$ is not split in x and y, Theroem 4.16 and Theorem 4.18 imply f does not have any telescoper of type (∂_x, D_y, D_z).

4.4. Existence criteria of the fourth class

We continue to address the fourth class of the existence problems of telescopers for rational functions in three variables. There are four cases in this class.

Problem 4.21. Let $\partial_x \in \{S_x, T_{q,x}\}$ and $\Theta_y \in \{\Delta_y, \Delta_{q,y}\}$. Given $f \in \mathbb{K}(x,y,z)$, decide whether there exists a nonzero operator $L \in \mathbb{K}(x)\langle \partial_x \rangle$ such that $L(f) = \Theta_y(g) + D_z(h)$ for some $g, h \in \mathbb{K}(x,y,z)$.

Let $(\partial_x, \theta_x) \in \{(S_x, \sigma_x), (T_{q,x}, \tau_{q,x})\}$ and $(\Theta_y, \theta_y) \in \{(\Delta_y, \sigma_y), (\Delta_{q,y}, \tau_{q,y})\}$. By the Ostrogradsky–Hermite reduction in z and the reduction formula (7) with $\theta = \theta_y$, we can decompose f as

$$f = \Theta_y(u) + D_z(v) + r, \text{ where } r = \sum_{i=1}^{I} \sum_{j=0}^{J_i} \frac{a_{i,j}}{\theta_x^j(d_i)} \tag{23}$$

with $a_{i,j} \in \mathbb{K}(x,y)[z]$ and $d_i \in \mathbb{K}[x,y,z]$ satisfying the condition $\deg_z(a_{i,j}) < \deg_z(d_i)$ and the d_i's are irreducible polynomials in distinct $\langle \theta_x, \theta_y \rangle$-orbits. Note that f has a telescoper of type $(\partial_x, \Theta_y, D_z)$ if and only if r does.

Lemma 4.22. Let $r \in \mathbb{K}(x, y, z)$ be as in (23). Then r has a telescoper of type $(\partial_x, \Theta_y, D_z)$ if and only if for each i with $1 \leq i \leq I$, the function $r_i = \sum_{j=0}^{J_i} \frac{a_{i,j}}{\theta_x^j(d_i)}$ has a telescoper of the same type.

Proof. The sufficiency part follows from Lemma 2.6. For the necessity, we assume that $L = \sum_{k=0}^{\rho} \ell_k \partial_x^k \in \mathbb{K}(x)\langle \partial_x \rangle$ with $\partial_x \in \{S_x, T_{q,x}\}$ and $\ell_0 \neq 0$ is a telescoper for r of type $(\partial_x, \Theta_y, D_z)$. Then

$$L(r) = \sum_{i=1}^{I} L(r_i) = \sum_{i=1}^{I} \left(\sum_{j=0}^{J_i+\rho} \frac{\sum_{k=0}^{j} \ell_k \theta_x^k(a_{i,j-k})}{\theta_x^j(d_i)} \right)$$

is (Θ_y, D_z)-exact, where $\ell_k = 0$ if $k > \rho$ and $a_{i,j} = 0$ if $j > J_i$. Since the d_i's are in distinct $\langle \theta_x, \theta_y \rangle$-orbits, $\theta_x^j(d_i)$ and $\theta_x^{j'}(d_{i'})$ are in distinct $\langle \theta_y \rangle$-orbits for any $j, j' \in \mathbb{Z}$ and $1 \leq i, i' \leq I$ with $i \neq i'$. By Lemma 3.9, we have $L(r_i)$ is (Θ_y, D_z)-exact for each i with $1 \leq i \leq I$. So each r_i has a telescoper of the same type. ∎

Now the existence problem is reduced to that for rational functions of the form

$$f = \sum_{i=0}^{I} \frac{a_i}{\theta_x^i(d)}, \tag{24}$$

where $a_i \in \mathbb{K}(x, y)[z], d \in \mathbb{K}[x, y, z]$ with $\deg_z(a_i) < \deg_z(d)$ and d is irreducible in z over $\mathbb{K}(x, y)$. We will proceed by a case distinction according to whether or not d satisfies the condition: there exist $c \in \mathbb{K} \setminus \{0\}$ and integers m, n with $m > 0$ such that

$$\theta_x^m(d) = c \cdot \theta_y^n(d). \tag{25}$$

Note that the constant c in (25) must be 1 if $(\theta_x, \theta_y) \in \{(\sigma_x, \sigma_y), (\sigma_x, \tau_{q,y})\}$ by the comparison of leading coefficients. When $(\theta_x, \theta_y) = (\tau_{q,x}, \sigma_y)$, we have $\tau_{q,x}^m(d) = c \cdot \sigma_y^n(d)$ which implies $d = x^{m_0} \cdot d_1$ for some $m_0 \in \mathbb{Z}$ and $d_1 \in \mathbb{K}(y, z)$. Then it is easy to see that $c = q^{mm_0}$. When $(\theta_x, \theta_y) = (\tau_{q,x}, \tau_{q,y})$, we claim that $c = q^s$ for some $s \in \mathbb{Z}$. To show this claim, we write $d = \sum_{i,j,k} c_{i,j,k} x^i y^j z^k$. Then the equality $\tau_{q,x}^m(d) = c \cdot \tau_{q,y}^n(d)$ implies that for all i, j, we have $c = q^{im-jn}$. Let $s = \gcd(m, n)$. Then $m = s\bar{m}$ and $n = s\bar{n}$. For different pairs (i_1, j_1) and (i_2, j_2) with $q^{i_1 m - j_1 n} = q^{i_2 m - j_2 n}$, we have $i_1 m - j_1 n = i_2 m - j_2 n$ since q is not a root of unity, which further implies that $(i_2, j_2) = (i_1, j_1) + \lambda(\bar{n}, \bar{m})$ for some nonzero $\lambda \in \mathbb{Z}$. Thus $d = x^{i_0} y^{j_0} \bar{d}$, where $i_0, j_0 \in \mathbb{Z}$ and $\bar{d} = \sum_{k=0}^{\rho} d_k(x^{\bar{n}} y^{\bar{m}}) z^k$ with $\rho \in \mathbb{N}$ and the d_k's being univariate polynomials over $\mathbb{K}$. Since $\tau_{q,x}^m(\bar{d}) = \tau_{q,y}^n(\bar{d})$, we have $c = q^{i_0 m - j_0 n}$. Combining the above discussions with [12, Proposition 1] yields a characterization of polynomials satisfying the condition (25).

Lemma 4.23. Let $d = \sum_{i=0}^{\rho} d_i z^i \in \mathbb{K}(x, y)[z]$ be a polynomial in z over $\mathbb{K}(x, y)$. If there exist $c \in \mathbb{K} \setminus \{0\}$ and $m, n \in \mathbb{Z}$ with $m > 0$ such that $\theta_x^m(d) = c \cdot \theta_y^n(d)$, then for each i with $0 \leq i \leq \rho$ we have

(1) if $(\theta_x, \theta_y) = (\sigma_x, \sigma_y)$, then $c = 1$ and d_i is integer-linear in x and y, i.e., $d_i = f(nx + my)$ for some $f \in \mathbb{K}(t)$;
(2) if $(\theta_x, \theta_y) = (\sigma_x, \tau_{q,y})$, then $c = 1$ and $d_i \in \mathbb{K}(y)$. If $n \neq 0$, we have $d_i \in \mathbb{K}$;
(3) if $(\theta_x, \theta_y) = (\tau_{q,x}, \sigma_y)$, then $d_i = x^{m_0} \cdot f_i(y)$ for some $m_0 \in \mathbb{Z}$ and $f_i(y) \in \mathbb{K}(y)$. If $n \neq 0$, we have $d_i = c_i \cdot x^{m_0}$ for some $c_i \in \mathbb{K}$;

(4) if $(\theta_x, \theta_y) = (\tau_{q,x}, \tau_{q,y})$, then $c = q^s$ for some $s \in \mathbb{Z}$ and d_i is q-integer-linear in x and y, i.e., $d_i = x^{n_0} y^{m_0} f_i(x^n y^m)$ for some $f_i \in \mathbb{K}(t)$ and $n_0, m_0 \in \mathbb{Z}$.

By the above characterization, the condition (25) can be checked by solving the bivariate case of Problems 4.3 and 4.4 in the pure shift and q-shift cases, respectively.

Lemma 4.24. Let $f \in \mathbb{K}(x, y, z)$ be of the form (24) and d does not satisfy the condition (25). Then f has a telescoper of type $(\partial_x, \Theta_y, D_z)$ if and only if f is (Θ_y, D_z)-exact.

Proof. The sufficiency is clear by definition. Assume that $L = \sum_{k=0}^{\rho} \ell_k \partial_x^k$ with $\ell_0 \neq 0$ is a telescoper for f of type $(\partial_x, \Theta_y, D_z)$. Then we have that

$$L(f) = \sum_{i=0}^{\rho+I} \left(\frac{\sum_{j=0}^{i} \ell_j \theta_x^j (a_{i-j})}{\theta_x^i(d)} \right)$$

is (Δ_y, D_z)-exact, where $\ell_j = 0$ if $j > \rho$ and $a_i = 0$ if $i > I$. Since d does not satisfy the condition (25), we have $\theta_x^i(d)$ and $\theta_x^{i'}(d)$ in distinct $\langle \theta_y \rangle$-orbits for all $i \neq i'$. By Lemma 3.9, for any i with $0 \leq i \leq \rho + I$, there exist $u_i, v_i \in \mathbb{K}(x, y, z)$ such that

$$\frac{\sum_{j=0}^{i} \ell_j \theta_x^j (a_{i-j})}{\theta_x^i(d)} = \Theta_y(u_i) + D_z(v_i). \tag{26}$$

To show that all fractions $a_i/\theta_x^i(d)$ are (Θ_y, D_z)-exact, we proceed by induction. The assertion is true for $i = 0$ since $a_0/d = \Theta_y(u_0/\ell_0) + D_z(v_0/\ell_0)$. Suppose that we have shown that $a_i/\theta_x^i(d)$ is (Θ_y, D_z)-exact for $i = 0, \ldots, s-1$ with $s \leq I$. By the equality (26) with $i = s$, we get

$$\frac{a_s}{\theta_x^s(d)} = \Theta_y \left(\frac{u_s}{\ell_0} \right) + D_z \left(\frac{v_s}{\ell_0} \right) - \sum_{j=1}^{s} \frac{\ell_j}{\ell_0} \theta_x^j \left(\frac{a_{s-j}}{\theta_x^{s-j}(d)} \right).$$

Since θ_x, θ_y and δ_z commute, by Lemma 3.9, we have $a/\theta_x^i(d)$ is (Θ_y, D_z)-exact for any $i \in \mathbb{N}$ if a/d is. By the induction hypothesis, we have $\frac{\ell_j}{\ell_0} \theta_x^j (a_{s-j}/\theta_x^{s-j}(d))$ is (Θ_y, D_z)-exact for all $1 \leq j \leq s$. Then so are $a_s/\theta_x^s(d)$ and f. ∎

We now deal with the case in which d satisfies condition (25). From now on, we will always assume that m is the smallest such positive integer such that $\theta_x^m(d) = c \cdot \theta_y^n(d)$ for some $n \in \mathbb{Z}$ and $c \in \mathbb{K} \setminus \{0\}$. By the reduction formula (7) with $\theta = \theta_y$, the existence problem is further reduced to that for rational functions of the form

$$f = \sum_{i=0}^{m-1} \frac{a_i}{\theta_x^i(d)}, \tag{27}$$

where $a_i \in \mathbb{K}(x, y)[z], d \in \mathbb{K}[x, y, z]$ with $\deg_z(a_i) < \deg_z(d)$ and d is irreducible in z over $\mathbb{K}(x, y)$.

The following lemma is similar to Lemma 5.3 in [18].

Lemma 4.25. Let $f \in \mathbb{K}(x, y, z)$ be of the form (27) and let d satisfy the condition (25). Then f has a telescoper of type $(\partial_x, \Theta_y, D_z)$ if and only if for each i with $0 \leq i \leq I$, the fraction $a_i/\theta_x^i(d)$ has a telescoper of the same type.

Proof. The sufficiency follows from Lemma 2.6. For the necessity direction, one can adapt the second part of the proof of [18, Lemma 5.3] to the setting of telescopers of type $(\partial_x, \Theta_y, D_z)$ literally by interpreting $\equiv_{y,z} 0$ as being (Θ_y, D_z)-exact. ∎

The above lemma further reduces the existence problem to that for simple fractions of the form

$$f = \frac{a}{bd}, \tag{28}$$

where $a, d \in \mathbb{K}[x, y, z], b \in \mathbb{K}[x, y]$ satisfy that $\gcd(a, bd) = 1$ and $\deg_z(a) < \deg_z(d)$, and d is irreducible and satisfies the condition (25). We will consider two cases according to whether d is in $\mathbb{K}[x, z]$ or not. If $d \in \mathbb{K}[x, z]$, then $\theta_y^i(d) = d$ for all $i \in \mathbb{N}$. The condition $\theta_x^m(d) = c \cdot \theta_y^n(d)$ implies that d is also free of x, i.e., $d \in \mathbb{K}[z]$. Thus $L \in \mathbb{K}(x)\langle\partial_x\rangle$ is a telescoper for f of type $(\partial_x, \Theta_y, D_z)$ if and only if $L(a/b) = \Theta_y(u)$ for some $u \in \mathbb{K}(x, y)[z]$ with $\deg_z(u) < \deg_z(d)$. Write $a = \sum_{i=0}^{\deg_z(d)-1} a_i z^i$ and $u = \sum_{i=0}^{\deg_z(d)-1} u_i z^i$. Then for each i with $0 \le i \le \deg_z(d) - 1$, we have $L(a_i/b) = \Theta_y(u_i)$, i.e., L is a telescoper for all a_i/b of type (∂_x, Θ_y). The existence problem is then reduced to that in the bivariate case, for which Theorem 4.1 applies. So it remains to deal with the case when d is not in $\mathbb{K}[x, z]$.

Lemma 4.26. Let $\tau := \theta_x^m \theta_y^{-n}$ with $m, n \in \mathbb{Z}$ and $m > 0$ and let $p \in \mathbb{K}[x, y]$ be an irreducible polynomial. If $\tau^k(p) = \lambda \cdot p$ for some nonzero $k \in \mathbb{Z}$ and nonzero $\lambda \in \mathbb{K}$, then $\tau(p) = \mu \cdot p$ for some nonzero $\mu \in \mathbb{K}$.

Proof. Write $p = \sum_{i,j} p_{i,j} x^i y^j$ with $p_{i,j} \in \mathbb{K}$. We will proceed by case distinctions.

(1) If $(\theta_x, \theta_y) = (\sigma_x, \sigma_y)$, then $\tau^k(p) = \lambda \cdot p$ implies that $\lambda = 1$ by comparing the leading coefficients. So $\sigma_x^{km}(p) = \sigma_y^{kn}(p)$. By Lemma 4.23, we have $p = r(knx + kmy)$ for some $r = \sum_{j=0}^{s} r_j z^j \in \mathbb{K}[z]$. Thus $p = \tilde{r}(nx + my)$ with $\tilde{r} = \sum_{j=0}^{s} r_j k^j z^j$, which implies that $\tau(p) = p$.

(2) If $(\theta_x, \theta_y) = (\sigma_x, \tau_{q,y})$, then $\tau^k(p) = \lambda \cdot p$ implies that $p \in \mathbb{K}[y]$ and moreover $p = c \cdot y$ for some $c \in \mathbb{K}$ if $n \neq 0$ by [13, Lemma 5.4], which leads to $\tau(p) = \mu \cdot p$ with $\mu = q^{-n}$.

(3) If $(\theta_x, \theta_y) = (\tau_{q,x}, \sigma_y)$, by [13, Lemma 5.4], $\tau^k(p) = \lambda \cdot p$ implies that $p \in \mathbb{K}[y]$ or $p = cx$ for some $c \in \mathbb{K}$. In the former case, we have $n = 0$ or $p \in \mathbb{K}$, hence $\tau(p) = p$. In the latter case, $\tau(p) = cq^m x = \mu \cdot p$ with $\mu = q^m$.

(4) If $(\theta_x, \theta_y) = (\tau_{q,x}, \tau_{q,y})$, then $\tau^k(p) = \lambda \cdot p$ implies that $p = (x^s y^t) \cdot r(x^{kn} y^{km})$ for some $s, t \in \mathbb{Z}$ and $r \in \mathbb{K}[z]$ by [27, Lemma 5.2]. So we have $\tau(p) = \mu \cdot p$ with $\mu = q^{sm-nt}$. This completes the proof. ∎

Lemma 4.27. Let $\tau := \theta_x^m \theta_y^{-n}$ with $m, n \in \mathbb{Z}$ and $m > 0$ and let $f = a/b$ with $a, b \in \mathbb{K}[x, y]$ and $\gcd(a, b) = 1$. If there exist $e_0, \ldots, e_r \in \mathbb{K}(x)$, not all zero, such that $\sum_{i=0}^{r} e_i \tau^i(f) = 0$, then $b = b_1 b_2$ with $b_1 \in \mathbb{K}[x]$ and $b_2 \in \mathbb{K}[x, y]$ satisfying that $\tau(b_2) = \lambda \cdot b_2$ for some nonzero $\lambda \in \mathbb{K}$.

Proof. Assume that $\sum_{i=0}^{r} e_i \tau^i(f) = 0$. Let b_1 and b_2 be the content and primitive part of b as a polynomial in y over $\mathbb{K}[x]$. If b_2 is a constant in $\mathbb{K}$, then the assertion holds since $\tau(b_2) = b_2$. We now assume that $b_2 \notin \mathbb{K}$. Then all of its irreducible factors have positive degree in y. Assume that there exists an irreducible factor p of b_2 such that $\tau(p) \neq c \cdot p$ for any $c \in \mathbb{K}$. Then for any integers $i \neq 0$, $\tau^i(p) \neq c_i \cdot p$ for any $c_i \in \mathbb{K}$ by Lemma 4.26.

Among all of such irreducible factors, we can always find one factor p such that $\tau^i(p) \nmid b_2$ for all integer $i < 0$. Then $\tau^i(p)$ is also irreducible for all $i \in \mathbb{Z}$ and $\gcd(\tau^i(p), \tau^j(p)) = 1$ if $i \neq j$. Let s be the largest integer such that $\tau^s(p) \mid b_2$. Then the irreducible polynomial $\tau^{r+s}(p)$ only divides the r-th transform $\tau^r(b)$ of b and not others, which implies that $\sum_{i=0}^{r} e_i \tau^i(f) \neq 0$ since p depends on y and the coefficients e_i are in $\mathbb{K}(x)$. This leads to a contradiction. So for each irreducible factor p of b_2 we have $\tau(p) = c \cdot p$ for some $c \in \mathbb{K}$. This implies that $\tau(b_2) = \lambda \cdot b_2$ for some $\lambda \in \mathbb{K}$. ∎

Lemma 4.28. Let $a \in \mathbb{K}(x)[y, z]$ and $b \in \mathbb{K}[x, y, z]$ be such that $b \neq 0$ and $\theta_x^m(b) = c \cdot \theta_y^n(b)$ for some $c \in \mathbb{K} \setminus \{0\}$ and $m, n \in \mathbb{Z}$ with $m > 0$. Then a/b has a telescoper of type $(\partial_x, \Theta_y, D_z)$.

Proof. Set $f = a/b$. It suffices to show that for sufficiently large $I \in \mathbb{N}$, there exist $\ell_0, \ldots, \ell_I \in \mathbb{K}(x)$, not all zero, and $g \in \mathbb{K}(x, y, z)$ such that $L(f) = \Theta_y(g)$ with $L = \sum_{i=0}^{I} \ell_i \partial_x^{im}$. By the reduction formula (7) with $\theta = \theta_y$, we have

$$\theta_x^{im}(f) = \frac{\theta_x^{im}(a)}{\theta_x^{im}(b)} = \frac{\theta_x^{im}(a)}{c^i \cdot \theta_y^{in}(b)} = \Theta_y(g_i) + \frac{\theta_y^{-in}\theta_x^{im}(a)}{c^i \cdot b}$$

for some $g_i \in \mathbb{K}(x, y, z)$. Note that the degrees of the polynomials $\theta_y^{-in}\theta_x^{im}(a)$ in y and z are the same as those of a. So all of the polynomials $\theta_y^{-in}\theta_x^{im}(a)$ lie in a finite dimensional linear space over $\mathbb{K}(x)$. Therefore, for sufficiently large I, there exist $\ell_0, \ldots, \ell_I \in \mathbb{K}(x)$, not all zero, such that $\sum_{i=0}^{I} \ell_i \theta_y^{-in}\theta_x^{im}(a) = 0$. This implies that L is a telescoper for f of type $(\partial_x, \Theta_y, D_z)$. ∎

Theorem 4.29. Let $f \in \mathbb{K}(x, y, z)$ be of the form (28). Assume that d is not in $\mathbb{K}[x, z]$. Then f has a telescoper of type $(\partial_x, \Theta_y, D_z)$ if and only if $b = b_1 b_2$ for some $b_1 \in \mathbb{K}[x]$ and $b_2 \in \mathbb{K}[x, y]$ satisfying $\theta_x^m(b_2) = \lambda \cdot \theta_y^n(b_2)$ for some nonzero $\lambda \in \mathbb{K}$.

Proof. The sufficiency part follows from Lemma 4.28. For the necessity, we assume that $L \in \mathbb{K}(x)\langle \partial_x \rangle$ is a telescoper for f of type $(\partial_x, \Theta_y, D_z)$. Write $L = L_0 + L_1 + \cdots + L_{m-1}$ with $L_i = \sum_{j=0}^{r_i} \ell_{i,j} \partial_x^{jm+i}$. Since $\theta_x^i(d)$ and $\theta_x^j(d)$ are in distinct $\langle \theta_y \rangle$-orbits for all $0 \leq i \neq j \leq m-1$, Lemma 3.9 implies that L_i is also a telescoper for f of the same type for each i with $0 \leq i \leq m-1$. A direct calculation yields

$$L_0(f) = \Theta_y(g_0) + \frac{A}{d},$$

where $g_0 \in \mathbb{K}(x, y, z)$, $A = \sum_{j=0}^{r_0} c^{-j} \ell_{0,j} \tau^j(a/b)$ with $\tau = \theta_x^m \theta_y^{-n}$ and $\tau(d) = c \cdot d$. By Lemma 3.9, we have $A = 0$ since $d \notin \mathbb{K}[x, z]$. Necessity follows from Lemma 4.27. ∎

Example 4.30. Let $f = \frac{x+z}{(x+y)(x+y+z^2)^2}$. Note that $d = x + y + z^2 \in \mathbb{K}[x, y, z] \setminus \mathbb{K}[x, z]$ satisfies the condition $\sigma_x(d) = \sigma_y(d)$. Since $b = x + y$ satisfies the same condition as d, Theorem 4.29 implies that f has a telescoper of type (S_x, Δ_y, D_z). In fact $L = -xS_x + (1 + x)$ is a telescoper for f since

$$L(f) = \Delta_y(r_1 \cdot f) + D_z(r_2 \cdot f) \text{ with } r_1 = -\frac{x + x^2 + xz}{x + z} \text{ and } r_2 = -\frac{x + y + z^2}{2(x + z)}.$$

Example 4.31. Let $f = \frac{x+z}{(2x+y)(x+y+z^2)^2}$. Since $b_1 = 2x + y$ does not satisfy the same condition as $d = x + y + z^2$, we know from Theorem 4.29 that f does not have any telescoper of type (S_x, Δ_y, D_z).

4.5. Existence criteria of the fifth class

We now consider the fifth class of existence problems in which both telescopers and (Θ_y, Θ_z) are involving (q-)shift operators. More precisely, we solve the following problem.

Problem 4.32. Let $\partial_x \in \{S_x, T_{q,x}\}$ and $(\Theta_y, \Theta_z) \in \{(\Delta_y, \Delta_z), (\Delta_{q,y}, \Delta_z), (\Delta_{q,y}, \Delta_{q,z})\}$. Given $f \in \mathbb{K}(x, y, z)$, determine if there exists a nonzero operator $L \in \mathbb{K}(x)\langle\partial_x\rangle$ such that $L(f) = \Theta_y(g) + \Theta_z(h)$ for some $g, h \in \mathbb{K}(x, y, z)$.

For $v \in \{x, y, z\}$, let $\theta_v = \sigma_v$ if $\Theta_v = \Delta_v$ or $\theta_v = \tau_{q,v}$ if $\Theta_v = \Delta_{q,v}$. By partial fraction decomposition w.r.t z and the transformation (7) with $\theta = \theta_y$ and subsequently with $\theta = \theta_z$, any rational function $f \in \mathbb{K}(x, y, z)$ can be decomposed into

$$f = \Theta_y(u) + \Theta_z(v) + \mu + \sum_{i=1}^{I} \sum_{j=1}^{J_i} \sum_{\ell=0}^{t_{i,j}} \frac{a_{i,j,\ell}}{\theta_x^\ell d_i^j}, \tag{29}$$

where $u, v \in \mathbb{K}(x, y, z), \mu \in \mathbb{K}(x, y), a_{i,j,\ell} \in \mathbb{K}(x, y)[z], d_i \in \mathbb{K}[x, y, z]$ with $\deg_z(a_{i,j,\ell}) < \deg_z(d_i)$, d_i's are irreducible polynomials in distinct $\langle\theta_x, \theta_y, \theta_z\rangle$-orbits, $\theta_x^\ell d_i$ and $\theta_x^{\ell'} d_i$ are not $\langle\theta_y, \theta_z\rangle$-equivalent for any $1 \le i \le I$, $0 \le \ell, \ell' \le t_{i,j}$ with $\ell \ne \ell'$. Then by similar discussions as in the proofs of Lemmas 5.2 and 5.3 in [18], we can obtain the following result.

Lemma 4.33. Let $f \in \mathbb{K}(x, y, z)$ be of the form (29). Then f has telescopers of type $(\partial_x, \Theta_y, \Theta_z)$ if and only if μ and all $\frac{a_{i,j,\ell}}{\theta_x^\ell d_i^j}$ with $1 \le i \le I, 1 \le j \le J_i$ and $0 \le \ell \le t_{i,j}$ have telescopers of the same type.

Notice that for $\mu \in \mathbb{K}(x, y)$, having telescopers of type $(\partial_x, \Theta_y, \Theta_z)$ and (∂_x, Θ_y) are equivalent. The existence problem of bivariate rational functions has been solved by Theorem 4.1. Thus Problem 4.32 for a general rational function has been reduced to that for a rational function of the form

$$f = \frac{b(x, y, z)}{c(x, y)d(x, y, z)^\lambda}, \tag{30}$$

where $\lambda \in \mathbb{N}\setminus\{0\}$, $c \in \mathbb{K}[x, y]$, $b, d \in \mathbb{K}[x, y, z]$, d is irreducible and $0 \le \deg_z(b) < \deg_z(d)$. Suppose $\alpha(x) \in \mathbb{K}(x) \setminus \{0\}$. It is easy to check that

$$\sum_{i=0}^{\rho} a_i(x)\partial_x^i(\alpha f) = \sum_{i=0}^{\rho} \left(a_i(x)\partial_x^i(\alpha)\right) \partial_x^i(f)$$

whenever $a_i(x) \in \mathbb{K}(x)$ and $f \in \mathbb{K}(x, y, z)$. This means the existence problem of f is equivalent to that of αf. As such we can assume in the form (30) that b, c, d are all primitive as polynomials in y and z. If f is (Θ_y, Θ_z)-exact, then 1 is a telescoper for f.

Lemma 4.34. Let $f \in \mathbb{K}(x, y, z)$ be of the form (30) and not (Θ_y, Θ_z)-exact. If f has a telescoper of type $(\partial_x, \Theta_y, \Theta_z)$, then

$$\theta_x^m(d) = q^s \theta_y^n \theta_z^k(d) \quad \text{for some } m, s, n, k \in \mathbb{Z} \text{ with } m > 0. \tag{31}$$

Proof. We prove the claim by contradiction. Suppose the condition (31) does not hold. Assume that $L = \sum_{i=0}^{I} a_i \partial_x^i \in \mathbb{K}(x)\langle \partial_x \rangle$ with $a_0 \neq 0$ is a telescoper for f. Then

$$L(f) = \sum_{i=0}^{I} \frac{a_i \theta_x^i(b)}{\theta_x^i(c)\theta_x^i(d^\lambda)} = \Theta_y(g) + \Theta_z(h)$$

for some $g, h \in \mathbb{K}(x, y, z)$. By assumption, we know $\theta_x^i d$'s are in distinct $\langle \theta_y, \theta_z \rangle$-orbits, Lemmas 3.6–3.8 show that for any $0 \leq i \leq I$, $\frac{a_i \theta_x^i(b)}{\theta_x^i(c)\theta_x^i(d^\lambda)}$ are (Θ_y, Θ_z)-exact. In particular,

$$\frac{a_0 b}{c d^\lambda} = \Theta_y(g_0) + \Theta_z(h_0) \text{ for some } g_0, h_0 \in \mathbb{K}(x, y, z).$$

As $a_0 \in \mathbb{K}(x) \setminus \{0\}$, we get $\frac{b}{cd^\lambda} = \Theta_y(\frac{g_0}{a_0}) + \Theta_z(\frac{h_0}{a_0})$ which contradicts the assumption that f is not (Θ_y, Θ_z)-exact. This completes the proof. ∎

Next, we will proceed by case distinction according to whether or not

$$\theta_y^{n_1}(d) = q^{s_1} \theta_z^{k_1}(d) \quad \text{for some } s_1, n_1, k_1 \in \mathbb{Z} \text{ with } n_1 > 0. \tag{32}$$

Theorem 4.35. Let $f \in \mathbb{K}(x, y, z)$ be of the form (30) and d satisfy the condition (31) but not the condition (32). Then f has a telescoper of type $(\partial_x, \Theta_y, \Theta_z)$ if and only if

$$\theta_x^{tm}(c) = q^{s_2} \theta_y^{tn}(c) \tag{33}$$

for (m, n) as in (31) and some $t, s_2 \in \mathbb{Z}$ with $t > 0$.

Proof. For the sufficiency, assume that c satisfies the condition (33). Then we will set $L = \sum_{i=0}^{I} a_i \partial_x^{itm}$, where $I \in \mathbb{N}$ and $a_i \in \mathbb{K}(x)$ are to be determined. Applying the reduction formula (7) yields

$$L(f) = \sum_{i=0}^{I} \frac{a_i q^{-is_2 - its} \theta_x^{itm}(b)}{\theta_y^{itn}(c)\theta_y^{itn}\theta_z^{itk}(d^\lambda)} = \Theta_y(u) + \Theta_z(v) + \frac{1}{cd^\lambda} \sum_{i=0}^{I} a_i q^{-is_2 - its} \theta_x^{itm} \theta_y^{-itn} \theta_z^{-itk}(b)$$

for some $u, v \in \mathbb{K}(x, y, z)$. Note that the degrees of the polynomials $\theta_x^{itm}\theta_y^{-itn}\theta_z^{-itk}(b)$ in y or z are the same as that of b. Thus all shifts of b lie in a finite dimensional linear space over $\mathbb{K}(x)$. If I is large enough, then there always exist $a_i \in \mathbb{K}(x)$, not all zero, such that $\sum_{i=0}^{I} a_i q^{-is_2 - its} \theta_x^{itm}\theta_y^{-itn}\theta_z^{-itk}(b) = 0$. As a result $L = \sum_{i=0}^{I} a_i \partial_x^{itm}$ is a telescoper for f.

For the necessity, assume $f = \frac{b(x,y,z)}{c(x,y)d(x,y,z)^\lambda}$ has a telescoper L_1 of type $(\partial_x, \Theta_y, \Theta_z)$. Let C_1 be the maximal factor of c satisfying the condition (33) and $C_2 = c/C_1$. If $C_2 \in \mathbb{K}$, then we are done. Now assume that $C_2 \notin \mathbb{K}$. Then $\deg_y(C_2) > 0$ since c is primitive with respect to y, z. It follows that there exist $B_1, B_2 \in \mathbb{K}[x, y, z]$ with $\deg_z(B_i) < \deg_z(d)$ and $\gcd(B_i, C_i) = 1$ for $i = 1, 2$, such that

$$f = \frac{1}{d^\lambda}\left(\frac{B_1}{C_1} + \frac{B_2}{C_2}\right).$$

Then $\frac{B_1}{C_1 d^\lambda}$ has a telescoper L_2 of type $(\partial_x, \Theta_y, \Theta_z)$ by the sufficiency. The least common left multiple of L_1 and L_2 is a telescoper for $\frac{B_2}{C_2 d^\lambda}$. Since d satisfies the condition (31), we can assume $L = \sum_{i=0}^{I} a_i \partial_x^{im} \in \mathbb{K}(x)\langle \partial_x \rangle$ with $a_0 a_I \neq 0$ to be a telescoper for $\frac{B_2}{C_2 d^\lambda}$.

Thus

$$L\left(\frac{B_2}{C_2 d^\lambda}\right) = \sum_{i=0}^{I} \frac{q^{-is} a_i \theta_x^{im}(B_2)}{\theta_x^{im}(C_2)\theta_y^{in}\theta_z^{ik}(d^\lambda)} = \Theta_y(u) + \Theta_z(v) + \sum_{i=0}^{I} \frac{q^{-is} a_i \theta_x^{im}\theta_y^{-in}\theta_z^{-ik}(B_2)}{\theta_x^{im}\theta_y^{-in}(C_2) d^\lambda} \tag{34}$$

for some $u, v \in \mathbb{K}(x, y, z)$. Notice that $L\left(\frac{B_2}{C_2 d^\lambda}\right)$ is (Θ_y, Θ_z)-exact and that d does not satisfy condition(32). Then Lemmas 3.6–3.8 lead to

$$\sum_{i=0}^{I} \frac{q^{-is} a_i \theta_x^{im}\theta_y^{-in}\theta_z^{-ik}(B_2)}{\theta_x^{im}\theta_y^{-in}(C_2)} = 0. \tag{35}$$

Let $\Lambda = \{c_j \in \mathbb{K}[x, y] \setminus \mathbb{K}[x] |\ c_j$ is an irreducible factor of $C_2\}$. Then Λ is nonempty and finite and none of c_j satisfies condition (33) by the maximality of C_1. By the method of proof by contradiction, one can prove that there exists a $c_\ell \in \Lambda$ such that $c_\ell \neq q^{s'}\theta_x^{im}\theta_y^{-in} c_j$ for any $c_j \in \Lambda$ and $s', i \in \mathbb{Z}$ with $i > 0$. This fact together with equation (35) and the constraint $\gcd(B_2, C_2) = 1$ implies that $B_2 = 0$. ∎

Lemma 4.36. Let $f \in \mathbb{K}(x, y, z)$ be of the form (30) and d satisfy conditions (31) and (32). Suppose

$$\theta_x^{m_2}(c) = q^{s_2}\theta_y^{n_2}(c) \quad \text{for some integers } m_2, s_2, n_2 \text{ with } m_2 > 0. \tag{36}$$

Then f has a telescoper of type $(\partial_x, \Theta_y, \Theta_z)$.

Proof. Since d satisfies both (31) and (32), without loss of generality, we assume m, n_1 are the smallest such positive integers. Let $m_0 = m m_2 n_1$ and $L = \sum_{i=0}^{I} a_i \partial_x^{i m_0}$, where $I \in \mathbb{N}$ and $a_i \in \mathbb{K}(x)$ are to be determined. Then

$$\begin{aligned} L(f) &= \sum_{i=0}^{I} \frac{a_i \theta_x^{i m_0}(b)}{\theta_x^{i m_0}(c)\theta_x^{i m_0}(d^\lambda)} = \sum_{i=0}^{I} \frac{a_i q^{-i m s_2 n_1 - i s m_2 n_1}\theta_x^{i m_0}(b)}{\theta_y^{i m n_2 n_1}(c)\theta_y^{i n m_2 n_1}\theta_z^{i k m_2 n_1}(d^\lambda)} = \sum_{i=0}^{I} \frac{a_i q^{\alpha}\theta_x^{i m_0}(b)}{\theta_y^{i m n_2 n_1}\theta_z^{\beta}(c d^\lambda)} \\ &= \Theta_y(u) + \Theta_z(v) + \frac{\sum_{i=0}^{I} a_i q^{\alpha}\theta_x^{i m_0}\theta_y^{-i m n_2 n_1}\theta_z^{-\beta}(b)}{c d^\lambda}, \end{aligned} \tag{37}$$

where $u, v \in \mathbb{K}(x, y, z)$, $\alpha = -i m s_2 n_1 - i s m_2 n_1 - i(m_2 n - m n_2)s_1$ and $\beta = i k m_2 n_1 + i(m_2 n - m n_2)k_1$. Since the ($q$-)shift operators do not change the degree of b, when I is large enough, we can find nontrivial solutions a_i such that

$$\sum_{i=0}^{I} a_i q^{\alpha}\theta_x^{i m_0}\theta_y^{-i m n_2 n_1}\theta_z^{-\beta}(b) = 0.$$

Then identity (37) leads to the fact that $L = \sum_{i=0}^{I} a_i \partial_x^{i m_0}$ is a telescoper for f. ∎

Theorem 4.37. Let f be of the form (30) and assume that d satisfies conditions (31) and (32). Then f has a telescoper of type $(\partial_x, \Theta_y, \Theta_z)$ if and only if f can be decomposed into the form

$$f = \frac{1}{d^\lambda}\left(\frac{B_1}{C_1} + \frac{B_2}{C_2}\right),$$

where $B_1, B_2 \in \mathbb{K}[x, y, z]$, $C_1, C_2 \in \mathbb{K}[x, y]$ satisfy the following two constraints: (1) C_1 satisfies the condition (36); (2) $B_2/(C_2 d^\lambda)$ is (Θ_y, Θ_z)-exact.

Proof. The sufficiency part follows from Lemma 4.36. For the necessity, let C_1 be the maximal factor of c satisfying the condition (36) and $C_2 = c/C_1$. If $C_2 \in \mathbb{K}$, then we are done. Now assume that $C_2 \notin \mathbb{K}$. Then $\deg_y(C_2) > 0$ since c is primitive with respect to y, z. It follows that there exist $B_1, B_2 \in \mathbb{K}[x, y, z]$ with $\deg_z(B_i) < \deg_z(d)$ and $\gcd(B_i, C_i) = 1$ for $i = 1, 2$, such that $f = \frac{1}{d^\lambda}\left(\frac{B_1}{C_1} + \frac{B_2}{C_2}\right)$. Next we will prove that $\frac{B_2}{C_2 d^\lambda}$ is (Θ_y, Θ_z)-exact. Note that $\frac{B_1}{C_1 d^\lambda}$ has a telescoper of the same type as f by Lemma 4.36. Then $\frac{B_2}{C_2 d^\lambda}$ has a telescoper $L = \sum_{i=0}^{I} a_i \partial_x^{im}$ with $a_0 a_I \neq 0$ and

$$L\left(\frac{B_2}{C_2 d^\lambda}\right) = \Theta_y(u) + \Theta_z(v) + \sum_{i=0}^{I} \frac{q^{-is} a_i \theta_x^{im} \theta_y^{-in} \theta_z^{-ik}(B_2)}{\theta_x^{im} \theta_y^{-in}(C_2) d^\lambda} \tag{38}$$

for some $u, v \in \mathbb{K}(x, y, z)$. Since $\deg_z(B_2) < \deg_z(d)$, the function $\sum_{i=0}^{I} \frac{q^{-is} a_i \theta_x^{im} \theta_y^{-in} \theta_z^{-ik}(B_2)}{\theta_x^{im} \theta_y^{-in}(C_2) d^\lambda}$ is (Θ_y, Θ_z)-exact and d satisfies condition (32), exactness criteria in Lemmas 3.6–3.8 yield that there exists $g \in \mathbb{K}(x, y)[z]$ such that

$$\sum_{i=0}^{I} q^{-is} a_i \theta_x^{im} \theta_y^{-in} \theta_z^{-ik}\left(\frac{B_2}{C_2}\right) = q^{-\lambda s_1} \theta_y^{n_1} \theta_z^{-k_1}(g) - g. \tag{39}$$

Let $\Lambda = \{c_j \in \mathbb{K}[x, y] \setminus \mathbb{K}[x] \mid c_j \text{ is an irreducible factor of } C_2\}$. Then Λ is nonempty and finite since $\deg_y(C_2) > 0$. Notice that none of c_j in Λ satisfies the condition (36). One can find a $c_\ell \in \Lambda$ such that $c_\ell \neq q^s \theta_x^{m_3} \theta_y^{n_3} c_j$ for any $c_j \in \Lambda$ and $s, m_3, n_3 \in \mathbb{Z}$ with $m_3 > 0$. Collecting all irreducible factors in C_2, which are $\langle\theta_y\rangle$-equivalent to c_ℓ, into D_1, we can decompose $\frac{B_2}{C_2}$ into $\frac{B_2}{C_2} = \frac{A_1}{D_1} + \frac{A}{D}$, where $A_1, A \in \mathbb{K}[x, y, z], D = C_2/D_1$. Rewrite $g = g_1 + g_2^\star$ where $g_1, g_2^\star \in \mathbb{K}(x, y)[z]$ and the denominator of g_1 contains exactly all irreducible factors in the denominator of g which are $\langle\theta_y\rangle$-equivalent to c_ℓ. Equation (39) and the choice of D_1 and g_1 imply $\frac{A_1}{D_1} = q^{-\lambda s_1} \theta_y^{n_1} \theta_z^{-k_1}(\frac{g_1}{a_0}) - \frac{g_1}{a_0}$, and hence

$$\sum_{i=0}^{I} q^{-is} a_i \theta_x^{im} \theta_y^{-in} \theta_z^{-ik}\left(\frac{A_1}{D_1}\right) = q^{-\lambda s_1} \theta_y^{n_1} \theta_z^{-k_1}(h_1) - h_1, \tag{40}$$

where $h_1 = \sum_{i=0}^{I} q^{-is} a_i \theta_x^{im} \theta_y^{-in} \theta_z^{-ik}(\frac{g_1}{a_0})$. Subtracting Equation (40) from (39), we obtain

$$\sum_{i=0}^{I} q^{-is} a_i \theta_x^{im} \theta_y^{-in} \theta_z^{-ik}\left(\frac{A}{D}\right) = q^{-\lambda s_1} \theta_y^{n_1} \theta_z^{-k_1}(g_1^\star) - g_1^\star \tag{41}$$

with $g_1^\star = g - h_1$. Repeating the above arguments for the equation (41), $\frac{B_2}{C_2}$ can be finally decomposed as $\frac{B_2}{C_2} = \frac{A_1}{D_1} + \frac{A_2}{D_2} + \cdots + \frac{A_T}{D_T}$ for some $A_i \in \mathbb{K}[x, y, z]$, $D_i \in \mathbb{K}[x, y]$ and $\frac{A_i}{D_i} = q^{-\lambda s_1} \theta_y^{n_1} \theta_z^{-k_1}(\frac{g_i}{a_0}) - \frac{g_i}{a_0}$ for any $1 \leq i \leq T$. Then we get

$$\frac{B_2}{C_2} = q^{-\lambda s_1} \theta_y^{n_1} \theta_z^{-k_1}\left(\frac{1}{a_0} \sum_{i=0}^{T} g_i\right) - \frac{1}{a_0} \sum_{i=0}^{T} g_i$$

and hence $\frac{B_2}{C_2 d^\lambda}$ is (Θ_y, Θ_z)-exact. This completes the proof. ∎

Example 4.38. Let $f = 1/(bd^2)$, where $b = x + y$ and $d = 3x + 2y + z$. Since d satisfies $\sigma_x(d) = \sigma_y\sigma_z(d)$ and $\sigma_y(d) = \sigma_z^2(d)$, and b satisfies $\sigma_x(b) = \sigma_y(b)$, Lemma 4.36 implies f has a telescoper of type $(S_x, \Delta_y, \Delta_z)$. In fact,

$$L(f) = \Delta_y(r_1 \cdot f) + \Delta_z(r_2 \cdot f) \text{ with } L = S_x - 1, r_1 = \frac{3x + 2y + z}{1 + 3x + 2y + z} \text{ and } r_2 = -\frac{2y + z}{3x}.$$

Example 4.39. Let $f = 1/(bd_1^2)$ with $b = x + y$ as in the Example 4.38 and $d_1 = 3x^2 + 2y + z$. Then f does not have any telescoper of type $(S_x, \Delta_y, \Delta_z)$ since d_1 does not satisfy the condition (31).

4.6. Existence criteria of the sixth class

We consider the last class of the existence problems of telescopers for rational functions in three variables.

Problem 4.40. Let $\partial_y \in \{S_y, T_{q,y}\}$ and $\Theta_y = \partial_y - 1$. Given $f \in \mathbb{K}(x, y, z)$, decide whether there exists a nonzero operator $L \in \mathbb{K}(x)\langle D_x\rangle$ such that $L(f) = \Theta_y(g) + D_z(h)$ for some $g, h \in \mathbb{K}(x, y, z)$.

Let $\theta_y = \sigma_y$ if $\Theta_y = \Delta_y$ and $\theta_y = \tau_{q,y}$ if $\Theta_y = \Delta_{q,y}$. By the Ostrogradsky–Hermite reduction and the reduction formula (7), we can decompose $f \in \mathbb{K}(x, y, z)$ as

$$f = \Theta_y(u) + D_z(v) + r \text{ with } r = \sum_{i=1}^{I} \frac{\alpha_i}{z - \beta_i}, \tag{42}$$

where $u, v \in \mathbb{K}(x, y, z)$ and $\alpha_i, \beta_i \in \overline{\mathbb{K}(x, y)}$ with $\alpha_i \neq 0$ and the β_i's are in distinct $\langle\theta_y\rangle$-orbits. Then f has a telescoper of type (D_x, Θ_y, D_z) if and only if r has a telescoper of the same type.

Lemma 4.41. For any $L = \sum_{j=0}^{\rho} \ell_j D_x^j \in \mathbb{K}(x)\langle D_x\rangle$ and $\alpha, \beta \in \overline{\mathbb{K}(x, y)}$, there exists $g \in \overline{\mathbb{K}(x, y)}(z)$ such that

$$L\left(\frac{\alpha}{z - \beta}\right) = \frac{L(\alpha)}{z - \beta} + D_z(g). \tag{43}$$

Proof. Let $\operatorname{res}_z(f, \beta)$ denote the residue of $f \in \mathbb{K}(x, y, z)$ at $z = \beta$ in z. The map $\operatorname{res}_z(\cdot, \beta)$ is $\mathbb{K}(x, y)$-linear and commutes with the operator D_x by [21, Proposition 3]. Then we have

$$\operatorname{res}_z\left(L\left(\frac{\alpha}{z - \beta}\right), \beta\right) = L\left(\operatorname{res}_z\left(\frac{\alpha}{z - \beta}, \beta\right)\right) = L(\alpha).$$

So all residues of $h := L(\alpha/(z - \beta)) - L(\alpha)/(z - \beta)$ at all of its poles are zero. By Proposition 2.2 in [22], we have h is D_z-exact, i.e., $h = D_z(g)$ for some $g \in \overline{\mathbb{K}(x, y)}(z)$. ∎

The next theorem reduces Problem 4.40 to the separation problem for algebraic functions (Problem 4.5) and the existence problem of telescopers in $\mathbb{K}(x, y)(\beta)$ with $\beta \in \overline{\mathbb{K}(x)}$.

Theorem 4.42. Let $f \in \mathbb{K}(x, y, z)$ be of the form (42). Then f has a telescoper of type (D_x, Θ_y, D_z) if and only if for each i with $1 \leq i \leq I$, either α_i is separable in x and y or $\beta_i \in \overline{\mathbb{K}(x)}$ and $\alpha_i \in \mathbb{K}(x, y)(\beta_i)$ has a telescoper of type (D_x, Θ_y).

Proof. If for each i with $1 \le i \le I$, either α_i is separable or $\beta_i \in \overline{\mathbb{K}(x)}$ and $\alpha_i \in \mathbb{K}(x,y)(\beta_i)$ has a telescoper of type (D_x, Θ_y), then there exists a nonzero $L_i \in \mathbb{K}(x)\langle D_x \rangle$ such that either $L_i(\alpha_i) = 0$ or $L_i(\alpha_i) = \Theta_y(\gamma_i)$ for some $\gamma_i \in \mathbb{K}(x,y)(\beta_i)$. By Lemma 4.41, we have

$$\begin{aligned} L_i\left(\frac{\alpha_i}{z-\beta_i}\right) &= D_z(g_i) + \frac{L_i(\alpha_i)}{z-\beta_i} = D_z(g_i) + \frac{\Theta_y(\gamma_i)}{z-\beta_i} \\ &= D_z(g_i) + \Theta_y\left(\frac{\gamma_i}{z-\beta_i}\right), \end{aligned}$$

where $g_i \in \overline{\mathbb{K}(x,y)}(z)$. So for each i with $1 \le i \le I$, the fraction $\alpha_i/(z-\beta_i)$ has a telescoper of type (D_x, Θ_y, D_z). Then f has a telescoper of the same type by Lemmas 2.6 and 3.4. To show the necessity, we assume that $L \in \mathbb{K}(x)\langle D_x \rangle$ is a telescoper for f of type (D_x, Θ_y, D_z). By Lemma 4.41, there exists $w \in \overline{\mathbb{K}(x,y)}(z)$ such that

$$\begin{aligned} L(f) &= \Theta_y(L(u)) + D_z(L(v) + w) + \sum_{i=1}^{I} \frac{L(\alpha_i)}{z-\beta_i} \\ &= \Theta_y(g) + D_z(h) \end{aligned}$$

for some $g, h \in \mathbb{K}(x,y,z)$. For each i with $1 \le i \le I$, either α_i is separable if $L(\alpha_i) = 0$ or $L(\alpha_i)/(z-\beta_i)$ is (Θ_y, D_z)-exact if $L(\alpha_i) \neq 0$. In the latter case we have $\beta_i \in \overline{\mathbb{K}(x)}$ and $L(\alpha_i) = \Theta_y(\gamma_i)$ for some $\gamma_i \in \mathbb{K}(x,y)(\beta_i)$ by Lemma 3.9. ∎

Remark 4.43. The separation problem for algebraic functions will be solved in the forthcoming paper [16]. The existence problem of telescopers of type (D_x, Θ_y) can be solved by using Theorem 4.1, whose statement is for functions in $\mathbb{K}(x,y)$, but its proof also works for functions in $\overline{\mathbb{K}(x)}(y)$. In particular, this covers the case in which the functions are in $\mathbb{K}(x,y)(\beta)$ with $\beta \in \overline{\mathbb{K}(x)}$.

Example 4.44. Let $f = 1/(z^2 - xy)$. Then

$$f = \frac{\alpha}{z-\beta} + \frac{-\alpha}{z+\beta},$$

where $\alpha = \frac{1}{2\sqrt{xy}}$ and $\beta = \sqrt{xy}$. Since α is separable in x and y, Theorem 4.42 implies that f has a telescoper of type (D_x, Θ_y, D_z). In fact, $L(f) = \Theta_y(0) + D_z(-z \cdot f)$, where $L = 2xD_x + 1$.

Example 4.45. Let $f = 1/((x+y)(z^2 - x - y))$. Then

$$f = \frac{\alpha}{z-\beta} + \frac{-\alpha}{z+\beta},$$

where $\alpha = \frac{1}{2(x+y)\sqrt{x+y}}$ and $\beta = \sqrt{x+y}$. Note that α is not separable in x and y since its successive derivatives $D_x^i(\alpha) = (-1)^i \prod_{j=0}^{i}(j+1/2)(x+y)^{-(i+3/2)}$ are linearly independent over $\mathbb{K}(x)$. Since β is not in $\overline{\mathbb{K}(x)}$, it follows from Theorem 4.42 that f has no telescoper of type (D_x, Θ_y, D_z).

5. Conclusion

In this paper, we present existence criteria for telescopers for rational functions in three variables. The criteria reduce the existence problem of telescopers for the trivariate inputs to that for the bivariate inputs and two related solvable problems: the (q-)shift equivalence testing problem and the separation problem. In the pure differential case, algorithms for constructing minimal telescopers for rational functions in three variables have been presented in [21, 8] using residues and reductions. This has also recently been extended to the pure shift case in [17] based on the existence criteria given in [18]. The first natural direction for future work is to implement the existence criteria presented in this paper and also develop efficient algorithms for computing telescopers if they exist in the other sixteen cases using the existence criteria from this paper. The next more challenging direction is to study the existence problem of telescopers for more general inputs, such as rational functions and hypergeometric terms in several variables. To this end, we first need to solve the multivariate summability problem for those inputs. In particular, it is already quite intriguing to extend the classical Gosper algorithm for indefinite hypergeometric summation [30] to the bivariate case.

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