Parallel Summation in P-Recursive Extensions

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ABSTRACT

We propose investigating a summation analog of the paradigm for parallel integration. We make some first steps towards an indefinite summation method applicable to summands that rationally depend on the summation index and a P-recursive sequence and its shifts. There is a distinction between so-called normal and so-called special polynomials. Under the assumption that the corresponding difference field has no unnatural constants, we are able to predict the normal polynomials appearing in the denominator of a potential closed form. We can also handle the numerator. Our method is incomplete so far as we cannot predict the special polynomials appearing in the denominator. However, we do have some structural results about special polynomials for the setting under consideration.

CCS CONCEPTS

• **Computing methodologies → Algebraic algorithms**.

KEYWORDS

symbolic summation; difference rings

ACM Reference Format:
Shaoshi Chen[a,b], Ruyong Feng[a,b], Manuel Kauers[c], Xiuyun Li[a,b,c]. 2024. Parallel Summation in P-Recursive Extensions. In . ACM, New York, NY, USA, 9 pages. https://doi.org/10.1145/nnnnnnn.nnnnnnn

1 INTRODUCTION

The main difference between the first and the second edition of Manuel Bronstein's classical textbook on symbolic integration [8] is an additional tenth chapter about parallel integration, which is based on his last paper [9] on the subject. Parallel integration is an alternative approach to the more widely known Risch algorithm for indefinite integration, whose careful description dominates the remainder of Bronstein's book. Parallel integration is also known as the Risch-Norman algorithm [12–14, 19, 20, 23, 33] and as poor-man's integrator [51].

Although the technique is not complete, i.e., it fails to find a closed form of certain integrals, it is an attractive alternative to a full implementation of the Risch algorithm, which is guaranteed to find a closed form whenever there is one. One advantage is that it is much easier to program. Indeed, Bronstein's Maple implementation [51] barely needs 100 lines of code. A second advantage is that it extends more easily to integrals of non-elementary functions. For example, it can find the evaluation

$$\int \frac{x^2+(x^2+2)W(x^2)}{x(1+W(x^2))^2}dx = \frac{1}{2}\frac{x^2}{W(x^2)} + \log(1+W(x^2))$$

involving the Lambert W function [11]. This is not only interesting because W is defined by a nonlinear equation, but also because there is a factor in the denominator of the closed form that is not already present in the integrand.

In a seminar talk that never led to a formal publication, Zimmermann observed that parallel integration can be combined with the concept of creative telescoping [50] in order to handle definite integrals involving a parameter, similar as done by Raab [37] with Risch's algorithm. A version of parallel integration for integrals involving algebraic functions was presented by Böttner in [6] and for integrals of Airy functions by Du and Raab in [15].

To our knowledge, the idea of parallel integration has not yet been translated to the setting of symbolic summation. The goal of the present paper is to do so. A summation example that is similar to the above integral can be given in terms of the logistic sequence t_n [17, Example 1.9, Chapter 1] satisfying the nonlinear recurrence equation $t_{n+1} = t_n(1-t_n)$ with $t_0 \in (0,1)$. Here we have the summation identity

$$\sum_{k=0}^{n-1} \frac{1}{1-t_k} = \sum_{k=0}^{n-1}\left(\frac{1}{t_{k+1}} - \frac{1}{t_k}\right) = \frac{1}{t_n} - \frac{1}{t_0},$$

and again, the denominator of the closed form contains a factor that is not already present in the summand.

On the other hand, the denominator of a closed form is not completely unpredictable. Like in parallel integration, we can distinguish the *special* and the *normal* part of a denominator. Based on this distinction, in Section 2 we show how the normal part of the denominator of a closed form depends on the normal part of the denominator of the corresponding summand. Unfortunately, we do not have a complete understanding of the special part, but we do have some results that limit the number of special polynomials (Sect. 2.2). More can be said if we focus on a more specific setting.

Sect. 2 is about the general paradigm of parallel summation, which in principle could be applied to many different specific settings. In Sect. 3 we restrict the attention to one such setting. We

S. Chen was partially supported by the National Key R&D Program of China (No. 2023YFA1009401), the NSFC grant (No. 12271511), CAS Project for Young Scientists in Basic Research (Grant No. YSBR-034), and the CAS Fund of the Youth Innovation Promotion Association (No. Y2022001). R. Feng was partially supported by the National Key R&D Program of China (No. 2023YFA1009401) and the National Key Research and Development Project 2020YFA0712300. M. Kauers was supported by the Austrian FWF grants PAT 9952223 and I6130-N. X. Li was partially supported by the Land Oberösterreich through the LIT-AI Lab .

https://doi.org/10.1145/nnnnnnn.nnnnnnn

consider summation problems of the form

$$\sum_{k=0}^{n} \operatorname{rat}(k, f(k), f(k+1), \dots, f(k+r-1)),$$

where rat is a multivariate rational function and f is defined by a linear recurrence of order r with polynomial coefficients. The task is to decide whether a given sum of this type can be written as a rational function in $n, f(n), \dots, f(n+r-1)$. While we are not (yet) able to solve this task in full generality, the idea of parallel summation provides a significant step towards such an algorithm. We can describe more precisely the structure of special polynomials in this case (Sect. 3.1), and we can effectively solve the σ-equivalence problem (Sect. 3.2), which implies that we can completely identify the normal part of the denominator of any closed form. Similar as in the differential case [8, 9], the special part has to be determined heuristically, unless we impose further restrictions on the setting (Sect. 4).

2 PARALLEL SUMMATION

Similar to parallel integration, the general idea of parallel summation is to avoid the recursive nature of summation algorithms such as Karr's algorithm by viewing the summand as an element of a field of multivariate rational functions over a ground field. We now set up the general algebraic foundation for parallel summation and list some related problems. Like in the situation of parallel integration, these problems are in general far from being solved.

Let A be a ring and $\sigma\colon A \longrightarrow A$ be an automorphism of A. We call the pair (A, σ) a *difference ring* and a *difference field* if A is a field. Note that the set $\{a \in A \mid \sigma(a) = a\}$ forms a subring of A which is called the *constant subring* of (A, σ), denoted by C_A. A difference ring (A^*, σ^*) is called a *difference extension* of (A, σ) if $A \subseteq A^*$ and $\sigma^* \mid_A = \sigma$. By abuse of notation, we will often write σ for the extended automorphism σ^* of A^*.

Problem 1 (Indefinite Summation Problem). *Let (A^*, σ) be a specific difference extension of (A, σ). Given $f \in A$, decide whether there exists $g \in A^*$ such that $f = \sigma(g) - g$. If such a g exists, f is said to be* summable *in A^*.*

Abramov's algorithm [1–3] solves the indefinite summation problem for rational functions. The indefinite hypergeometric summation problem was solved by Gosper's algorithm in [21] and the more general P-recursive case without denominators was solved by Abramov-van Hoeij's algorithm [4, 5]. As a discrete analogue of Risch's algorithm for elementary integration, Karr's algorithm [26, 28] solves the indefinite summation problem in a so-called $\Pi\Sigma$-extension of a given difference field. Karr's algorithm has been implemented and improved by Schneider in [38, 41, 42] with applications in physics [43]. The ideas of Karr have also been extended to higher order equations [7, 24, 39, 40].

The difference fields employed in Karr's algorithm are univariate rational function fields $K(t)$ with potentially complicated ground fields K. The summation problem is solved in such fields following Abramov's two-step approach: first find a $v \in K[t]$ such that every solution $g \in K(t)$ must have a denominator that divides v, and then find a $u \in K[t]$ such that $g = u/v$ is a solution. In order to find such u and v, certain subproblems have to be solved for the ground field K, and if K is again a univariate rational function field (perhaps still with a complicated ground field), then the procedure is applied recursively. The recursion ends when the constant field is reached.

Parallel summation also follows Abramov's two-step approach, but instead of using univariate rational function fields with potentially complicated ground fields, it allows the use of multivariate rational function fields, which may then have simpler ground fields. The summand is thus given as an element of a difference field of the form $F = K(t_0, \dots, t_{n-1})$. In the first step, we seek a $v \in K[t_0, \dots, t_{n-1}]$ such that every solution $g \in F$ must have a denominator that divides v, and in the second step, we then find a polynomial $u \in K[t_0, \dots, t_{n-1}]$ such that $g = u/v$ is a solution.

Thus, while Karr's algorithm handles the generators of a difference field $K(t_0)(t_1)\cdots(t_{n-1})$ one after the other, the parallel approach handles them "in parallel". How exactly this is done, this depends on the automorphism σ of the difference field. There is little hope to obtain an algorithm that can execute both steps for arbitrary difference fields of the form $K(t_0, \dots, t_{n-1})$. This is the same as in the integration case, where it can only be guaranteed under strong restrictions on the differential field that the method does not overlook any solutions, and where the method is still of interest as a valuable heuristic tool in situations where these strong restrictions do not apply.

In this paper, we restrict the attention to the ground field $K = C(x)$ where C is a field of characteristic zero. Together with the C-automorphism $\sigma\colon K \longrightarrow K$ defined by $\sigma(x) = x + 1$, we have that (K, σ) is a difference field and its constant subfield is C. Let $R := K[t_0, \dots, t_{n-1}]$ and $F := K(t_0, \dots, t_{n-1})$. The central problem of parallel summation is as follows.

Problem 2. *Given $f \in F$, decide whether there exists $g \in F$ such that $f = \sigma(g) - g$.*

For a general C-automorphism σ of F, the following example shows that the difference field (F, σ) may contain new constants that are not in C.

Example 3. *Let $F = C(x, t_0, t_1)$ with the C-automorphism σ satisfying $\sigma(x) = x + 1$, $\sigma(t_0) = t_1$, and $\sigma(t_1) = t_0 + t_1$. Then $p = (t_1^2 - t_0^2 - t_0 t_1)^2$ is a new constant in F.*

In general, deciding the existence of new constants is a difficult problem. It is equivalent to finding algebraic relations among sequences. See [31] for how to do this for C-finite sequences and Karr's algorithm [27] for how to solve it for $\Pi\Sigma$-fields.

The following example shows that in general, the denominator of g may have factors that are not related to any of the factors of the denominator of f.

Example 4. *Let $F = C(x, t_0, t_1)$ and σ is the C-automorphism defined by $\sigma(x) = x + 1$, $\sigma(t_0) = 2t_0 + xt_1$ and $\sigma(t_1) = 2t_1$. Then*

$$\sigma\left(\frac{t_0}{t_1}\right) - \frac{t_0}{t_1} = \frac{2t_0 + xt_1}{2t_1} - \frac{t_0}{t_1} = \frac{x}{2}.$$

Similar phenomena can be expected if $\sigma(t_i)$ is not a polynomial for some i. We are interested in predicting denominators of closed forms when $\sigma(t_i)$ is a polynomial for every i. This motivates the following hypothesis.

Hypothesis 5. *The constant field of (F, σ) is the field C and σ is also a C-automorphism of R.*

To solve Problem 2, one first needs to estimate the possible irreducible polynomials in the denominator of g. To this end, we now extend the notion of special polynomials in parallel integration to the summation setting.

Definition 6. *A polynomial $P \in R$ is said to be* special *if there exist $i \in \mathbb{Z} \setminus \{0\}$ such that $P \mid \sigma^i(P)$ and it is said to be* normal *if $\gcd(P, \sigma^i(P)) = 1$ for all $i \in \mathbb{Z} \setminus \{0\}$. A polynomial $P \in R$ is said to be* factor-normal *if all of its irreducible factors are normal. Two polynomials $P, Q \in R$ are said to be σ-equivalent if there exist $m \in \mathbb{Z}$ and $u \in K$ such that $P = u \cdot \sigma^m(Q)$.*

By the above definition, any nonzero element in K is both special and normal and an irreducible polynomial in R is either special or normal. The product of special polynomials is also special. If two normal polynomials are not σ-equivalent, then their product is still normal.

Concerning special and normal polynomials, there are two basic and natural questions: firstly, how to decide if a given irreducible polynomial is special or normal? Secondly, how to decide whether two polynomials are σ-equivalent or not? We will answer these questions in next section for the difference field generated by P-recursive sequences.

2.1 Local dispersions and denominator bounds

Abramov in [1] introduced the notion of dispersions for rational summation. It is a discrete analogue of the multiplicity. We define a local version of Abramov's dispersions in R at an irreducible normal polynomial, following [10]. Let $p, Q \in R$ with p being an irreducible normal polynomial. If $\sigma^i(p) \mid Q$ for some $i \in \mathbb{Z}$, the *local dispersion* of Q at p, denoted by $\operatorname{disp}_p(Q)$, is defined as the maximal integer distance $|i-j|$ with $i, j \in \mathbb{Z}$ satisfying $\sigma^i(p) \mid Q$ and $\sigma^j(p) \mid Q$; otherwise we define $\operatorname{disp}_p(Q) = -\infty$. Conventionally, we set $\operatorname{disp}_p(0) = +\infty$. The (global) *dispersion* of Q, denoted by $\operatorname{disp}(Q)$, is defined as

$$\max\{\operatorname{disp}_p(Q) \mid p \text{ is an irreducible normal polynomial in } R\}.$$

Note that $\operatorname{disp}(Q) = -\infty$ if $Q \in R \setminus \{0\}$ has no irreducible normal factor. For a rational function $f = a/b \in F$ with $a, b \in R$, $\deg(b) \geq 1$ and $\gcd(a, b) = 1$, we also define $\operatorname{disp}_p(f) = \operatorname{disp}_p(b)$ and $\operatorname{disp}(f) = \operatorname{disp}(b)$. The set $\{\sigma^i(p) \mid i \in \mathbb{Z}\}$ is called the σ-orbit at p, denoted by $[p]_\sigma$. Note that $\operatorname{disp}_p(Q) = \operatorname{disp}_q(Q)$ if $q \in [p]_\sigma$. So we can define the local dispersion and dispersion of a rational function at a σ-orbit.

The following lemma shows how the local dispersions and dispersions change under the action of the difference operator Δ, which is defined by $\Delta(f) = \sigma(f) - f$ for any $f \in F$.

Lemma 7. *Let $f = a/b \in F$ with $a, b \in R$ and $\gcd(a, b) = 1$ and let $p \in R$ be an irreducible normal factor of b. Then $\operatorname{disp}_p(\Delta(f)) = \operatorname{disp}_p(f) + 1$ and $\operatorname{disp}(\Delta(f)) = \operatorname{disp}(f) + 1$.*

Proof. Let $d = \operatorname{disp}_p(b)$. Without loss of generality, we may assume that $p \mid b$ but $\sigma^i(p) \nmid b$ for any $i < 0$. Since $\gcd(a, b) = 1$ and σ is a C-automorphism of $K[t_0, \ldots, t_{n-1}]$, $\gcd(\sigma^i(a), \sigma^i(b)) = 1$ for any $i \in \mathbb{Z}$. We now write

$$\sigma(f) - f = \frac{\sigma(a)b - a\sigma(b)}{b\sigma(b)} = \frac{A}{B},$$

where $A, B \in K[t_0, \ldots, t_{n-1}]$ and $\gcd(A, B) = 1$. Since $p \mid b$ but $p \nmid a\sigma(b)$, we have $p \nmid (\sigma(a)b - a\sigma(b))$ and then $p \nmid A$. By the definition of local dispersions, $\sigma^d(p) \mid b$ but $\sigma^{d+1}(p) \nmid b$. Since $\gcd(a, b) = 1$, we have $\sigma^d(p) \nmid a$ and then $\sigma^{d+1}(p) \nmid \sigma(a)$. Then $\sigma^{d+1}(p) \nmid \sigma(a)b$, which implies $\sigma^{d+1}(p) \nmid (\sigma(a)b - a\sigma(b))$ and also $\sigma^{d+1}(p) \nmid A$. So $p \mid B$ and $\sigma^{d+1}(p) \mid B$, which implies that $\operatorname{disp}_p(B) \geq d + 1$. Since $B \mid b\sigma(b)$, we have $\operatorname{disp}_p(B) \leq \operatorname{disp}_p(b\sigma(b)) = d + 1$. Therefore, $\operatorname{disp}_p(B) = d + 1$. Since the equality $\operatorname{disp}_p(B) = d + 1$ holds for all irreducible normal factors, we have $\operatorname{disp}(\Delta(f)) = \operatorname{disp}(f) + 1$. ∎

By the above lemma, we get that f is not σ-summable in F if $\operatorname{disp}(f) = 0$. If we know how to detect the σ-equivalence in R, then we can write a given polynomial $P \in R$ as $P = P_s \cdot P_n$, all irreducible factors of $P_s \in R$ are special and all irreducible factors of $P_n \in R$ are normal. We call (P_s, P_n) the *splitting factorization* of P and P_n the *normal part* of P.

Theorem 8. *Let $f \in F$ and $v_n \in R$ be the normal part of the denominator of f. If $f = \sigma(g) - g$ for some $g \in F$, then the normal part of the denominator of g divides the polynomial*

$$\gcd\left(\prod_{i=0}^{d} \sigma^i(v_n), \prod_{i=0}^{d} \sigma^{-i-1}(v_n)\right),$$

where $d := \operatorname{disp}(g) = \operatorname{disp}(f) - 1$.

Proof. Write $f = u/v \in F$ with $u, v \in R$ and $gcd(u, v) = 1$. Assume that the splitting factorization of v is $(v_s, v_n) \in R^2$. If $f = \sigma(g) - g$ for some $g \in F$, we also write $g = p/q$ with $p, q \in R$ and $gcd(p, q) = 1$ and let (q_s, q_n) be the splitting factorization of q. By Lemma 7, we have $d := \operatorname{disp}(q_n) = \operatorname{disp}(v_n) - 1$. We now show

$$q_n \mid \gcd\left(\prod_{i=0}^{d} \sigma^i(v_n), \prod_{i=0}^{d} \sigma^{-i-1}(v_n)\right). \tag{1}$$

We first show that $q_n \mid \prod_{i=0}^{d} \sigma^i(v_n)$. The equality $f = \sigma(g) - g$ implies that

$$g = \frac{v\sigma(g) - u}{v} \tag{2}$$

Applying σ to both sides of the above equation yields

$$\sigma(g) = \frac{\sigma(v)\sigma^2(g) - \sigma(u)}{\sigma(v)}.$$

Substituting $\sigma(g)$ in the equation (2) yields

$$g = \frac{1}{v}\left(v \cdot \frac{\sigma(v)\sigma^2(g) - \sigma(u)}{\sigma(v)} - u\right)$$

After d repetitions of the above process, we get

$$g = \frac{a \cdot \sigma^{d+1}(g) - b}{v\sigma(v) \cdots \sigma^d(v)}$$

for some $a, b \in R$. The denominator of the g is $q = q_s q_n$, while the denominator of the right-hand side of the above equality is a divisor of $V := v\sigma(v) \cdots \sigma^d(v)\sigma^{d+1}(q)$. Then $q_n \mid V$. Let (V_s, V_n) be the splitting factorization of V. Then $V_n = v_n\sigma(v_n) \cdots \sigma^d(v_n)\sigma^{d+1}(q_n)$

and $q_n \mid V_n$. Since $\operatorname{disp}(q_n) = d$, we have $\gcd(q_n, \sigma^{d+1}(q_n)) = 1$. Hence we have $q_n \mid v_n\sigma(v_n)\cdots\sigma^d(v_n)$. The proof of the divisibility $q_n \mid \prod_{i=0}^{d}\sigma^{-i-1}(v_n)$ is analogous. So the divisibility (1) holds. ∎

Example 9. *Let $F = C(x)(t_0, t_1)$ with a C-automorphism defined by $\sigma(x) = x+1, \sigma(t_0) = t_1$ and $\sigma(t_1) = -6t_0 + 5t_1$. Consider the equation*

$$f = \frac{636t_0^3 + 443t_0^2t_1 - 1428t_0t_1^2 + 565t_1^3}{2592(3t_0-2t_1)^2(t_0-t_1)^2(2t_0-t_1)(t_0+t_1)} = \sigma(y) - y.$$

We now decide whether this equation has a solution in F. Firstly, we can detect that the irreducible factor $2t_0 - t_1$ is special and other irreducible factors are normal. Then the normal part of the denominator of f is $B := (3t_0-2t_1)^2(t_0-t_1)^2(t_0+t_1)$ with dispersion $d = 2$. By Theorem 8, the normal part of the denominator of any solution g divides the polynomial $36(t_1+t_0)^3(t_1-t_0)^2$. Then we can make an ansatz for g as

$$g = \frac{U}{36(t_1+t_0)^3(t_1-t_0)^2(2t_0-t_1)},$$

where $U \in C(x)[t_0, t_1]$ satisfying the recurrence equation

$$(t_1+t_0)^3\sigma(U) - 72(t_1-t_0)(2t_1-3t_0)^2U = b,$$

where $b = (t_1+t_0)^2 6(t_1-t_0)(636t_0^3 + 443t_0^2t_1 - 1428t_0t_1^2 + 565t_1^3)$. We can bound the degree of U which is 3. Then we get $U = t_0^3 + 4t_0^2 + 5t_0t_1^2 + 2t_1^3$ by solving a linear difference system for rational solutions. So we have the rational solution

$$g = \frac{2t_1 + t_0}{36(t_1-2t_0)(t_0+t_1)(t_1-t_0)^2}.$$

We will discuss how to estimate special factors in the denominator of g in next section for the difference fields generated by P-recursive sequences.

2.2 Number of irreducible special polynomials

In Section 2.1, we have already outlined the procedure for computing the normal part of the denominator of g satisfying $\sigma(g) - g = f$. The challenge that remains is to handle the special part. As demonstrated in Example 4, a peculiar situation arises where the denominator of g contains a special polynomial that does not already appear in the denominator of f. Hence, to determine the denominator of g, it becomes necessary to identify all irreducible special polynomials. However, the computation of all special polynomials remains an unresolved issue at present. In this subsection, we aim to establish that the number of irreducible special polynomials in R that do not pairwise differ by elements of K^* is bounded by the number of generators of R over $C(x)$. This provides a crucial insight into the limited diversity of irreducible special polynomials. In Section 3.1, under certain assumptions, we will unveil the structure of irreducible special polynomials. We begin with the following lemma that is a direct consequence of Theorem 2.1.12 on page 114 of [32].

Lemma 10. *Suppose that $\mathcal{F}$ is a σ-field of characteristic zero with algebraically closed field C of constants, and $f \in \mathcal{F}$ satisfying that $\sigma^\ell(f) = f$ for some $\ell > 0$. Then $f \in C$.*

Corollary 11. *Suppose that $p_1, p_2, \ldots, p_m$ are special polynomials that are linearly independent over K, $\alpha_2, \ldots, \alpha_m \in K$. Then $p_1 + \alpha_2p_2 + \cdots + \alpha_mp_m$ is a special polynomial if and only if $\alpha_2 = \cdots = \alpha_m = 0$.*

Proof. It suffices to show the necessary part. Suppose that $p_1 + \alpha_2p_2 + \cdots + \alpha_mp_m$ is a special polynomial. Then there is a positive integer ℓ and $\gamma, \beta_1, \ldots, \beta_m \in K$ such that $\sigma^\ell(p_i) = \beta_ip_i$ and

$$\sigma^\ell(p_1 + \alpha_2p_2 + \cdots + \alpha_mp_m) = \gamma(p_1 + \alpha_2p_2 + \cdots + \alpha_mp_m).$$

A straightforward calculation reveals that $\beta_1 = \gamma$, and $\sigma^\ell(\alpha_i)\beta_i = \beta_1\alpha_i$ for all $2 \le i \le m$. This implies that $\sigma^\ell(\alpha_ip_i) = \beta_1\alpha_ip_i$ and consequently, $\sigma^\ell(\alpha_ip_i/p_1) = \alpha_ip_i/p_1$, for all $2 \le i \le m$. According to Lemma 10, $\alpha_ip_i/p_1 \in C$. For each $2 \le i \le m$, as p_i and p_1 are linearly independent over K, it follows that $\alpha_i = 0$. ∎

Proposition 12. *Suppose that $p_1, \ldots, p_m$ are irreducible special polynomials that are pairwise not shift equivalent. Denote by ℓ_i the smallest positive integer such that $p_i \mid \sigma^{\ell_i}(p_i)$. Then the $\sigma^j(p_i), i = 1, \ldots, m, j = 0, \ldots, \ell_i - 1$ are algebraically independent over K.*

Proof. Set $N = \operatorname{lcm}(\ell_1, \ldots, \ell_m)$. Then $\sigma^N(\sigma^j(p_i)) = \alpha_{i,j}\sigma^j(p_i)$ for some $\alpha_{i,j} \in K$. Suppose on the contrary that the $\sigma^j(p_i), i = 1, \ldots, m, j = 0, \ldots, \ell_i - 1$ are algebraically dependent over K. Due to the difference analogue of Kolchin-Ostrowski theorem (see [22, 34]), there are integers $d_{i,j}, i = 1, \ldots, m, j = 0, \ldots, \ell_i - 1$, not all zero, and $\beta \in K^*$ such that

$$\prod_{i=1}^{m}\prod_{j=0}^{\ell_i-1}\alpha_{i,j}^{d_{i,j}} = \frac{\sigma^N(\beta)}{\beta}.$$

Since $\sigma^N(\prod_{i,j}\sigma^j(p_i)^{d_{i,j}}) = \prod_{i,j}\alpha_{i,j}^{d_{i,j}}\prod_{i,j}\sigma^j(p_i)^{d_{i,j}}$, we have that

$$\sigma^N\left(\frac{\prod_{i,j}\sigma^j(p_i)^{d_{i,j}}}{\beta}\right) = \frac{\prod_{i,j}\sigma^j(p_i)^{d_{i,j}}}{\beta}.$$

Due to Lemma 10, $\prod_{i,j}\sigma^j(p_i)^{d_{i,j}} = c\beta$ for some $c \in C$. Denote $S_1 = \{(i,j) \mid d_{i,j} > 0\}$ and $S_2 = \{(i,j) \mid d_{i,j} < 0\}$. Since the $d_{i,j}$ are not all zero and $t_0, \ldots, t_{n-1}$ are algebraically independent over K, neither S_1 nor S_2 is empty. This leads to

$$\prod_{(i,j)\in S_1}\sigma^j(p_i)^{d_{i,j}} = c\beta\prod_{(i,j)\in S_2}\sigma^j(p_i)^{-d_{i,j}}.$$

Choose $(i_1, j_1) \in S_1$. Then $\sigma^{j_1}(p_{i_1})$ divides $\prod_{(i,j)\in S_2}\sigma^j(p_i)^{-d_{i,j}}$ and thus there exists $(i_2, j_2) \in S_2$ such that $\sigma^{j_1}(p_{i_1})$ divides $\sigma^{j_2}(p_{i_2})$. As both $\sigma^{j_1}(p_{i_1})$ and $\sigma^{j_2}(p_{i_2})$ are irreducible, $\sigma^{j_1}(p_{i_1}) = \gamma\sigma^{j_2}(p_{i_2})$ for some $\gamma \in K$. If $i_1 = i_2$ then $0 \le j_1 \ne j_2 \le \ell_{i_1} - 1$. Without loss of generality, assume $j_2 > j_1$. Then $\sigma^{j_2-j_1}(p_{i_1}) = p_{i_1}/\sigma^{-j_1}(\gamma)$, which contradicts the minimality of ℓ_{i_1}. If $i_1 \ne i_2$ then p_{i_1} and p_{i_2} are shift equivalent, contradicting the initial assumption. ∎

Since F is a field generated over K by n indeterminates, the transcendence degree of F over K is equal to n. So we have the following corollary.

Corollary 13. *Suppose that $p_1, \ldots, p_m$ are irreducible special polynomials that are pairwise not shift equivalent. Then $\sum_{i=1}^{m}\ell_i \le n$, where ℓ_i is the smallest positive integer such that $p_i \mid \sigma^{\ell_i}(p_i)$.*

3 THE P-RECURSIVE CASE

P-recursive sequences, introduced by Stanley [44], satisfy linear recurrence equations with polynomial coefficients. The generating function of a P-recursive sequence is a D-finite function, which satisfies a linear differential equations with polynomial coefficients. This class of sequences has been extensively studied in combinatorics [45, 49] and symbolic computation [29, 30] together with its generating functions. In this section, we will focus on parallel summation in difference fields generated by P-recursive sequences.

Let F be the field $C(x)(t_0, \dots, t_{n-1})$ with a C-automorphism σ satisfying that $\sigma(x) = x + 1$, $\sigma(t_0) = t_1, \dots, \sigma(t_{n-2}) = t_{n-1}$, and

$$\sigma(t_{n-1}) = a_0 t_0 + \cdots + a_{n-1} t_{n-1},$$

where $a_0, \dots, a_{n-1} \in C(x)$ and $a_0 \neq 0$. So σ is a C-automorphism of the ring $R = C(x)[t_0, \dots, t_{n-1}]$. We still assume in this section that the constant field of (F, σ) is the field C, even though this condiction is not easy to check for a given field. In order to study the indefinite summation problem in F, we first address two basic questions on special and normal polynomials. In Section 3.1, we prove some structural properties on special polynomials under certain assumptions. In Section 3.2, we will answer the question of deciding whether two irreducible polynomials in R are σ-equivalent or not.

3.1 Degrees of irreducible special polynomials

In the P-recursive case, by Lemma 14 below, computing all special polynomials of degree m is equivalent to computing all hypergeometric solutions of the mth symmetric power of the system

$$\sigma(Y) = AY \tag{3}$$

or $\sigma^s(Y) = A_{(s)} Y$ for some $s > 1$, where

$$A = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ a_0 & a_1 & a_2 & \dots & a_{n-1} \end{pmatrix}$$

and $A_{(s)} = \sigma^{s-1}(A) \dots \sigma(A) A$. Algorithms for computing all hypergeometric solutions of a given linear difference equation are known, for example, refer to [35]. Corollary 13 establishes the existence of a degree bound for all irreducible special polynomials. However, by the absence of a known degree bound, the computation of all irreducible special polynomials remains an unresolved challenge. In this subsection, we will prove that when $\sum_{i=1}^m \ell_i = n$ with ℓ_i as defined in Corollary 13, all irreducible special polynomials are linear in $t_0, t_1, \dots, t_{n-1}$. Consequently, in this specific case, the degree of all irreducible special polynomials is exactly equal to 1 and thus we can compute all irreducible special polynomials.

Lemma 14. *All special polynomials are homogeneous.*

Proof. Suppose that p is a special polynomial and is not homogeneous. Write $p = \sum_{i=0}^m p_i$ where p_i is the ith homogeneous part of p and $p_m \neq 0$. Assume that $\sigma^\ell(p) = \alpha p$ for some nonzero $\alpha \in C(x)$. Then $\sigma^\ell(p) = \sum_{i=0}^m \sigma^\ell(p_i) = \alpha p = \sum_{i=0}^m \alpha p_i$. Note that $\sigma^\ell(p_i)$ is also homogeneous of degree i. We have that $\sigma^\ell(p_i) = \alpha p_i$ for all $0 \le i \le m$. Since p is not homogeneous, there is an i_0 such that $p_{i_0} \neq 0$. Hence $\sigma^\ell(p_{i_0}/p_m) = p_{i_0}/p_m$. According to Lemma 10, $p_{i_0}/p_m \in C$, which contradicts the fact that the numerator and denominator of p_{i_0}/p_m have different degrees. ∎

We start with the C-finite case. In this case, we will demonstrate that the degree of all irreducible special polynomials is always equal to 1, without requiring the assumption that $\sum_{i=1}^m \ell_i = n$.

Proposition 15. *Suppose that $A \in \mathrm{GL}_n(C)$. Then all irreducible special polynomials are linear in $t_0, t_1, \dots, t_{n-1}$.*

Proof. Let $B \in \mathrm{GL}_n(C)$ such that $BAB^{-1} = \mathrm{diag}(J_1, J_2, \dots, J_\ell)$, where J_i is a Jordan block of order n_i. We claim that $n_i = 1$ for all $1 \le i \le \ell$. Without loss of generality, assume that $n_1 > 1$ and α_1 is the eigenvalue of J_1. Set $\bar{T} = (\bar{t}_0, \dots, \bar{t}_{n-1})^t = B(t_0, \dots, t_{n-1})^t$. Then $\sigma(\bar{T}) = BAB^{-1}\bar{T}$. Therefore $\sigma(\bar{t}_{n_1-1}) = \alpha_1 \bar{t}_{n_1-1} + \bar{t}_{n_1}$ and $\sigma(\bar{t}_{n_1}) = \alpha_1 \bar{t}_{n_1}$. From these, it follows that $\sigma(\frac{\bar{t}_{n_1-1}}{\bar{t}_{n_1}}) = \frac{\bar{t}_{n_1-1}}{\bar{t}_{n_1}} + \frac{1}{\alpha_1}$. Consequently,

$$\sigma\left(\frac{\bar{t}_{n_1-1}}{\bar{t}_{n_1}} - \frac{x}{\alpha_1}\right) = \frac{\bar{t}_{n_1-1}}{\bar{t}_{n_1}} - \frac{x}{\alpha_1}.$$

In other words, $\bar{t}_{n_1-1}/\bar{t}_{n_1} - x/\alpha_1 \in C$, which is a contradiction with Hypothesis 5. This proves our claim. Therefore $BAB^{-1} = \mathrm{diag}(\alpha_1, \dots, \alpha_n)$, where $\alpha_i \in C$. Finally, suppose that p is an irreducible special polynomial. Note that p can be expressed as a polynomial in $\bar{t}_0, \bar{t}_1, \dots, \bar{t}_{n-1}$. Corollary 11 implies that p is a monomial in $\bar{t}_0, \bar{t}_1, \dots, \bar{t}_{n-1}$. Hence $p = \beta \bar{t}_i$ for some $0 \le i \le n-1$ and $\beta \in K$, and so it is linear in $t_0, t_1, \dots, t_{n-1}$. ∎

Remark 16. *In the proof of Proposition 15, the special polynomials $\bar{t}_0, \dots, \bar{t}_{n-1}$ are pairwise not shift equivalent and then the condition $\sum_{i=1}^m \ell_i = n$ is automatically satisfied. In fact, suppose $\sigma(\bar{t}_{i_1}) = \beta \bar{t}_{i_2}$ for some $0 \le i_1 \neq i_2 \le n-1$ and $\beta \in K$. Since $\sigma(\bar{t}_{i_1}) = \alpha \bar{t}_{i_1}$ for some $\alpha \in K$, it follows that $\bar{t}_1, \bar{t}_2$ are linearly dependent over K, which contradicts the fact that $\bar{t}_0, \dots, \bar{t}_{n-1}$ are algebraically independent over K.*

Before proceeding to the general case, let's recall some fundamental results from difference Galois theory. For detailed information, readers can refer to Chapter 1 of [47]. Let $\mathcal{R}$ be the Picard-Vessiot ring for $\sigma(Y) = AY$ over K, where A is given as in (3). In $\mathcal{R}$, there exist idempotents $e_0, e_1, \dots, e_{s-1}$ such that

$$\mathcal{R} = \mathcal{R}_0 \oplus \mathcal{R}_1 \oplus \cdots \oplus \mathcal{R}_{s-1},$$

where $\mathcal{R}_i = e_i \mathcal{R}$ and $\mathcal{R}_i$ is a domain. Moreover, $\mathcal{R}_i$ serves as the Picard-Vessiot ring for $\sigma^s(Y) = A_{(s)} Y$ over K with

$$A_{(s)} = \sigma^{s-1}(A) \dots \sigma(A) A.$$

Let G be the Galois group of $\sigma(Y) = AY$ over K and H be the Galois group of $\sigma^s(Y) = A_{(s)} Y$ over K. By Corollary 1.17 [47, p.13] we have $[G : H] = s$ and consequently, H contains G°, the identity component of G. On the other hand, due to Proposition 1.20, $\mathcal{R}_i$ is a trivial H-torsor which implies that H is connected since $\mathcal{R}_i$ is a domain. Hence $H = G^\circ$. In the following, it will be shown that the hypothesis $\sum_{i=1}^m \ell_i = n$ implies that the group G° is a torus. Consequently, the system $\sigma^s(Y) = A_{(s)} Y$ is equivalent to a diagonal system, indicating that appropriate linear combinations of $t_0, t_1, \dots, t_{n-1}$ are special polynomials. Indeed, they are all irreducible special polynomials.

Lemma 17. *Suppose that $p_1, \dots, p_m$ are special polynomials. Then there exists a fundamental matrix $\mathcal{Z} \in \mathrm{GL}_n(\mathcal{R})$ of $\sigma(Y) = AY$ such that $p_i(\mathcal{Z}_j)$ is invertible in $\mathcal{R}$ for all $1 \le i \le m, 1 \le j \le n$, where $\mathcal{Z}_j$ denotes the jth column of $\mathcal{Z}$.*

Proof. Let N be a positive integer such that $p_i \mid \sigma^N(p_i)$ for all $1 \le i \le m$ and let $q = (\prod_{i=1}^m p_i)\sigma(\prod_{i=1}^m p_i) \cdots \sigma^{N-1}(\prod_{i=1}^m p_i)$. Then $\sigma(q) = \alpha q$ for some $\alpha \in K$. We first show the lemma for q.

Let $\mathcal{Z} \in \mathrm{GL}_n(\mathcal{R})$ be a fundamental matrix of $\sigma(Y) = AY$. We claim that there exists an $M \in \mathrm{GL}_n(C)$ such that $q((\mathcal{Z}M)_j) \neq 0$. Let $\mathbf{u} = (u_1, \dots, u_n)^t$ be a vector with indeterminate entries. Since $\mathcal{Z}$ is invertible, as a polynomial in $\mathcal{R}[\mathbf{u}]$, $q(\mathcal{Z}\mathbf{u}) \neq 0$. Write $q(\mathcal{Z}\mathbf{u}) = \sum_{j=1}^d f_j(\mathbf{u})\mathbf{m}_j$, where $f_j(\mathbf{u}) \in C[\mathbf{u}]$ and $\mathbf{m}_1, \dots, \mathbf{m}_d \in \mathcal{R}$ are linearly independent over C. As $q(\mathcal{Z}\mathbf{u}) \neq 0$, at least one of $f_1(\mathbf{u}), \dots, f_d(\mathbf{u})$ is not zero, say $f_{j_1}(\mathbf{u}) \neq 0$. Set U to be the Zariski open subset of C^n consisting of all $\mathbf{a}$ in C^n such that $f_{j_1}(\mathbf{a}) \neq 0$. Then $U \times \cdots \times U$ is a non-empty Zariski open subset of $C^{n\times n}$, where the direct product takes n times. Let $M \in U \times \cdots \times U$ be such that $\det(M) \neq 0$. Such M exists because $U \times \cdots \times U$ is Zariski dense in $C^{n\times n}$. Then for each column $\mathbf{c}$ of M, it follows that $f_{j_1}(\mathbf{c}) \neq 0$, and thus $q(\mathcal{Z}\mathbf{c}) \neq 0$. Since M is invertible, $\mathcal{Z}M$ is also a fundamental matrix of $\sigma(Y) = AY$. This proves our claim.

Let $\mathbf{c}$ be a column of M. Then

$$\sigma(q(\mathcal{Z}\mathbf{c})) = q^\sigma(A\mathcal{Z}\mathbf{c}) = \sigma(q)(\mathcal{Z}\mathbf{c}) = \alpha q(\mathcal{Z}\mathbf{c})$$

where q^σ denotes the polynomial obtained by applying σ to the coefficients of q. Hence, $q(\mathcal{Z}\mathbf{c})$ generates a nonzero σ-ideal in $\mathcal{R}$. Since $\mathcal{R}$ is σ-simple, this ideal must be equal to $\mathcal{R}$ and thus $q(\mathcal{Z}\mathbf{c})$ is invertible in $\mathcal{R}$. Finally, for each $1 \le i \le m$, since $q(\mathcal{Z}\mathbf{c}) = p_i(\mathcal{Z}\mathbf{c})h_i$ for some $h_i \in \mathcal{R}$, $p_i(\mathcal{Z}\mathbf{c})$ is invertible in $\mathcal{R}$. ∎

Lemma 18. *Suppose that there exist irreducible special polynomials $p_1, \dots, p_m$ that are pairwise not shift equivalent, satisfying that $\sum_{i=1}^m \ell_i = n$, where ℓ_i is the smallest positive integer such that $p_i \mid \sigma^{\ell_i}(p_i)$. Then $\dim(G) = n$.*

Proof. It suffices to show that $\dim(G^\circ) = n$. Set

$$q_{\ell_0+\ell_1+\cdots+\ell_i+j} = \sigma^{j-1}(p_{i+1})$$

where $0 \le i \le m-1$, $1 \le j \le \ell_i$, and $\ell_0 = 0$. By Lemma 17, there exists a fundamental matrix $\mathcal{Z} \in \mathrm{GL}_n(\mathcal{R})$ of $\sigma(Y) = AY$ such that $q_i(\mathcal{Z}_j)$ is invertible in $\mathcal{R}$ for all $1 \le i, j \le n$, where $\mathcal{Z}_j$ denotes the jth column of $\mathcal{Z}$. Consider $\mathcal{R}_0$, the Picard-Vessiot ring for $\sigma^s(Y) = A_{(s)}Y$ over K. Let $\mathcal{F}$ be the field of fractions of $\mathcal{R}_0$. Since G° is the Galois group of $\sigma^s(Y) = A_{(s)}Y$ over K, $\dim(G^\circ) = \mathrm{tr.deg}(\mathcal{F}/K)$. Note that $e_0\mathcal{Z}$ is a fundamental matrix of $\sigma^s(Y) = A_{(s)}Y$ and $e_0 q_i(\mathcal{Z}_j) = q_i(e_0\mathcal{Z}_j)$ is invertible in $\mathcal{R}_0$. Set $N = \mathrm{lcm}(\ell_1, \dots, \ell_m)$. Then $q_i \mid \sigma^{sN}(q_i)$ for all $1 \le i \le n$. Suppose that $\sigma^{sN}(q_i) = \alpha_i q_i$ with $\alpha_i \in K$. Then for each $1 \le i \le n$, it holds that $\sigma^{sN}(q_i(e_0\mathcal{Z}_j)) = \alpha_i q_i(e_0\mathcal{Z}_j)$ for all $1 \le j \le n$.

We claim that for each $1 \le j \le n$, $q_1(e_0\mathcal{Z}_j), \dots, q_n(e_0\mathcal{Z}_j)$ are algebraically independent over K. Assume on the contrary that $q_1(e_0\mathcal{Z}_j), \dots, q_n(e_0\mathcal{Z}_j)$ are algebraically dependent over K. Using an argument similar to that in the proof of Proposition 12, we obtain integers d_i, not all zero, and a nonzero $\beta \in K$ such that

$$\sigma^{sN}\left(\frac{\prod_i q_i^{d_i}}{\beta}\right) = \frac{\prod_i q_i^{d_i}}{\beta}.$$

Due to Lemma 10, $\prod_i q_i^{d_i} = c\beta$ for some $c \in C$. In other words, $q_1, \dots, q_n$ are algebraically dependent over K. This contradicts the conclusion of Proposition 12. The claim is established.

Now, for $1 \le j_1 \neq j_2 \le n$, we have that

$$\sigma^{sN}(q_i(e_0\mathcal{Z}_{j_1})/q_i(e_0\mathcal{Z}_{j_2})) = q_i(e_0\mathcal{Z}_{j_1})/q_i(e_0\mathcal{Z}_{j_2}).$$

By Lemma 10 (replacing σ with σ^s), $q_i(e_0\mathcal{Z}_{j_1}) = c_{i,j_1,j_2} q_i(e_0\mathcal{Z}_{j_2})$ for all $1 \le i \le n$, where $c_{i,j_1,j_2} \in C$. Denote by $\tilde{\mathcal{F}}$ the subfield of $\mathcal{F}$ generated by all $q_i(e_0\mathcal{Z}_j)$ over K. Then the previous discussion implies that $\mathrm{tr.deg}(\tilde{\mathcal{F}}/K) = n$. Note that for each $1 \le j \le n$, every entry of $e_0\mathcal{Z}_j$ is algebraic over $K(q_1(e_0\mathcal{Z}_j), \dots, q_n(e_0\mathcal{Z}_j))$ (and thus algebraic over $\tilde{\mathcal{F}}$), because $q_1(e_0\mathcal{Z}_j), \dots, q_n(e_0\mathcal{Z}_j)$ are algebraically independent over K and they are polynomial in the entries of $e_0\mathcal{Z}_j$. Hence $\mathcal{F}$ is a finite algebraic extension of $\tilde{\mathcal{F}}$, as $\mathcal{F} = K(e_0\mathcal{Z})$. So $\mathrm{tr.deg}(\mathcal{F}/K) = n$ and then $\dim(G^\circ) = n$. Consequently, $\dim(G) = n$. ∎

Theorem 19. *Under the same assumption as in Lemma 18, all irreducible special polynomials are linear in $t_0, t_1, \dots, t_{n-1}$.*

Proof. Let q_i, α_i be as in the proof of Lemma 18. We first show that G° is a torus. Due to Lemma 18, $\dim(G^\circ) = n$. Hence the rank of $X(G^\circ)$, the group of characters of G° (which is a free abelian group), is at most n. As $\sigma^{sN}(q_i(e_0\mathcal{Z}_1)) = \alpha_i q_i(e_0\mathcal{Z}_1)$, for each $g \in G^\circ$, $\sigma^{sN}(g(q_i(e_0\mathcal{Z}_1))) = \alpha_i g(q_i(e_0\mathcal{Z}_1))$. Hence $g(q_i(e_0\mathcal{Z}_1)) = \chi_i(g)g(q_i(e_0\mathcal{Z}_1))$, where $\chi_i(g) \in C$. In other words, $q_i(e_0\mathcal{Z}_1)$ induces a character $\chi_i \in X(G^\circ)$. Suppose that there are integers d_i, not all zero, such that $\prod_i \chi_i^{d_i} = \mathrm{id}$, where id is the unitary of $X(G^\circ)$. Then for all $g \in G^\circ$,

$$g\left(\prod_i q_i(e_0\mathcal{Z}_1)^{d_i}\right) = \prod_i \chi_i^{d_i}(g)\prod_i q_i(e_0\mathcal{Z}_1)^{d_i} = \prod_i q_i(e_0\mathcal{Z}_1)^{d_i}.$$

The Galois correspondence (see, for example, Lemma 1.28 on page 20 of [47]) implies that $\prod_i q_i(e_0\mathcal{Z}_1)^{d_i} \in K$, which contradicts the fact that $q_1(e_0\mathcal{Z}_1), \dots, q_n(e_0\mathcal{Z}_1)$ are algebraically independent over K. Therefore the rank of $X(G^\circ)$ is exactly equal to n. Let $\tilde{\chi}_1, \dots, \tilde{\chi}_n$ be a base of $X(G^\circ)$, as a free abelian group. Consider the morphism

$$\begin{aligned} \phi\colon G^\circ &\longrightarrow \mathrm{GL}_1(C)^n \\ g &\longmapsto (\tilde{\chi}_1(g), \dots, \tilde{\chi}_n(g)). \end{aligned}$$

Then ϕ is surjective with finite kernel because $\dim(G^\circ) = n$. Moreover, due to Lemma B.20 of [18], $\ker(\phi)$ is generated as an algebraic group by all unipotent elements of G°. Note that if $h \in \mathrm{GL}_n(C)$ is unipotent then h is of finite order if and only if $h = I$, the identity matrix. So $\ker(\phi) = \{I\}$ and ϕ is an isomorphism. This proves that G° is a torus.

By Theorem 2.1 of [24], there exists a $T \in \mathrm{GL}_n(K)$ such that

$$\sigma^s(T)A_{(s)}T^{-1} = \mathrm{diag}(b_1, \dots, b_n),$$

where $b_i \in K$. Set $(\bar{t}_0, \dots, \bar{t}_{n-1})^t = T(t_0, \dots, t_{n-1})^t$. Then $\sigma^s(\bar{t}_i) = b_i\bar{t}_i$ for all $0 \le i \le n-1$. In other words, $\bar{t}_0, \dots, \bar{t}_{n-1}$ are special polynomials. Finally, using an argument similar to the proof of Proposition 15, it follows that every irreducible special polynomial is linear in $t_0, t_1, \dots, t_{n-1}$. ∎

3.2 The σ-Equivalence Problem

We now present a method for deciding whether two irreducible polynomials $p, q \in R = K[t_0, \dots, t_{r-1}]$ are σ-equivalent or not.

If one of p and q is special, say p, then there exists a minimal positive integer m (not greater than n by Corollary 13) such that $p \mid \sigma^m(p)$. To decide whether p and q are σ-equivalent, it suffices to check whether q is associate over K to one of elements in the set $\{p, \sigma(p), \dots, \sigma^{m-1}(p)\}$. It remains to consider the case in which both p and q are normal.

We now assume that $p, q \in R$ are irreducible and normal in R. We want to decide whether there exist $i \in \mathbb{Z}$ and $u \in K \setminus \{0\}$ such that $\sigma^i(p) = uq$. Observe first that there can be at most one such i. For, if i, i' and u, u' are such that $\sigma^i(p) = uq$ and $\sigma^{i'}(p) = u'q$, then $\sigma^i(p)/\sigma^{i'}(p) = u/u'$, so $p \mid \sigma^{i'-i}(p)$, and since p is not special, we must have $i = i'$. Observe also that for a given candidate $i \in \mathbb{Z}$, it is easy to check whether there exists a u with $\sigma^i(p) = uq$. Therefore, it suffices to determine a finite list of candidates for i.

Let $L, M \in K[S]$ be the (unique) monic minimal order annihilating operators of p and q, respectively. Let s be their order. Note that p and q cannot be shift equivalent if the orders of L and M are distinct. Write $L = S^s + \ell_{s-1}S^{s-1} + \dots + \ell_0$ and $M = S^s + m_{s-1}S^{s-1} + \dots + m_0$. By the minimality of the order of L and M, we have $\ell_0, m_0 \neq 0$.

For every $i \in \mathbb{Z}$, the monic minimal order annihilating operator of $\sigma^i(p)$ is

$$L^{(i)} := S^s + \sigma^i(\ell_{s-1})S^{s-1} + \dots + \sigma^i(\ell_0),$$

and for every $u \in K \setminus \{0\}$, the monic minimal order annihilating operator of $\frac{1}{u}q$ is

$$\frac{1}{\sigma^s(u)}Mu = S^s + m_{s-1}\frac{\sigma^{s-1}(u)}{\sigma^s(u)}S^{s-1} + \dots + m_0\frac{u}{\sigma^s(u)}.$$

A necessary condition for a pair (i, u) to be a solution to the shift equivalence problem is that $L^{(i)} = \frac{1}{\sigma^s(u)}Mu$. Therefore, for every such pair we must have

$$\frac{\sigma^i(\ell_k)}{m_k} = \frac{\sigma^k(u)}{\sigma^s(u)}$$

simultaneously for all $k \in \{0, \dots, s\}$.

Observe that L must have at least three terms. If it had only two terms, we would have $L = S^s + \ell_0$. This means $\sigma^s(p) = -\ell_0 p$, and this is a contradiction to p not being special. We can therefore assume that L has at least three terms. We may further assume that the coefficient of S^k in L is nonzero if and only if the coefficient of S^k in M is nonzero, because if this is not the case, then the shift equivalence problem has no solution.

Lemma 20. *Under these circumstances, we have*

$$\sigma^i\Big(\frac{\ell_k/\sigma^s(\ell_k)}{\sigma^k(\ell_0)/\sigma^s(\ell_0)}\Big) = \frac{m_k/\sigma^s(m_k)}{\sigma^k(m_0)/\sigma^s(m_0)} \tag{4}$$

for every $k \in \{1, \dots, s-1\}$ such that $\ell_k \neq 0$.

Proof. From

$$(a)\quad \frac{\sigma^i(\ell_0)}{m_0} = \frac{u}{\sigma^s(u)} \quad\text{and}\quad (b)\quad \frac{\sigma^i(\ell_k)}{m_k} = \frac{\sigma^k(u)}{\sigma^s(u)}$$

we obtain

$$(c)\quad \frac{\sigma^i(\ell_0)}{m_0}\frac{m_k}{\sigma^i(\ell_k)} = \frac{u}{\sigma^k(u)}$$

Apply σ^k to (a) and σ^r to (c) to obtain

$$(a')\quad \frac{\sigma^i(\sigma^k(\ell_0))}{\sigma^k(m_0)} = \frac{\sigma^k(u)}{\sigma^{s+k}(u)} \quad\text{and}$$

$$(c')\quad \frac{\sigma^i(\sigma^s(\ell_0))}{\sigma^s(m_0)}\frac{\sigma^s(m_k)}{\sigma^i(\sigma^s(\ell_k))} = \frac{\sigma^s(u)}{\sigma^{s+k}(u)}.$$

Dividing (a') by (c') gives

$$\frac{\sigma^i(\sigma^k(\ell_0))\sigma^s(m_0)\sigma^i(\sigma^s(\ell_k))}{\sigma^k(m_0)\sigma^i(\sigma^s(\ell_0))\sigma^s(m_k)} = \frac{\sigma^k(u)}{\sigma^s(u)}.$$

Finally, divide (b) by this equation to obtain

$$\frac{\sigma^k(m_0)\sigma^i(\sigma^s(\ell_0))\sigma^s(m_k)\sigma^i(\ell_k)}{\sigma^i(\sigma^k(\ell_0))\sigma^s(m_0)\sigma^i(\sigma^s(\ell_k))m_k} = 1.$$

The claim follows from here. ∎

Unless both sides of Equation (4) are constant, we get at most one candidate for i and are done. It remains to consider the case when both sides are constant for every k with $\ell_k \neq 0$ (and $m_k \neq 0$). In this case, the constant can only be 1, because ℓ_0, ℓ_k, m_0, m_k are rational functions and σ does not change leading terms.

If both sides of (4) are equal to 1 then

$$\sigma^s(m_0)M\frac{1}{m_0} = \sum_{k=0}^{s}\frac{m_k}{\sigma^k(m_0)/\sigma^s(m_0)}S^k = \sum_{k=0}^{s}\sigma^s(m_k)S^k = M^{(s)}.$$

Therefore, if $i \in \mathbb{Z}$ and $u \in K$ are such that $L^{(i)} = \frac{1}{\sigma^s(u)}Mu$, then we also have $L^{(i+s)} = \frac{1}{\sigma^{2s}(u)}M^{(s)}\sigma^s(u) = \frac{1}{\sigma^s(\sigma^s(u)m_0)}M\sigma^s(u)m_0$.

This means that in terms of operators, the shift equivalence problem may have more than one solution in the situation under consideration. In the former cases, where there was at most one i with $L^{(i)} = \frac{1}{\sigma^s(u)}Mu$, this i is then the only candidate for which we can possibly have $\sigma^i(p) = uq$. In the present situation, where there are infinitely many i's that solve the problem on the level of operators, it remains to determine which of them (if any) solves the original problem in terms of p and q.

Lemma 21. *If $\sigma^s(m_0)M = M^{(s)}m_0$, then there is an operator $T \in C[S]$ with constant coefficients such that M is a right factor of the symmetric product $T \otimes (S^s - m_0)$.*

Proof. The condition $\sigma^s(m_0)M = M^{(s)}m_0$ means that for any solution q of M, also $\frac{1}{m_0}\sigma^s(q)$ is a solution of M. But the solutions of M form a C-vector space of dimension at most s, so for every solution q of M, the elements

$$q,\ \frac{1}{m_0}\sigma^s(q),\ \frac{1}{m_0\sigma^s(m_0)}\sigma^{2s}(q),\ \dots,\ \Big(\prod_{i=0}^{s-1}\frac{1}{\sigma^{si}(m_0)}\Big)\sigma^{s^2}(q)$$

are linearly dependent over C.

Therefore, every solution of M is also a solution of an operator of the form

$$S^{s^2} + c_{s(s-1)}\sigma^{s(s-1)}(m_0)S^{s(s-1)} + \cdots$$
$$\cdots + c_1\Big(\prod_{i=1}^{s-1}\sigma^{si}(m_0)\Big)S^s + c_0\Big(\prod_{i=0}^{s-1}\sigma^{si}(m_0)\Big)$$

for certain constants $c_0, c_s, \dots, c_{s(s-1)}$.

This operator can be factored as a symmetric product. Up to (irrelevant) left-multiplication by an element of K, it is equal to

$$(S^{s^2} + c_{s(s-1)} S^{s(s-1)} + \cdots + c_1 S^s + c_0) \otimes (S^s - m_0).$$

This completes the proof. ∎

This means that every solution of M, in particular q, can be interpreted as a product of a C-finite and an s-hypergeometric quantity, i.e., a quantity annihilated by an operator of the form $S^s - r$ for some rational function r. (We do not claim that these factors are elements of R.)

We can reason analogously for L and find that every solution of L, in particular p, can be interpreted as a product of a C-finite and an s-hypergeometric quantity, with the s-hypergeometric part annihilated by $S^s - \ell_0$.

If the s-hypergeometric factors are not also C-finite, then their comparison leads to at most one candidate $i \in \mathbb{Z}$ such that $\sigma^i(p)/q \in K$. In this comparison, we must take into account that in the factorization of p and q into a C-finite and an s-hypergeometric part, exponential terms λ^x and polynomials in x can be freely moved from one factor to the other.

For the comparison, we use the Gosper-Petkovšek form [36] of ℓ_0 and m_0:

$$\ell_0 = \lambda \frac{\sigma(a)}{a} \frac{b}{c}, \qquad m_0 = \tilde{\lambda} \frac{\sigma(\tilde{a})}{\tilde{a}} \frac{\tilde{b}}{\tilde{c}}.$$

We ignore $\lambda, \tilde{\lambda}, a, \tilde{a}$, as they correspond to the exponential and polynomial part, respectively, and check if there is an $i \in \mathbb{Z}$ such that $\sigma^i(b/c) = \tilde{b}/\tilde{c}$. If so, then this i is the only candidate for which we may have $\sigma^i(p)/q \in K$.

It remains to consider the case when p and q both are C-finite. By Theorem 4.1 in [46], there exist pairwise distinct $\lambda_1, \dots, \lambda_t \in C$, pairwise distinct $\mu_1, \dots, \mu_{t'} \in C$ and polynomials $a_1, \dots, a_t, b_1, \dots, b_{t'} \in C[x]$ such that

$$p = a_1(n)\lambda_1^n + \cdots + a_t(n)\lambda_t^n$$

$$q = b_1(n)\mu_1^n + \cdots b_{t'}(n)\mu_{t'}^n,$$

The requirement $\sigma^i(p) = uq$ translates into

$$\sigma^i(a_1)\lambda_1^{i+n} + \cdots + \sigma(a_s)\lambda_s^{i+n} = ub_1\mu_1^n + \cdots + ub_t\mu_t^n.$$

There is no solution unless $\{\lambda_1, \dots, \lambda_t\} = \{\mu_1, \dots, \mu_{t'}\}$, so we may assume that $t = t'$ and $\lambda_k = \mu_k$ for all $1 \le k \le t$. Then we need

$$\lambda_k^i \sigma^i(a_k) = ub_k$$

for all k. From any two such equations, say the kth and the ℓth, we get the constraint

$$\left(\frac{\lambda_k}{\lambda_\ell}\right)^i \sigma^i \left(\frac{a_k}{a_\ell}\right) = \frac{b_k}{b_\ell}$$

If a_k/a_ℓ is not a constant or λ_k/λ_ℓ is not a root of unity, then there is at most one solution i. If a_k/a_ℓ is a constant and λ_k/λ_ℓ is a root of unity for every choice of k and ℓ, then p can be written as a product of a_1 and λ_1^x and a C-linear combination of powers of roots of unity. This implies that p/a_1 is a special polynomial, which conflicts with the assumption that p is normal.

In conclusion, we have proven the correctness of the following algorithm.

Algorithm 22. *INPUT: $p, q \in R = C(x)[t_0, \dots, t_{r-1}]$ irreducible and normal*
OUTPUT: $i \in \mathbb{Z}$ such that either $\sigma^i(p)/q \in C(x)$ or p and q are not shift-equivalent.

Compute monic minimal annihilating operators $L, M \in C(x)[S]$ of p and q.
If there is a k such that the coefficient of S^k is zero in one of the two operators but nonzero in the other, return 0.
Let s be the order of L (and M).
For every $k \in \{1, \dots, s-1\}$ with $\ell_k \neq 0$, do:
  If at least one of the two rational functions $a := \frac{\ell_k/\sigma^s(\ell_k)}{\sigma^k(\ell_0)/\sigma^s(\ell_0)}$, $b := \frac{m_k/\sigma^s(m_k)}{\sigma^k(m_0)/\sigma^s(m_0)}$ is not in C
    Return $i \in \mathbb{Z}$ such that $\sigma^i(a) = b$, or 0 if no such i exists
Compute the Gosper-Petkovšek form $\lambda \frac{\sigma(a)}{a} \frac{b}{c}$ of ℓ_0 and the Gosper-Petkovšek form $\tilde{\lambda} \frac{\sigma(\tilde{a})}{\tilde{a}} \frac{\tilde{b}}{\tilde{c}}$ of m_0.
If $b/c \neq 1$ or $\tilde{b}/\tilde{c} \neq 1$ then
  Return $i \in \mathbb{Z}$ such that $\sigma^i(b/c) = \tilde{b}/\tilde{c}$, or 0 if no such i exists
Compute the constants $\lambda_1, \dots, \lambda_s \in C$, polynomials $a_1, \dots, a_s$, $b_1, \dots, b_s \in C[x]$, as above.
For $k = 1, \dots, s$, do:
  For $\ell = 1, \dots, k-1$, do:
    If a_k/a_ℓ is not a constant or λ_k/λ_ℓ is not a root of unity then
      Return $i \in \mathbb{Z}$ such that $(\frac{\lambda_k}{\lambda_\ell})^i \sigma^i(\frac{a_k}{a_\ell}) = \frac{b_k}{b_\ell}$, or 0 if no such i exists.

4 THE C-FINITE CASE

The problem of indefinite summation in the C-finite case has been investigated in [16, 25, 48] under specific assumptions. Let A be defined as in (3) and assume that $A \in \mathrm{GL}_n(C)$. In [25], the authors introduce a method for computing rational solutions of the equation $u\sigma(y) - vy = w$, where $u, v, w \in R$, under the assumption that $n = 2$ and A has two eigenvalues λ_1, λ_2 such that λ_1/λ_2 is not a root of unity. However, their method is not complete as they are unable to bound the multiplicities of irreducible special polynomials appearing in the denominator of solutions. In [16], assuming that A is a diagonalizable matrix, the authors characterize all new constants in F and present an algorithm for computing one rational solution of the equation $c\sigma(y) - y = f$, where $c \in C^*$ and $f \in F$. According to the proof of Proposition 15, if F contains no new constants, then A will always be diagonalizable, and thus the C-finite case under our assumption is reduced to the case considered in [16].

Acknowledgement. The authors thank Michael Singer for many discussions about summation in finite terms.

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