Constructing minimal telescopers for rational functions in three discrete variables*

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Abstract

We present a new algorithm for constructing minimal telescopers for rational functions in three discrete variables. This is the first discrete reduction-based algorithm that goes beyond the bivariate case. The termination of the algorithm is guaranteed by a known existence criterion of telescopers. Our approach has the important feature that it avoids the potentially costly computation of certificates. Computational experiments are also provided so as to illustrate the efficiency of our approach.

1 Introduction

Creative telescoping is a powerful tool used to find closed form solutions for definite sums and definite integrals. The method constructs a recurrence (resp. differential) equation satisfied by the definite sum (resp. integral) with closed form solutions over a specified domain resulting in formulas for the sum or integral. Methods for finding such closed form solutions are available for many special functions, with examples given in [2, 33, 4, 8]. Even when no closed form exists the telescoping method often remains useful. For example the resulting recurrence or differential equation enables one to determine asymptotic expansions and derive other interesting facts about the original sum or integral.

In the case of summation, specialized to the trivariate case, in order to compute a sum of the form

$$\sum_{y=a_1}^{b_1}\sum_{z=a_2}^{b_2} f(x,y,z),$$

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the main task of creative telescoping consists in finding $c_0, \ldots, c_\rho$, rational functions (or polynomials) in x, not all zero, and two functions $g(x,y,z), h(x,y,z)$ in the same domain as f such that

$$c_0 f + c_1 \sigma_x(f) + \cdots + c_\rho \sigma_x^\rho(f) = (\sigma_y(g) - g) + (\sigma_z(h) - h), \tag{1.1}$$

where σ_x, σ_y and σ_z denote shift operators in x, y and z, respectively. The number ρ may or may not be part of the input. If $c_0, \ldots, c_\rho$ and g, h are as above, then $L = c_0 + \cdots + c_\rho \sigma_x^\rho$ is called a *telescoper* for f and (g,h) is a *certificate* for L. Additional details, along with classical algorithms for computing telescopers and certificates, can be found in [34, 37, 38].

Over the past two decades, a number of generalizations and refinements of creative telescoping have been developed. At the present time reduction-based methods have gained the most support as they are both efficient and have the important feature of being able to find a telescoper for a given function without also computing a corresponding certificate. This is desirable in the situation where only the telescoper is of interest and its size is much smaller than the size of the certificate. This is often the case, for example, where the right hand side of (1.1) is known to collapse to zero.

The reduction-based approach was first developed in the differential case for rational functions [11], and later generalized to rational functions in several variables [14], to hyperexponential functions [12], to algebraic functions [18] and to D-finite functions [20, 36, 13]. In the shift case a reduction-based approach was developed for hypergeometric terms [17, 27] and to multiple binomial sums [15] (a subclass of the sums of proper hypergeometric terms).

In the case of discrete functions having more than two variables no complete reduction-based creative telescoping algorithm has been known so far. The goal of the present paper is to take the first step towards filling the gap, namely to further extend the approach to the trivariate rational case where f, g, h are all rational functions in (1.1). This is a natural follow up to the recent work of Chen, Hou, Labahn and Wang [16] which solved the existence problem of telescopers for rational functions in three discrete variables.

The basic idea of the general reduction-based approach, formulated for the shift trivariate rational case, is as follows. Let C be a field of characteristic zero. Assume that there is some $C(x)$-linear map $\operatorname{red}(\cdot) : C(x,y,z) \mapsto C(x,y,z)$ with the property that for all $f \in C(x,y,z)$, there exist $g, h \in C(x,y,z)$ such that $f - \operatorname{red}(f) = \sigma_y(g) - g + \sigma_z(h) - h$. In other words, $f - \operatorname{red}(f)$ is summable with respect to y, z. Such a map is called a *reduction* with $\operatorname{red}(f)$ considered as the *remainder* of f with respect to the reduction $\operatorname{red}(\cdot)$. Then in order to find a telescoper for f, we can iteratively compute $\operatorname{red}(f), \operatorname{red}(\sigma_x(f)), \operatorname{red}(\sigma_x^2(f)), \ldots$ until we find a nontrivial linear dependence over the field $C(x)$. Once we have such a dependence, say

$$c_0 \operatorname{red}(f) + \cdots + c_\rho \operatorname{red}(\sigma_x^\rho(f)) = 0$$

for $c_i \in C(x)$ not all zero, then by linearity, $\operatorname{red}(c_0 f + \cdots + c_\rho \sigma_x^\rho(f)) = 0$, that is, $c_0 f + \cdots + c_\rho \sigma_x^\rho(f) = \sigma_y(g) - g + \sigma_z(h) - h$ for some $g, h \in C(x,y,z)$. This yields a telescoper $c_0 + \cdots + c_\rho \sigma_x^\rho$ for f.

To guarantee the termination of the above process, one possible way is to show that, for every summable function f, we have $\operatorname{red}(f) = 0$. If this is the case and $L = c_0 + \cdots + c_\rho \sigma_x^\rho$ is a telescoper for f, then $L(f)$ is summable by the definition. So $\operatorname{red}(L(f)) = 0$, and again by the linearity, $\operatorname{red}(f), \ldots, \operatorname{red}(\sigma_x^\rho(f))$ are linear dependent over $C(x)$. This means that we will not miss any telescoper and that the method will terminate provided that a telescoper is known to exist. This approach was taken in [17]. It requires us to know exactly under what kind of conditions a telescoper exists, so-called the *existence problem of telescopers*, and, when these conditions are fulfilled, then it is guaranteed to find one of minimal order ρ. Such existence problems have been well studied in the case of bivariate hypergeometric terms [5] and more recently in the triviariate rational case [16].

A second, alternate way to ensure termination, used for example in [11, 12], is to show that, for a given function f, the remainders $\operatorname{red}(f), \operatorname{red}(\sigma_x(f)), \operatorname{red}(\sigma_x^2(f)), \ldots$ form a finite-dimensional $C(x)$-vector space. Then, as soon as ρ exceeds this finite dimension, one can be sure that a telescoper of order at most ρ will be found. This also implies that every bound for the dimension gives rise to an upper bound for the minimal order of telescopers. This approach provides an independent proof for the existence of a telescoper. However, since such an upper order bound is only of theoretical interest and will not affect the practical efficiency of the algorithms, in this paper we will confine ourselves with the first approach for termination and leave the second approach for future research.

Our starting point is thus to find a suitable reduction for trivariate rational functions. In particular we present a reduction $\operatorname{red}(\cdot)$ which satisfies the following properties: (i) $\operatorname{red}(f) = 0$ whenever $f \in C(x, y, z)$ is summable and (ii) $\operatorname{red}(f)$ is minimal in certain sense. One issue with this reduction, similar to that encountered in [17], is the difficulty that $\operatorname{red}(\cdot)$ is not a $C(x)$-linear map in general. To overcome this we follow the ideas of [17]. Namely, we introduce the idea of congruence modulo summable rational functions and show that $\operatorname{red}(\cdot)$ becomes $C(x)$-linear when it is viewed as a residue class. Using the existence criterion of telescopers established in [16], we are then able to design a creative telescoping algorithm from $\operatorname{red}(\cdot)$ as described in the previous paragraphs.

The remainder of the paper proceeds as follows. The next section gives some preliminary material needed for this paper, particularly a review of a reduction method due to Abramov. In Section 3 we extend Abramov's reduction method to the bivariate case by incorporating a primary reduction. In Section 4 we show that the reduction remainders introduced in the previous section are well-behaved with respect to taking linear combinations, followed in Section 5 by a new algorithm for constructing telescopers for trivariate rational functions based on the bivariate extension of Abramov's reduction method. In Section 6 we provide some experimental tests of our new algorithm. The paper ends with a conclusion and topics for future research.

2 Preliminaries

Throughout the paper we let C denote a field of characteristic zero, with $\mathbb{F} = C(x)$ and $\mathbb{F}(y, z)$ the field of rational functions in y, z over $\mathbb{F}$. We denote by σ_y and σ_z the *shift operators* of $\mathbb{F}(y, z)$, where for any $f \in \mathbb{F}(y, z)$ we have

$$\sigma_y(f(x, y, z)) = f(x, y + 1, z) \quad \text{and} \quad \sigma_z(f(x, y, z)) = f(x, y, z + 1).$$

Let G be the free abelian group generated by the shift operators σ_y and σ_z. For any $\tau \in G$ a polynomial $p \in \mathbb{F}[y, z]$ is said to be τ*-free* if $\gcd(p, \tau^\ell(p)) = 1$ for all nonzero $\ell \in \mathbb{Z}$. A rational function $f \in \mathbb{F}(y, z)$ is called τ*-summable* if $f = \tau(g) - g$ for some $g \in \mathbb{F}(y, z)$. The τ*-summability problem* is then to decide whether a given rational function in $\mathbb{F}(y, z)$ is τ-summable or not. Rather than merely giving a negative answer in case the function is not τ-summable, one could instead seek solutions for a more general problem, namely the τ*-decomposition problem*, with the intent to make the nonsummable part as small as possible. Precisely speaking, the τ-decomposition problem, for a given rational function $f \in \mathbb{F}(y, z)$, asks for an additive decomposition of the form $f = \tau(g) - g + r$, where $g, r \in \mathbb{F}(y, z)$ and r is minimal in certain sense such that f would be τ-summable if and only if $r = 0$. It is readily seen that any solution to the decomposition problem tackles the corresponding summability problem as well.

In the case where $\tau = \sigma_y$, the decomposition problem was first solved by Abramov in [1] with refined algorithms in [3, 32, 9, 24]. All these algorithms can be viewed as discrete analogues of the Ostrogradsky-Hermite reduction for rational integration. We refer to any of these algorithms as the reduction of Abramov.

Theorem 2.1 (Reduction of Abramov)**.** *Let f be a rational function in $\mathbb{F}(y,z)$. Then the reduction of Abramov finds $g \in \mathbb{F}(y,z)$ and $a, b \in \mathbb{F}[y,z]$ with $\deg_y(a) < \deg_y(b)$ and b being σ_y-free such that*

$$f = \sigma_y(g) - g + \frac{a}{b}.$$

Moreover, if f admits such a decomposition then

- *f is σ_y-summable if and only if $a = 0$;*
- *b has the lowest possible degree in y when $\gcd(a,b) = 1$. That is, if there exist a second $g' \in \mathbb{F}(y,z)$ and $a', b' \in \mathbb{F}[y,z]$ such that $f = \sigma_y(g') - g' + a'/b'$, then $\deg_y(b') \geq \deg_y(b)$.*

Generalizing to the bivariate case, we consider the (σ_y, σ_z)-summability problem of deciding whether a given rational function $f \in \mathbb{F}(y,z)$ can be written in the form $f = \sigma_y(g) - g + \sigma_z(h) - h$ for $g, h \in \mathbb{F}(y,z)$. If such a form exists, we say that f is (σ_y, σ_z)-*summable*, abbreviated as summable in certain instances. The (σ_y, σ_z)-decomposition problem is then to decompose a given rational function $f \in \mathbb{F}(y,z)$ into the form

$$f = \sigma_y(g) - g + \sigma_z(h) - h + r,$$

where $g, h, r \in \mathbb{F}(y,z)$ and r is minimal in certain sense. Moreover, f is (σ_y, σ_z)-summable if and only if $r = 0$.

Recall that an irreducible polynomial $f \in \mathbb{F}[y,z]$ is called (y,z)-*integer linear* over the $\mathbb{F}$ if it can be written in the form $f = p(\alpha y + \beta z)$ for a polynomial $p(Z) \in \mathbb{F}[Z]$ and integers $\alpha, \beta \in \mathbb{Z}$. One may assume without loss of generality that $\beta \geq 0$ and α, β are coprime. A polynomial in $\mathbb{F}[y,z]$ is called (y,z)-*integer linear* over $\mathbb{F}$ if all its irreducible factors are (y,z)-integer linear over $\mathbb{F}$ while a rational function in $\mathbb{F}(y,z)$ is called (y,z)-*integer linear* over $\mathbb{F}$ if its numerator and denominator are both (y,z)-integer linear over $\mathbb{F}$. For simplicity, we just say a rational function is (y,z)-integer linear over $\mathbb{F}$ of (α,β)-type if it is equal to $p(\alpha y + \beta z)$ for some $p(Z) \in \mathbb{F}(Z)$ and α, β are coprime integers with $\beta \geq 0$. Algorithms for determining integer linearity are found in [6, 31, 25].

In the context of creative telescoping, we will also need to consider the variable x and its shift operator σ_x, which for every $f \in \mathbb{F}(y,z)$ maps $f(x,y,z)$ to $f(x+1,y,z)$. Recall that two polynomials $p, q \in C[x,y,z]$ are called (x,y,z)-*shift equivalent*, denoted by $p \sim_{x,y,z} q$, if there exist three integers ℓ, m, n such that $p = \sigma_x^\ell \sigma_y^m \sigma_z^n(q)$. We generalize this notion to the domain $\mathbb{F}[y,z]$ by saying that two polynomials $p, q \in \mathbb{F}[y,z]$ are (x,y,z)-shift equivalent if $p = \sigma_x^\ell \sigma_y^m \sigma_z^n(q)$ for integers ℓ, m, n. When $\ell = 0$ then p is also called (y,z)-*shift equivalent* to q, denoted by $p \sim_{y,z} q$. Clearly, $\sim_{x,y,z}$, as well as $\sim_{y,z}$, is an equivalence relation.

Let $\mathbb{F}(y,z)[\mathrm{S}_x, \mathrm{S}_y, \mathrm{S}_z]$ be the ring of linear recurrence operators in x, y, z over $\mathbb{F}(y,z)$. Here $\mathrm{S}_x, \mathrm{S}_y, \mathrm{S}_z$ commute with each other, and $\mathrm{S}_v(f) = \sigma_v(f)\,\mathrm{S}_v$ for any $f \in \mathbb{F}(y,z)$ and $v \in \{x,y,z\}$. The application of an operator $P = \sum_{i,j,k \geq 0} p_{ijk}\, \mathrm{S}_x^i\, \mathrm{S}_y^j\, \mathrm{S}_z^k$ in $\mathbb{F}(y,z)[\mathrm{S}_x, \mathrm{S}_y, \mathrm{S}_z]$ to a rational function $f \in \mathbb{F}(y,z)$ is then defined as

$$P(f) = \sum_{i,j,k \geq 0} p_{ijk} \sigma_x^i \sigma_y^j \sigma_z^k(f).$$

Definition 2.2. *Let f be a rational function in $\mathbb{F}(y,z)$. A nonzero linear recurrence operator $L \in \mathbb{F}[\mathrm{S}_x]$ is called a* telescoper *for f if $L(f)$ is (σ_y, σ_z)-summable, or equivalently, if there exist rational functions $g, h \in \mathbb{F}(y,z)$ such that*

$$L(f) = (\mathrm{S}_y - 1)(g) + (\mathrm{S}_z - 1)(h),$$

where 1 denotes the identity map of $\mathbb{F}(y,z)$. We call (g,h) a corresponding certificate for L. The order *of a telescoper is defined to be its degree in S_x. A telescoper of minimal order for f is called a* minimal telescoper *for f.*

In this paper, choosing the pure lexicographic order $y \prec z$, we say a polynomial in $\mathbb{F}[y,z]$ is *monic* if its highest term with respect to y, z has coefficient one. For a nonzero polynomial $p \in \mathbb{F}[y,z]$, its degree and leading coefficient with respect to the variable $v \in \{y,z\}$ are denoted by $\deg_v(p)$ and $\mathrm{lc}_v(p)$, respectively. We will follow the convention that $\deg_v(0) = -\infty$.

3 Bivariate extension of the reduction of Abramov

In this section, we demonstrate how to solve the bivariate decomposition problem (and thus also the bivariate summability problem) using the reduction of Abramov. To this end, let us first recall some key results on the bivariate summability from [26].

Based on a theoretical criterion given in [19], Hou and Wang [26] developed a practical algorithm for solving the (σ_y, σ_z)-summability problem. The proof found in [26, Lemma 3.1] contains a reduction algorithm with inputs and outputs specified as follows.

Primary reduction. Given a rational function $f \in \mathbb{F}(y,z)$, compute rational functions $g, h, r \in \mathbb{F}(y,z)$ such that

$$f = (\mathrm{S}_y - 1)(g) + (\mathrm{S}_z - 1)(h) + r \tag{3.1}$$

and r is of the form

$$r = \sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{a_{ij}}{b_{ij} d_i^j} \tag{3.2}$$

with $m, n_i \in \mathbb{N}$, $a_{ij}, d_i \in \mathbb{F}[y,z]$ and $b_{ij} \in \mathbb{F}[y]$ satisfying that

- $\deg_z(a_{ij}) < \deg_z(d_i)$,
- d_i is a monic irreducible factor of the denominator of r and of positive degree in z,
- $d_i \not\sim_{y,z} d_{i'}$ whenever $i \neq i'$ for $1 \leq i, i' \leq m$.

Let f be a rational function in $\mathbb{F}(y,z)$ and assume that applying the primary reduction to f yields (3.1). Deciding if f is (σ_y, σ_z)-summable then amounts to checking the summability of r. By [26, Lemma 3.2], this is equivalent to checking the summability of each simple fraction $a_{ij}/(b_{ij} d_i^j)$. Thus the bivariate summability problem for a general rational function is reduced to determining the summability of several simple fractions, which in turn can be addressed by the following.

Theorem 3.1 ([26, Theorem 3.3])**.** *Let $f = a/(b\, d^j)$, where $a, d \in \mathbb{F}[y,z]$, $b \in \mathbb{F}[y]$, $j \in \mathbb{N} \setminus \{0\}$ and d irreducible with $0 \leq \deg_z(a) < \deg_z(d)$. Then f is (σ_y, σ_z)-summable if and only if*

(i) d is (y,z)-integer linear over $\mathbb{F}$ of (α, β)-type,

$$\begin{array}{ccc} R_1 & \xrightarrow{\psi} & R_2 \\ \sigma_1 \downarrow & \circlearrowright & \downarrow \sigma_2 \\ R_1 & \xrightarrow[\psi]{} & R_2 \end{array} \qquad \begin{array}{ccc} \mathbb{F}(y,z) & \xrightarrow{\phi_{\alpha,\beta}} & \mathbb{F}(y,z) \\ \tau = \sigma_y^{\beta}\sigma_z^{-\alpha} \downarrow & \circlearrowright & \downarrow \sigma_y \\ \mathbb{F}(y,z) & \xrightarrow[\phi_{\alpha,\beta}]{} & \mathbb{F}(y,z) \end{array}$$

Figure 1: Commutative diagrams for difference homomorphisms/isomorphisms.

(ii) there exists $q \in \mathbb{F}(y)[z]$ *with* $\deg_z(q) < \deg_z(d)$ *so that*

$$\frac{a}{b} = \sigma_y^{\beta}\sigma_z^{-\alpha}(q) - q. \tag{3.3}$$

Since d is irreducible, the first condition is easily recognized by comparing coefficients once d is known. In [26, §4], the second condition is checked by finding polynomial solutions of a system of linear recurrence equations in one variable based on a universal denominator derived from the m-fold Gosper representation. Any polynomial solution of the system gives rise to a desired q for (3.3). For the rest of this section, we show how to detect the second condition via the reduction of Abramov, without solving any auxiliary recurrence equations. Before that, we need some notions.

Let R be a ring and $\sigma : R \to R$ be a homomorphism of R. The pair (R, σ) is called a *difference ring*. An element $r \in R$ is called a *constant* of the difference ring (R, σ) if $\sigma(r) = r$. The set of all constants in R forms a subring of R, called the *subring of constants*. If R is a field, the (R, σ) is called a difference field. Let (R_1, σ_1) and (R_2, σ_2) be two difference rings and $\psi : R_1 \to R_2$ be a homomorphism. If $\sigma_2 \circ \psi = \psi \circ \sigma_1$, that is, the left diagram in Figure 1 commutes, we call ψ a *difference homomorphism*, or a *difference isomorphism* if ψ is an isomorphism. Two difference rings are then said to be *isomorphic* if there exists a difference isomorphism between them.

Let α, β be two integers with β nonzero. We define an $\mathbb{F}$-homomorphism $\phi_{\alpha,\beta} : \mathbb{F}(y,z) \to \mathbb{F}(y,z)$ by

$$\phi_{\alpha,\beta}(y) = \beta y \quad \text{and} \quad \phi_{\alpha,\beta}(z) = \beta^{-1}z - \alpha y.$$

It is readily seen that $\phi_{\alpha,\beta}$ is an $\mathbb{F}$-isomorphism with inverse $\phi_{\alpha,\beta}^{-1}$ given by

$$\phi_{\alpha,\beta}^{-1}(y) = \beta^{-1}y \quad \text{and} \quad \phi_{\alpha,\beta}^{-1}(z) = \beta z + \alpha y.$$

We call $\phi_{\alpha,\beta}$ the *map for* (α, β)*-shift reduction.*

Proposition 3.2. *Let* $\alpha, \beta \in \mathbb{Z}$ *with* $\beta \neq 0$ *and* $\tau = \sigma_y^{\beta}\sigma_z^{-\alpha}$*. Then* $\phi_{\alpha,\beta}$ *is a difference isomorphism between the difference fields* $(\mathbb{F}(y,z), \tau)$ *and* $(\mathbb{F}(y,z), \sigma_y)$*.*

Proof. Since $\phi_{\alpha,\beta}$ is an $\mathbb{F}$-isomorphism, it remains to show that $\sigma_y \circ \phi_{\alpha,\beta} = \phi_{\alpha,\beta} \circ \tau$, namely the right diagram in Figure 1 commutes. This is confirmed by the observations that

$$\sigma_y(\phi_{\alpha,\beta}(f(y,z))) = \sigma_y(f(\beta y, \beta^{-1}z - \alpha y)) = f(\beta y + \beta, \beta^{-1}z - \alpha y - \alpha)$$

and

$$\phi_{\alpha,\beta}(\tau(f(y,z))) = \phi_{\alpha,\beta}(f(y+\beta, z-\alpha)) = f(\beta y + \beta, \beta^{-1}z - \alpha y - \alpha)$$

for any $f \in \mathbb{F}(y,z)$. $\square$

Corollary 3.3. *Let $f \in \mathbb{F}(y,z)$ and assume the conditions of Proposition 3.2. Then f is τ-summable if and only if $\phi_{\alpha,\beta}(f)$ is σ_y-summable.*

Proof. By Proposition 3.2, $\phi_{\alpha,\beta}$ is a difference isomorphism between $(\mathbb{F}(y,z),\tau)$ and $(\mathbb{F}(y,z),\sigma_y)$. It follows that

$$f = \tau(g) - g \quad \Longleftrightarrow \quad \phi_{\alpha,\beta}(f) = \phi_{\alpha,\beta}(\tau(g) - g) = \sigma_y(\phi_{\alpha,\beta}(g)) - \phi_{\alpha,\beta}(g)$$

for any $g \in \mathbb{F}(y,z)$. The assertion follows. □

The problem of deciding whether a rational function $f \in \mathbb{F}(y)[z]$ satisfies the equation (3.3), that is, the $\sigma_y^{\beta}\sigma_z^{-\alpha}$-summability problem for f, is now reduced to the σ_y-summability problem for $\phi_{\alpha,\beta}(f)$. In fact, there is also a one-to-one correspondence between additive decompositions of f with respect to $\sigma_y^{\beta}\sigma_z^{-\alpha}$ and additive decompositions of $\phi_{\alpha,\beta}(f)$ with respect to σ_y. In particular, we have the following proposition.

Proposition 3.4. *Let $f \in \mathbb{F}(y,z)$ and assume the conditions of Proposition 3.2. Suppose also that $\phi_{\alpha,\beta}(f)$ admits the decomposition*

$$\phi_{\alpha,\beta}(f) = \sigma_y(\tilde{g}) - \tilde{g} + \frac{\tilde{a}}{\tilde{b}}, \tag{3.4}$$

where $\tilde{g} \in \mathbb{F}(y)[z]$ and $\tilde{a},\tilde{b} \in \mathbb{F}[y,z]$ with $\deg_y(\tilde{a}) < \deg_y(\tilde{b})$ and $\tilde{b}$ being σ_y-free. Let $g = \phi_{\alpha,\beta}^{-1}(\tilde{g})$, $a = \phi_{\alpha,\beta}^{-1}(\tilde{a})$ and $b = \phi_{\alpha,\beta}^{-1}(\tilde{b})$. Then

$$f = \tau(g) - g + \frac{a}{b} \tag{3.5}$$

and g, a, b satisfy the conditions

(i) a can be written as $\sum_{i=0}^{\deg_z(a)} \hat{a}_i \cdot (\alpha y + \beta z)^i$ for $\hat{a}_i \in \mathbb{F}[y]$ with $\deg_y(\hat{a}_i) < \deg_y(b)$,

(ii) b is σ_y^{β}-free,

with f being τ-summable if and only if $a = 0$. Moreover, if $\gcd(a,b) = 1$ then b has the lowest possible degree in y.

Proof. By Proposition 3.2, $\sigma_y \circ \phi_{\alpha,\beta} = \phi_{\alpha,\beta} \circ \tau$ and thus $\phi_{\alpha,\beta}^{-1} \circ \sigma_y = \tau \circ \phi_{\alpha,\beta}^{-1}$. Then applying $\phi_{\alpha,\beta}^{-1}$ to both sides of (3.4) yields (3.5). Since $\phi_{\alpha,\beta}(f)$ is σ_y-summable if and only if $\tilde{a} = 0$ according to Theorem 2.1, one sees from Corollary 3.3 that f is τ-summable if and only if $a = 0$. Moreover, if $\gcd(a,b) = 1$ then $\gcd(\tilde{a},\tilde{b}) = 1$. By Theorem 2.1, $\tilde{b}$ has the lowest possible degree in y. The minimality of b then follows from that of $\tilde{b}$ because $\deg_y(b) = \deg_y(\tilde{b})$.

It remains to show that the conditions (i)-(ii) hold. By the definition of $\phi_{\alpha,\beta}^{-1}$, we have that $a = \phi_{\alpha,\beta}^{-1}(\tilde{a}) = \tilde{a}(\beta^{-1}y, \alpha y + \beta z) \in \mathbb{F}[y,z]$. It is then evident that there exist the $\hat{a}_i \in \mathbb{F}[y]$ so that a can be written into the required form described in condition (i). Further observe that $\deg_y(\tilde{a}) = \max_i \deg_y(\hat{a}_i)$. Thus the first condition follows by $\deg_y(\tilde{a}) < \deg_y(\tilde{b}) = \deg_y(b)$. Since $\tilde{b}$ is σ_y-free, we have that $\gcd(\tilde{b}, \sigma_y^{\ell}(\tilde{b})) = 1$ for any nonzero $\ell \in \mathbb{Z}$. Notice that $\phi_{\alpha,\beta}^{-1} \circ \sigma_y^{\ell} = \tau^{\ell} \circ \phi_{\alpha,\beta}^{-1}$ and $b \in \mathbb{F}[y]$, Therefore, $\gcd(b, \tau^{\ell}(b)) = \gcd(b, \sigma_y^{\beta\ell}(b)) = 1$ by the definition of τ. This implies that b is σ_y^{β}-free, proving the second condition. □

Remark 3.5. *With the notations and assumptions of Proposition 3.4, one is able to further reduce the size of the polynomial a so that condition (i) is replaced by the condition $\deg_y(a) < \deg_y(b)$. This is true since every polynomial is τ-summable, which is in turn implied by Corollary 3.3 and the fact that every polynomial is σ_y-summable. This reduction, however, does not affect the applicability of later algorithms.*

Proposition 3.4 motivates us to introduce the notions of remainder fractions and remainders, in order to characterize nonsummable rational functions concretely.

Definition 3.6. *A fraction $a/(b\,d^j)$ with $a, d \in \mathbb{F}[y,z]$, $b \in \mathbb{F}[y]$ and $j \in \mathbb{N}\setminus\{0\}$ is called a* remainder fraction *if*

- $\deg_z(a) < \deg_z(d)$;
- *d is monic, irreducible and of positive degree in z;*
- *conditions (i)-(ii) in Proposition 3.4 hold in case d is (y,z)-integer linear over $\mathbb{F}$ of (α,β)-type.*

Definition 3.7. *Let f be a rational function in $\mathbb{F}(y,z)$. Then $r \in \mathbb{F}(y,z)$ is called a (σ_y,σ_z)-*remainder *of f if $f - r$ is (σ_y,σ_z)-summable and r can be written as*

$$r = \sum_{i=1}^{m}\sum_{j=1}^{n_i} \frac{a_{ij}}{b_{ij}d_i^j}, \tag{3.6}$$

where $m, n_i \in \mathbb{N}$, $a_{ij}, d_i \in \mathbb{F}[y,z]$, $b_{ij} \in \mathbb{F}[y]$ with each $a_{ij}/(b_{ij}d_i^j)$ being a remainder fraction, d_i being a factor of the denominator of r, and $d_i \nsim_{y,z} d_{i'}$ whenever $i \neq i'$ and $1 \leq i, i' \leq m$. For brevity, we just say that r is a (σ_y,σ_z)-remainder if f is clear from the context. Sometimes we also just say remainder for short unless there is a danger of confusion. We refer to (3.6), along with the attached conditions, as the remainder form *of r.*

The uniqueness of partial fraction decompositions (in this case with respect to z) implies that the remainder form for a given (σ_y,σ_z)-remainder is unique up to reordering and multiplication by units of $\mathbb{F}$. Evidently, every single remainder fraction, or part of summands in (3.6), is a (σ_y,σ_z)-remainder. Remainders not only helps us to recognize summability, but also describes the "minimum" gap between a given rational function and summable rational functions, as shown in the next two propositions.

Proposition 3.8. *Let $r \in \mathbb{F}(y,z)$ be a nonzero (σ_y,σ_z)-remainder with the form (3.6). Then each nonzero $a_{ij}/(b_{ij}d_i^j)$ for $1 \leq i \leq m$ and $1 \leq j \leq n_i$, as well as r itself, is not (σ_y,σ_z)-summable.*

Proof. Since r is a (σ_y,σ_z)-remainder, each $a_{ij}/(b_{ij}d_i^j)$ is a remainder fraction. For a particular nonzero $a_{ij}/(b_{ij}d_i^j)$, namely $a_{ij} \neq 0$, we claim that it is not (σ_y,σ_z)-summable. If d_i is not (y,z)-integer linear over $\mathbb{F}$, then the simple fraction is not (σ_y,σ_z)-summable by Theorem 3.1. Otherwise, assume that d_i is (y,z)-integer linear over $\mathbb{F}$ of (α,β)-type. Since $a_{ij}/(b_{ij}d_i^j)$ is a remainder fraction, Definition 3.6 reads that a_{ij}, when viewed as a polynomial in $(\alpha y + \beta z)$, has coefficients of degrees in y less than $\deg_y(b_{ij})$ and that b_{ij} is σ_y^β-free. By Proposition 3.4, a_{ij}/b_{ij} is not $\sigma_y^\beta\sigma_z^{-\alpha}$-summable. The claim then follows by Theorem 3.1.

In either case, we have that $a_{ij}/(b_{ij}d_i^j)$ is not (σ_y,σ_z)-summable. Since r is nonzero, at least one of the $a_{ij}/(b_{ij}d_i^j)$ is nonzero. By [26, Lemma 3.2], r is therefore not (σ_y,σ_z)-summable. $\square$

Proposition 3.9. *Let $r \in \mathbb{F}(y,z)$ be a nonzero (σ_y, σ_z)-remainder with the form (3.6), in which a_{ij} and $b_{ij}d_i^j$ are further assumed to be coprime. Assume that there exists another $r' \in \mathbb{F}(y,z)$ such that $r' - r$ is (σ_y, σ_z)-summable. Write r' in the form*

$$r' = p' + \sum_{i=1}^{m'} \sum_{j=1}^{n'_i} \frac{a'_{ij}}{b'_{ij} {d'_i}^j},$$

where $m', n'_i \in \mathbb{N}$, $p' \in \mathbb{F}(y)[z]$, $a'_{ij}, d'_i \in \mathbb{F}[y,z]$ and $b'_{ij} \in \mathbb{F}[y]$ with $\deg_z(a'_{ij}) < \deg_z(d'_i)$ and d'_i being monic irreducible factor of the denominator of r' and of positive degree in z. For each $1 \le i \le m$, define

$$\Lambda_i = \{i' \in \mathbb{N} \mid 1 \le i' \le m' \text{ and } d'_{i'} = \sigma_y^{\lambda_{i'}} \sigma_z^{\mu_{i'}}(d_i) \text{ for } \lambda_{i'}, \mu_{i'} \in \mathbb{Z}\}.$$

Then Λ_i is nonempty for any $1 \le i \le m$. Moreover, $m \le m'$, $n_i \le n'_{i'}$ for all $i' \in \Lambda_i$, $\deg_y(b_{ij}) \le \sum_{i' \in \Lambda_i} \deg_y(b'_{i'j})$ for each $1 \le i \le m$ and $1 \le j \le n_i$, and the degree in z of the denominator of r is no more than that of r'.

Proof. Since $r' - r$ is (σ_y, σ_z)-summable, all the rational function $\sum_{i' \in \Lambda_i} a'_{i'j}/(b'_{i'j} {d'_{i'}}^j) - a_{ij}/(b_{ij} d_i^j)$ are (σ_y, σ_z)-summable by [26, Lemma 3.2], and then so are the

$$\sum_{i' \in \Lambda_i} \frac{\sigma_y^{-\lambda_{i'}} \sigma_z^{-\mu_{i'}}(a'_{i'j})}{\sigma_y^{-\lambda_{i'}}(b'_{i'j}) d_i^j} - \frac{a_{ij}}{b_{ij} d_i^j}. \tag{3.7}$$

Since r is a nonzero (σ_y, σ_z)-remainder, we conclude from Proposition 3.8 that each nonzero $a_{ij}/(b_{ij}d_i^j)$ is not (σ_y, σ_z)-summable. Notice that for each $1 \le i \le m$, there is at least one integer j with $1 \le j \le n_i$ such that $a_{ij} \neq 0$. It then follows from the summability of (3.7) that every Λ_i is nonempty, namely every d_i is (y,z)-shift equivalent to some $d'_{i'}$ for $1 \le i' \le m'$, and that $n_i \le n'_{i'}$ for any $i' \in \Lambda_i$. Notice that the d_i are pairwise (y,z)-shift inequivalent. Thus the Λ_i are pairwise disjoint, which implies that $m \le m'$. Accordingly, the degree in z of the denominator of r is no more than that of r'.

It remains to show the inequality for the degree of each b_{ij}. For each $1 \le i \le m$ and $1 \le j \le n_i$, by Theorem 3.1, the summability of (3.7) either yields

$$\sum_{i' \in \Lambda_i} \frac{\sigma_y^{-\lambda_{i'}} \sigma_z^{-\mu_{i'}}(a'_{i'j})}{\sigma_y^{-\lambda_{i'}}(b'_{i'j})} = \sigma_y^{\beta} \sigma_z^{-\alpha}(q) - q + \frac{a_{ij}}{b_{ij}} \quad \text{for some } q \in \mathbb{F}(y)[z],$$

if d_i is (y,z)-integer linear over $\mathbb{F}$ of (α, β)-type or otherwise yields

$$\sum_{i' \in \Lambda_i} \frac{\sigma_y^{-\lambda_{i'}} \sigma_z^{-\mu_{i'}}({a'}_{i'j})}{\sigma_y^{-\lambda_{i'}}(b'_{i'j})} = \frac{a_{ij}}{b_{ij}}.$$

The assertion is evident in the latter case. For the former case, the assertion then follows by the minimality of b_{ij} from Proposition 3.4, because $a_{ij}/(b_{ij}d_i^j)$ is a remainder fraction. □

With everything in place, we now present a bivariate extension of the reduction of Abramov, which addresses the (σ_y, σ_z)-decomposition problem.

Bivariate reduction of Abramov. Given a rational function $f \in \mathbb{F}(y, z)$, compute three rational functions $g, h, r \in \mathbb{F}(y, z)$ such that r is a (σ_y, σ_z)-remainder of f and

$$f = (\mathrm{S}_y - 1)(g) + (\mathrm{S}_z - 1)(h) + r. \tag{3.8}$$

1. apply the primary reduction to f to find $g, h \in \mathbb{F}(y, z)$, $m, n_i \in \mathbb{N}$, $a_{ij}, d_i \in \mathbb{F}[y, z]$ and $b_{ij} \in \mathbb{F}[y]$ such that (3.1) holds.

2. for $i = 1, \ldots, m$ do

   if d_i is (y, z)-integer linear over $\mathbb{F}$ of (α_i, β_i)-type then

   2.1 compute $\tilde{a}_{ij}/\tilde{b}_{ij} = \phi_{\alpha_i,\beta_j}(a_{ij}/b_{ij})$ with ϕ_{α_i,β_i} being the map for (α_i, β_i)-shift reduction;

   2.2 for $j = 1, \ldots, n_i$ do

   2.2.1 apply the reduction of Abramov to $\tilde{a}_{ij}/\tilde{b}_{ij}$ with respect to y to get $\tilde{q}_{ij}, \tilde{r}_{ij} \in \mathbb{F}(y)[z]$ such that
$$\frac{\tilde{a}_{ij}}{\tilde{b}_{ij}} = \sigma_y(\tilde{q}_{ij}) - \tilde{q}_{ij} + \tilde{r}_{ij}.$$

   2.2.2 apply $\phi^{-1}_{\alpha_i,\beta_i}$ to both sides of the previous equation to get
$$\frac{a_{ij}}{b_{ij}} = \sigma_y^{\beta_i}\sigma_z^{-\alpha_i}(q_{ij}) - q_{ij} + r_{ij}, \tag{3.9}$$
   where $q_{ij} = \phi^{-1}_{\alpha_i,\beta_i}(\tilde{q}_{ij})$ and $r_{ij} = \phi^{-1}_{\alpha_i,\beta_i}(\tilde{r}_{ij})$.

   2.2.3 update a_{ij} and b_{ij} to be the numerator and denominator of r_{ij}, respectively.

   2.3 update
$$g = g + \sum_{j=1}^{n_i}\sum_{k=0}^{\beta_i-1} \sigma_y^k \sigma_z^{-\alpha_i}\left(\frac{q_{ij}}{d_i^j}\right) \quad \text{and} \quad h = h + \begin{cases} \sum_{j=1}^{n_i}\sum_{k=1}^{\alpha_i} \sigma_z^{-k}\left(-\frac{q_{ij}}{d_i^j}\right) & \alpha_i \geq 0 \\ \sum_{j=1}^{n_i}\sum_{k=0}^{-\alpha_i-1} \sigma_z^{k}\left(\frac{q_{ij}}{d_i^j}\right) & \alpha_i < 0 \end{cases}.$$

3. set $r = \sum_{i=1}^{m}\sum_{j=1}^{n_i} a_{ij}/(b_{ij}d_i^j)$, and return g, h, r.

Theorem 3.10. *Let f be a rational function in $\mathbb{F}(y, z)$. Then the bivariate reduction of Abramov computes two rational functions $g, h \in \mathbb{F}(y, z)$ and a (σ_y, σ_z)-remainder $r \in \mathbb{F}(y, z)$ such that (3.8) holds. Moreover, f is (σ_y, σ_z)-summable if and only if $r = 0$.*

Proof. Applying the primary reduction to f yields (3.1). For any nonzero $a_{ij}/(b_{ij}d_i^j)$ obtained in step 1, if d_i is not (y, z)-integer linear over $\mathbb{F}$ then we know from Theorem 3.1 that $a_{ij}/(b_{ij}d_i^j)$ is not (σ_y, σ_z)-summable and is thus already a remainder fraction. Otherwise, there exist coprime integers α_i, β_i with $\beta_i > 0$ so that $d_i = p_i(\alpha_i y + \beta_i z)$ for some $p_i(Z) \in \mathbb{F}[Z]$. By Proposition 3.4, for each $1 \leq j \leq n_i$, steps 2.2.1-2.2.2 correctly find q_{ij} and r_{ij} such that (3.9) holds and r_{ij}/d_i^j is a remainder fraction. After step 2.2, plugging all (3.9) into (3.1) then gives (with a slight abuse of notation):

$$f = (\mathrm{S}_y - 1)(g) + (\mathrm{S}_z - 1)(h) + \sum_{i:\, d_i = p_i(\alpha_i y + \beta_i z)} \sum_{j=1}^{n_i} \frac{\sigma_y^{\beta_i} \sigma_z^{-\alpha_i}(q_{ij}) - q_{ij}}{d_i^j} + r,$$

where the index i runs through all (y,z)-integer linear d_i's and $r = \sum_{i=1}^m \sum_{j=1}^{n_i} a_{ij}/(b_{ij} d_i^j)$ is a (σ_y, σ_z)-remainder by Definition 3.7. The assertions then follow from Proposition 3.8 and the observation that

$$\frac{\sigma_y^{\beta_i} \sigma_z^{-\alpha_i}(q_{ij}) - q_{ij}}{d_i^j} = (\mathrm{S}_y - 1)\left(\sum_{k=0}^{\beta_i - 1} \sigma_y^k \sigma_z^{-\alpha_i}\left(\frac{q_{ij}}{d_i^j}\right)\right) + \begin{cases} (\mathrm{S}_z - 1)\left(\sum_{k=1}^{\alpha_i} \sigma_z^{-k}\left(-\frac{q_{ij}}{d_i^j}\right)\right) & \text{if } \alpha_i \geq 0 \\ (\mathrm{S}_z - 1)\left(\sum_{k=0}^{-\alpha_i - 1} \sigma_z^{k}\left(\frac{q_{ij}}{d_i^j}\right)\right) & \text{if } \alpha_i < 0 \end{cases} \tag{3.10}$$

for any $d_i = p_i(\alpha_i y + \beta_i z)$. □

Example 3.11. *Consider the rational function f admitting the partial fraction decomposition $f = f_1 + f_2 + f_3$ with*

$$f_1 = \frac{x^2 z + 1}{\underbrace{(x+y)(x+z)^2 + 1}_{d_1}}, \quad f_2 = \frac{(x^2 + xy + 3x - 3)z - x - y + 3}{(x+y)(x+y+3)(\underbrace{(x+2y+3z)^2 + 1}_{d_2})} \quad \text{and} \quad f_3 = \frac{1}{\underbrace{x - y + z}_{d_3}}.$$

Note that d_1, d_2, d_3 are (y,z)-shift inequivalent to each other. Hence f remains unchanged after applying the primary reduction. Since d_1 is not (y,z)-integer linear, we leave f_1 untouched and proceed to deal with f_2. Notice that d_2 is (y,z)-integer linear of $(2,3)$-type. Then applying the reduction of Abramov to $\phi_{2,3}(f_2 d_2)$ yields

$$\phi_{2,3}(f_2 d_2) = (\mathrm{S}_y - 1)\left(\frac{z - 6xy^2 - 2x^2 y + 2x}{3(x+3y)}\right) + \frac{\frac{1}{3}xz + \frac{2}{3}x^2 + 1}{x + 3y},$$

which, when applied by $\phi_{2,3}^{-1}$, leads to

$$f_2 d_2 = (\mathrm{S}_y^2 \mathrm{S}_z^{-2} - 1)(q_2) + \frac{\frac{1}{3}x(2y+3z) + \frac{2}{3}x^2 + 1}{x+y} \quad \text{with } q_2 = \frac{3(2y+3z) - 2xy^2 - 2x^2 y + 6x}{9(x+y)}.$$

Using (3.10), we decompose f_2 as

$$f_2 = (\mathrm{S}_y - 1)\left(\sum_{k=0}^{2} \sigma_y^k \sigma_z^{-2}\left(\frac{q_2}{d_2}\right)\right) + (\mathrm{S}_z - 1)\left(\sum_{k=1}^{2} \sigma_z^{-k}\left(-\frac{q_2}{d_2}\right)\right) + \underbrace{\frac{\frac{1}{3}x(2y+3z) + \frac{2}{3}x^2 + 1}{(x+y)((x+2y+3z)^2 + 1)}}_{r}.$$

One sees that r is a (σ_y, σ_z)-remainder of f_2, and thus f_2 is not (σ_y, σ_z)-summable by Theorem 3.10. Along the same lines as above, we have

$$f_3 = (\mathrm{S}_y - 1)\left(\frac{y}{x - y + z + 1}\right) + (\mathrm{S}_z - 1)\left(\frac{y}{x - y + z}\right),$$

implying that f_3 is (σ_y, σ_z)-summable. Combining everything together, f is finally decomposed as

$$f = (\mathrm{S}_y - 1)(g) + (\mathrm{S}_z - 1)(h) + f_1 + r$$

with $g = \sum_{k=0}^{2} \sigma_y^k \sigma_z^{-2}\left(\frac{q_2}{d_2}\right) + y/(x-y+z+1)$ and $h = \sum_{k=1}^{2} \sigma_z^{-k}\left(-\frac{q_2}{d_2}\right) + y/(x-y+z)$. Therefore, f is not (σ_y, σ_z)-summable by Theorem 3.10. We will use f as a running example in this paper.

4 Linearity of remainders

As mentioned in the introduction, we expect our reduction algorithm to induce a linear map, that is, the sum of two remainders was expected to also be a remainder. Unfortunately, this is not always the case in our setting, because some requirements in Definition 3.7 may be broken by the addition among remainders, as seen in the following examples. This prevents us from applying the bivariate reduction of Abramov developed in the previous section to construct a telescoper in a direct way as was done in the differential case. However, observe that a rational function in $\mathbb{F}(y, z)$ may have more than one (σ_y, σ_z)-remainder and any two of them differ by a (σ_y, σ_z)-summable rational function. This suggests a possible way to circumvent the above difficulty. That is, choosing proper members from the residue class modulo summable rational functions of the given remainders so as to make the linearity become true. The goal of this section is to show that this direction always works and it can be accomplished algorithmically. We note that a similar idea was used in [17, §5].

Example 4.1. *Let $r = f_1$ and $s = \sigma_x(f_1)$ with f_1 being given in Example 3.11. Then r and s are both (σ_y, σ_z)-remainders since both denominators d_1 and $\sigma_x(d_1)$ are not (y, z)-integer linear. However their sum is not a (σ_y, σ_z)-remainder since d_1 is (y, z)-shift equivalent to $\sigma_x(d_1)$, namely $d_1 = \sigma_y^{-1}\sigma_z^{-1}\sigma_x(d_1)$. Nevertheless, we can find another (σ_y, σ_z)-remainder t of s such that $r + t$ has this property. For example, let*

$$t = (\mathrm{S}_y - 1)\left(-\sigma_y^{-1}(s)\right) + (\mathrm{S}_z - 1)\left(-\sigma_y^{-1}\sigma_z^{-1}(s)\right) + s = \frac{(x+1)^2(z-1)+1}{(x+y)(x+z)^2+1},$$

and then

$$r + t = \frac{(2x^2+2x+1)z - x^2 - 2x + 1}{(x+y)(x+z)^2+1}$$

is a (σ_y, σ_z)-remainder by definition. Alternatively, one can compute a (σ_y, σ_z)-remainder $\tilde{t}$ of r, say

$$\tilde{t} = (\mathrm{S}_y - 1)(r) + (\mathrm{S}_z - 1)(\sigma_y(r)) + r = \frac{x^2(z+1)+1}{(x+y+1)(x+z+1)^2+1}$$

so that

$$\tilde{t} + s = \frac{(2x^2+2x+1)z + x^2 + 2}{(x+y+1)(x+z+1)^2+1}$$

is a (σ_y, σ_z)-remainder.

Example 4.2. *Let*

$$r = \frac{\frac{1}{3}x(2y+3z)+\frac{2}{3}x^2+1}{(x+y)((x+2y+3z)^2+1)} \quad \textit{and} \quad s = \frac{(\frac{1}{3}x+1)(2y+3z)+\frac{2}{3}(x+1)^2+2x+\frac{13}{3}}{(x+y+5)((x+2y+3z+1)^2+1)}.$$

Then both r and s are (σ_y,σ_z)-remainders, but again their sum is not since $(x+2y+3z)^2+1$ is (y,z)-shift equivalent to $(x+2y+3z+1)^2+1$. As in Example 4.1, we find a (σ_y,σ_z)-remainder

$$\tilde{s} = \frac{a/b}{(x+2y+3z)^2+1} \quad \textit{with} \quad \frac{a}{b} = \frac{(\frac{1}{3}x+1)(2y+3z)+\frac{2}{3}x^2+3x+4}{x+y+6}$$

such that $s-\tilde{s}$ is (σ_y,σ_z)-summable. However, the sum $r+\tilde{s}$ is still not a (σ_y,σ_z)-remainder since $(x+y)(x+y+6)$ is not σ_y^β-free (here $\beta=3$ in this case). Notice that

$$\frac{a}{b} = (\mathrm{S}_y^3\,\mathrm{S}_z^{-2}-1)\left(\sum_{k=1}^{2}\sigma_y^{-3k}\sigma_z^{2k}\left(\frac{a}{b}\right)\right) + \frac{(\frac{1}{3}x+1)(2y+3z)+\frac{2}{3}x^2+3x+4}{x+y},$$

so (3.10) enables us to find a new (σ_y,σ_z)-remainder

$$t = \frac{(\frac{1}{3}x+1)(2y+3z)+\frac{2}{3}x^2+3x+4}{(x+y)((x+2y+3z)^2+1)}$$

such that $s-t$ is (σ_y,σ_z)-summable and

$$r+t = \frac{(\frac{2}{3}x+1)(2y+3z)+\frac{4}{3}x^2+3x+5}{(x+y)((x+2y+3z)^2+1)}$$

is a (σ_y,σ_z)-remainder. Another possible choice is to find a (σ_y,σ_z)-remainder $\tilde{r}$ of r such that $\tilde{r}+s$ is a (σ_y,σ_z)-remainder.

In order to achieve the linearity of remainders, we need to develop two lemmas. The first one is an immediate result of Theorem 5.6 in [17] based on Proposition 3.4.

Lemma 4.3. *Let $\alpha,\beta\in\mathbb{Z}$ with $\beta\neq 0$. Let $a\in\mathbb{F}[y,z]$ and $b\in\mathbb{F}[y]\setminus\{0\}$ with b being σ_y^β-free. Then for any given σ_y^β-free polynomial $b^*\in\mathbb{F}[y]$, one finds $q\in\mathbb{F}(y)[z]$, $a'\in\mathbb{F}[y,z]$ and $b'\in\mathbb{F}[y]$ with a' of the form $\sum_{i=0}^{\deg_z(a')}\hat{a}'_i(\alpha y+\beta z)^i$ for $\hat{a}'_i\in\mathbb{F}[y]$ satisfying $\deg_y(\hat{a}'_i)<\deg_y(b')$, b' being σ_y^β-free and $\gcd(b^*,\sigma_y^{\beta\ell}(b'))=1$ for any nonzero $\ell\in\mathbb{Z}$, such that*

$$\frac{a}{b} = (\mathrm{S}_y^\beta\,\mathrm{S}_z^{-\alpha}-1)(q)+\frac{a'}{b'}.$$

Proof. Let $\tau=\sigma_y^\beta\sigma_z^{-\alpha}$. Then b is τ-free, since $b\in\mathbb{F}[y]$ and it is σ_y^β-free. By Proposition 3.2, we have that $\sigma_y\circ\phi_{\alpha,\beta}=\phi_{\alpha,\beta}\circ\tau$. Thus $\sigma_y^{\ell'}\circ\phi_{\alpha,\beta}=\phi_{\alpha,\beta}\circ\tau^{\ell'}$ for any integer ℓ'. It follows that $\gcd(\phi_{\alpha,\beta}(b),\sigma_y^{\ell'}(\phi_{\alpha,\beta}(b)))=\gcd(\phi_{\alpha,\beta}(b),\phi_{\alpha,\beta}(\tau^{\ell'}(b)))=1$ for any nonzero $\ell'\in\mathbb{Z}$, implying that $\phi_{\alpha,\beta}(b)$ is σ_y-free. Similarly, one shows that $\phi_{\alpha,\beta}(b^*)$ is also σ_y-free.

Now by [17, Theorem 5.6], there exist $\tilde{q}\in\mathbb{F}(y)[z]$, $\tilde{a}\in\mathbb{F}[y,z]$ and $\tilde{b}\in\mathbb{F}[y]$ with $\deg_y(\tilde{a})<\deg_y(\tilde{b})$, $\tilde{b}$ being σ_y-free and $\gcd(\phi_{\alpha,\beta}(b^*),\sigma_y^\ell(\tilde{b}))=1$ for any nonzero $\ell\in\mathbb{Z}$ such that

$$\phi_{\alpha,\beta}\left(\frac{a}{b}\right) = \sigma_y(\tilde{q})-\tilde{q}+\frac{\tilde{a}}{\tilde{b}}.$$

Let $q = \phi_{\alpha,\beta}^{-1}(\tilde{q})$, $a' = \phi_{\alpha,\beta}^{-1}(\tilde{a})$ and $b' = \phi_{\alpha,\beta}^{-1}(\tilde{b})$. Then $\gcd(b^*, \sigma^{\beta\ell}(b')) = 1$ for any nonzero $\ell \in \mathbb{Z}$, since $\phi_{\alpha,\beta}^{-1} \circ \sigma_y^\ell = \tau^\ell \circ \phi_{\alpha,\beta}^{-1}$ and $\tau(b') = \sigma_y^\beta(b')$. The assertion thus follows from Proposition 3.4.

□

Lemma 4.4. *Let $a/(b\,d^j)$ with $a, d \in \mathbb{F}[y,z]$, $b \in \mathbb{F}[y]$ and $j \in \mathbb{N} \setminus \{0\}$ be a remainder fraction. Then for any integer pair (λ, μ), one finds $g, h \in \mathbb{F}(y,z)$ such that*

$$\frac{a}{b\,d^j} = (\mathrm{S}_y - 1)(g) + (\mathrm{S}_z - 1)(h) + \frac{\sigma_y^\lambda \sigma_z^\mu(a)}{\sigma_y^\lambda(b)\sigma_y^\lambda\sigma_z^\mu(d)^j}. \tag{4.1}$$

Furthermore,

(i) if d is not (y,z)-integer linear over $\mathbb{F}$, then $\sigma_y^\lambda\sigma_z^\mu(a)/(\sigma_y^\lambda(b)\sigma_y^\lambda\sigma_z^\mu(d)^j)$ is a remainder fraction;

(ii) if d is (y,z)-integer linear over $\mathbb{F}$ of (α,β)-type, then for any given σ_y^β-free polynomial $b^* \in \mathbb{F}[y]$, one further finds $g', h' \in \mathbb{F}(y,z)$, $a' \in \mathbb{F}[y,z]$ and $b' \in \mathbb{F}[y]$ with $a'/(b'\sigma_y^\lambda\sigma_z^\mu(d)^j)$ being a remainder fraction and $\gcd(b^*, \sigma_y^{\beta\ell}(b')) = 1$ for any nonzero $\ell \in \mathbb{Z}$, such that

$$\frac{a}{b\,d^j} = (\mathrm{S}_y - 1)(g') + (\mathrm{S}_z - 1)(h') + \frac{a'}{b'\sigma_y^\lambda\sigma_z^\mu(d)^j}. \tag{4.2}$$

Proof. The equation (4.1) follows by iteratively applying the formulas

$$\frac{s}{t} = (\mathrm{S}_v - 1)\left(-\sum_{j=0}^{i-1}\sigma_v^j\left(\frac{s}{t}\right)\right) + \frac{\sigma_v^i(s)}{\sigma_v^i(t)} = (\mathrm{S}_v - 1)\left(\sum_{j=1}^{i}\sigma_v^{-j}\left(\frac{s}{t}\right)\right) + \frac{\sigma_v^{-i}(s)}{\sigma_v^{-i}(t)}$$

for any $s, t \in \mathbb{F}[y,z]$, $i \in \mathbb{N}$ and $v \in \{y, z\}$.

To see (i) note that $\deg_z(a) < \deg_z(d)$ and d is monic, irreducible and of positive degree in z, as $a/(b\,d^j)$ is a remainder fraction. Shifting polynomials in $\mathbb{F}[y,z]$ with respect to y or z preserves these properties. Since $\sigma_y^\lambda\sigma_z^\mu(d)$ is again not (y,z)-integer linear over $\mathbb{F}$, the assertion holds by Definition 3.6.

For (ii) applying Lemma 4.3 to $\sigma_y^\lambda\sigma_z^\mu(a)/\sigma_y^\lambda(b)$ in (4.1) with respect to b^* yields

$$\frac{\sigma_y^\lambda\sigma_z^\mu(a)}{\sigma_y^\lambda(b)} = (\mathrm{S}_y^\beta\,\mathrm{S}_z^{-\alpha} - 1)(q) + \frac{a'}{b'},$$

where $q \in \mathbb{F}(y)[z]$, $a' \in \mathbb{F}[y,z]$ and $b' \in \mathbb{F}[y]$ with $a'/(b'\sigma_y^\lambda\sigma_z^\mu(d)^j)$ being a remainder fraction and $\gcd(b^*, \sigma_y^{\beta\ell}(b')) = 1$ for any nonzero $\ell \in \mathbb{Z}$. Moreover, (3.10) enables one to compute $g', h' \in \mathbb{F}(y,z)$ so that

$$\frac{\sigma_y^\lambda\sigma_z^\mu(a)}{\sigma_y^\lambda(b)\sigma_y^\lambda\sigma_z^\mu(d)^j} = (\mathrm{S}_y - 1)(g') + (\mathrm{S}_z - 1)(h') + \frac{a'}{b'\sigma_y^\lambda\sigma_z^\mu(d)^j}.$$

Plugging this into (4.1) and updating $g' = g + g'$, $h' = h + h'$ gives (4.2). □

We are now ready to give an algorithm that provides a feasible way to obtain the linearity.

Remainder linearization. Given two (σ_y,σ_z)-remainders $r,s\in\mathbb{F}(y,z)$, compute $g,h\in\mathbb{F}(y,z)$ and a (σ_y,σ_z)-remainder $t\in\mathbb{F}(y,z)$ such that

$$s=(\mathrm{S}_y-1)(g)+(\mathrm{S}_z-1)(h)+t$$

with $r+t$ being a (σ_y,σ_z)-remainder.

1. write r and s in the remainder forms

$$r=\sum_{i=1}^{\bar m}\sum_{j=1}^{\bar n_i}\frac{\bar a_{ij}}{\bar b_{ij}\bar d_i^j}\quad\text{and}\quad s=\sum_{i=1}^{m}\sum_{j=1}^{n_i}\frac{a_{ij}}{b_{ij}d_i^j}. \tag{4.3}$$

2. set $g=h=0$.

   for $i=1,\dots,m$ do

   if there exists $k\in\{1,2,\dots,\bar m\}$ such that $\bar d_k=\sigma_y^{\lambda}\sigma_z^{\mu}(d_i)$ for some $\lambda,\mu\in\mathbb{Z}$, then

   2.1 for $j=1,\dots,n_i$, apply Lemma 4.4 to $a_{ij}/(b_{ij}d_i^j)$ to find $g_{ij},h_{ij}\in\mathbb{F}(y,z)$, $a'_{ij}\in\mathbb{F}[y,z]$ and $b'_{ij}\in\mathbb{F}[y]$ with $a'_{ij}/(b'_{ij}\bar d_k^j)$ being a remainder fraction and

   – $\gcd(\bar b_{kj},\sigma_y^{\beta\ell}(b'_{ij}))=1$ for any nonzero $\ell\in\mathbb{Z}$ if d_i is (y,z)-integer linear over $\mathbb{F}$ of (α,β)-type,

   – $a'_{ij}=\sigma_y^{\lambda}\sigma_z^{\mu}(a_{ij})$ and $b'_{ij}=\sigma_y^{\lambda}(b_{ij})$ otherwise,

   such that

$$\frac{a_{ij}}{b_{ij}d_i^j}=(\mathrm{S}_y-1)(g_{ij})+(\mathrm{S}_z-1)(h_{ij})+\frac{a'_{ij}}{b'_{ij}\bar d_k^j}; \tag{4.4}$$

   and update a_{ij}, b_{ij} and d_i to be a'_{ij}, b'_{ij} and $\bar d_k$, respectively.

   2.2 update $g=g+\sum_{j=1}^{n_i}g_{ij}$ and $h=h+\sum_{j=1}^{n_i}h_{ij}$.

3. set $t=\sum_{i=1}^{m}\sum_{j=1}^{n_i}a_{ij}/(b_{ij}d_i^j)$, and return g,h,t.

Theorem 4.5. *Let r and s be two (σ_y,σ_z)-remainders. Then the remainder linearization correctly finds two rational functions $g,h\in\mathbb{F}(y,z)$ and a (σ_y,σ_z)-remainder $t\in\mathbb{F}(y,z)$ such that*

$$s=(\mathrm{S}_y-1)(g)+(\mathrm{S}_z-1)(h)+t. \tag{4.5}$$

and c_1r+c_2t is a (σ_y,σ_z)-remainder for all $c_1,c_2\in\mathbb{F}$.

Proof. Since both r and s are (σ_y,σ_z)-remainders, they can be written into the remainder forms (4.3). Define

$$\begin{aligned}\Lambda_s&=\{i\in\mathbb{N}\mid 1\le i\le m\text{ and }d_i=\sigma_y^{-\lambda_i}\sigma_z^{-\mu_i}(\bar d_k)\text{ for some }\lambda_i,\mu_i\in\mathbb{Z}\text{ and }1\le k\le\bar m\},\\ \Lambda_s^c&=\{1,2,\dots,m\}\setminus\Lambda_s,\\ \text{and}\quad\Lambda_r^c&=\{k\in\mathbb{N}\mid 1\le k\le\bar m\text{ and }\bar d_k\not\sim_{y,z}d_i\text{ for all }1\le i\le m\}.\end{aligned}$$

For each $i \in \Lambda_s$, denote by k_i the integer so that $1 \leq k_i \leq \bar{m}$ and $\bar{d}_{k_i} \sim_{y,z} d_i$. Let $\{d'_1, \ldots, d'_m\}$ be a set of polynomials in $\mathbb{F}[y]$ with $d'_i = \bar{d}_{k_i}$ if $i \in \Lambda_s$ and $d'_i = d_i$ if $i \in \Lambda_s^c$. Then $d'_1, \ldots, d'_m$ are pairwise (y,z)-shift inequivalent since s is a (σ_y, σ_z)-remainder. For all $i \in \Lambda_s$ and all $1 \leq j \leq n_i$, one sees from Lemma 4.4 that step 2.1 correctly finds the $g_{ij}, h_{ij}, a'_{ij}, b'_{ij}$ satisfying described conditions such that (4.4) holds. It then follows from Definition 3.7 that $t = \sum_{i=1}^{m} \sum_{j=1}^{n_i} a'_{ij}/(b'_{ij} d_i'^j)$ is a (σ_y, σ_z)-remainder. Substituting all (4.4) into (4.3), together with step 2.2, immediately yields the equation (4.5).

Let $c_1, c_2 \in \mathbb{F}$. Then it remains to prove that $c_1 r + c_2 t$ is a (σ_y, σ_z)-remainder. A straightforward calculation yields that

$$c_1 r + c_2 t = \sum_{i \in \Lambda_s} \sum_{j=1}^{\max\{\bar{n}_{k_i}, n_i\}} \frac{a^*_{ij}}{b^*_{ij} d_i'^j} + \sum_{k \in \Lambda_r^c} \sum_{j=1}^{n_k} \frac{c_1 \bar{a}_{kj}}{\bar{b}_{kj} \bar{d}_k^j} + \sum_{i \in \Lambda_s^c} \sum_{j=1}^{n_i} \frac{c_2 a'_{ij}}{b'_{ij} d_i'^j},$$

in which b^*_{ij} is the least common multiple of $\{\bar{b}_{k_i j}, b'_{ij}\}$ and $a^*_{ij} = c_1 \bar{a}_{k_i j}(b^*_{ij}/\bar{b}_{k_i j}) + c_2 a'_{ij}(b^*_{ij}/b'_{ij})$ with $(\bar{a}_{k_i j}, \bar{b}_{k_i j})$ (resp. (a'_{ij}, b'_{ij})) being specified to be $(0,1)$ in case $j > \bar{n}_{k_i}$ (resp. $j > n_i$). Observe that polynomials in $\{\bar{d}_1, \bar{d}_2, \ldots, \bar{d}_{\bar{m}}\} = \{\bar{d}_{k_i} \mid i \in \Lambda_s\} \cup \{\bar{d}_k \mid k \in \Lambda_r^c\}$, as well as those in $\{d'_1, d'_2, \ldots, d'_m\} = \{d'_i \mid i \in \Lambda_s\} \cup \{d'_i \mid i \in \Lambda_s^c\}$, are pairwise (y,z)-shift inequivalent, as both r and t are (σ_y, σ_z)-remainders. Since $d'_i = \bar{d}_{k_i}$ for $i \in \Lambda_s$ and $d'_i = d_i$ for $i \in \Lambda_s^c$, polynomials in

$$\{d'_i \mid i \in \Lambda_s\} \cup \{\bar{d}_k \mid k \in \Lambda_r^c\} \cup \{d'_i \mid i \in \Lambda_s^c\}$$

are pairwise (y,z)-shift inequivalent as well by the definition of Λ_r^c. Since r and t are both (σ_y, σ_z)-remainders, each $c_1 \bar{a}_{kj}/(\bar{b}_{kj} \bar{d}_k^j)$ for $k \in \Lambda_r^c$ and $1 \leq j \leq n_k$, as well as each $c_2 a'_{ij}/(b'_{ij} d_i'^j)$ for $i \in \Lambda_s^c$ and $1 \leq j \leq n_i$, is a remainder fraction. Thus it amounts to showing that for $i \in \Lambda_s$ and $1 \leq j \leq \max\{\bar{n}_{k_i}, n_i\}$, each $a^*_{ij}/(b^*_{ij} d_i'^j)$ is also a remainder fraction with the conclusion then following by Definition 3.7. Let $i \in \Lambda_s$ and $1 \leq j \leq \max\{\bar{n}_{k_i}, n_i\}$. Obviously, $a^*_{ij} \in \mathbb{F}[y,z]$ and $\deg_z(a^*_{ij}) \leq \max\{\deg_z(\bar{a}_{k_i j}), \deg_z(a'_{ij})\} < \deg_z(d'_i)$. If d_i (and then $d'_i = \bar{d}_{k_i}$) is (y,z)-integer linear over $\mathbb{F}$ of (α, β)-type, then $\bar{a}_{k_i j}$ (resp. a'_{ij}) can be viewed as a polynomial in $(\alpha y + \beta z)$ with coefficients all having degrees in y less than $\deg_y(\bar{b}_{k_i j})$ (resp. $\deg_y(b'_{ij})$). Notice that b^*_{ij} is the least common multiple of $\{\bar{b}_{k_i j}, b'_{ij}\}$. Thus a^*_{ij} can be viewed as a polynomial in $(\alpha y + \beta z)$ with coefficients all having degrees in y less than $\deg_y(b^*_{ij})$. Moreover, b^*_{ij} is σ_y^β-free as $\bar{b}_{k_i j}, b'_{ij}$ both are and $\gcd(\bar{b}_{k_i j}, \sigma_y^{\beta \ell}(b'_{ij})) = 1$ for any nonzero $\ell \in \mathbb{Z}$ by step 2.2. Therefore, each $a^*_{ij}/(b^*_{ij} d_i'^j)$ is a remainder fraction by definition. This concludes the proof. □

5 Telescoping via reduction

Recall that a telescoper L, for a given rational function $f \in \mathbb{F}(y,z)$, is a nonzero operator in $\mathbb{F}[S_x]$ such that $L(f)$ is (σ_y, σ_z)-summable. For discrete trivariate rational functions, telescopers do not always exist. Recently, a criterion for determining the existence of telescopers in this case was presented by Chen, Hou, Labahn and Wang in [16]. We summarize this in the following theorem. In order to adapt it into our setting, we will consider primitive parts of polynomials in $\mathbb{F}[y]$. Recall that the primitive part of $p \in \mathbb{F}[y]$ with respect to y, denoted by $\mathrm{prim}_y(p)$, is the primitive part with respect to y of the numerator (with respect to x) of p. Then $\mathrm{prim}_y(p)$ is a primitive polynomial in $C[x,y]$ whose coefficients with respect to y have no nonconstant common divisors in $C[x]$.

Theorem 5.1 (Existence criterion)**.** *Let f be a rational function in $\mathbb{F}(y,z)$. Assume that applying the bivariate reduction of Abramov to f yields (3.8), where $g,h,r \in \mathbb{F}(y,z)$ and r is a (σ_y,σ_z)-remainder with the remainder form (3.6). Then f has a telescoper if and only if for each $1 \le i \le m$ and $1 \le j \le n_i$,*

(i) there exists a positive integer ξ_i such that $\sigma_x^{\xi_i}(d_i) = \sigma_y^{\zeta_i}\sigma_z^{\eta_i}(d_i)$ for some integers ζ_i, η_i;

(ii) and b_{ij} is (x,y)-integer linear over C, in particular, $\sigma_x^{\xi_i}(\mathrm{prim}_y(b_{ij})) = \sigma_y^{\zeta_i}(\mathrm{prim}_y(b_{ij}))$ if d_i is not (y,z)-integer linear over $\mathbb{F}$.

With termination guaranteed by the above criterion, we now use the bivariate reduction of Abramov to develop a telescoping algorithm in the spirit of the general reduction-based approach.

Algorithm ReductionCT. Given a rational function $f \in \mathbb{F}(y,z)$, compute a minimal telescoper $L \in \mathbb{F}[\mathrm{S}_x]$ for f and a corresponding certificate $(g,h) \in \mathbb{F}(y,z)^2$ when telescopers exist.

1. apply the bivariate reduction of Abramov to f to find $g_0, h_0 \in \mathbb{F}(y,z)$ and a (σ_y,σ_z)-remainder $r_0 \in \mathbb{F}(y,z)$ such that
$$f = (\mathrm{S}_y - 1)(g_0) + (\mathrm{S}_z - 1)(h_0) + r_0. \tag{5.1}$$
2. if $r_0 = 0$ then set $L = 1$, $(g,h) = (g_0,h_0)$ and return.
3. if conditions (i)-(ii) in Theorem 5.1 are not satisfied, then return "No telescopers exist".
4. set $R = c_0 r_0$, where c_0 is an indeterminate.

   for $\ell = 1, 2, \ldots$ do

   4.1 apply the remainder linearization to $\sigma_x(r_{\ell-1})$ with respect to R to find $g_\ell, h_\ell \in \mathbb{F}(y,z)$ and a (σ_y,σ_z)-remainder $r_\ell \in \mathbb{F}(y,z)$ such that
   $$\sigma_x(r_{\ell-1}) = (\mathrm{S}_y - 1)(g_\ell) + (\mathrm{S}_z - 1)(h_\ell) + r_\ell, \tag{5.2}$$
   and that $R + c_\ell r_\ell$ is a (σ_y,σ_z)-remainder, where c_ℓ is an indeterminate.

   4.2 update $R = R + c_\ell r_\ell$ and update $g_\ell = g_\ell + \sigma_x(g_{\ell-1})$, $h_\ell = h_\ell + \sigma_x(h_\ell)$ so that
   $$\sigma_x^\ell(f) = (\mathrm{S}_y - 1)(g_\ell) + (\mathrm{S}_z - 1)(h_\ell) + r_\ell. \tag{5.3}$$

   4.3 if there exist nontrivial $c_0, \ldots, c_\ell \in \mathbb{F}$ such that $R = 0$, then set $L = \sum_{i=0}^{\ell} c_i \mathrm{S}_x^i$ and $(g,h) = (\sum_{i=0}^{\ell} c_i g_i, \sum_{i=0}^{\ell} c_i h_i)$, and return.

Theorem 5.2. *Let f be a rational function in $\mathbb{F}(y,z)$. Then the algorithm* **ReductionCT** *terminates and returns a minimal telescoper for f when such a telescoper exists.*

Proof. By Theorems 3.10 and 5.1, steps 2-3 are correct. Observe from Definition 3.7 that $\sigma_x(r_0)$ is a (σ_y,σ_z)-remainder as r_0 is. By Theorem 4.5, step 4.1 correctly finds $g_1, h_1 \in \mathbb{F}(y,z)$ and a (σ_y,σ_z)-remainder $r_1 \in \mathbb{F}(y,z)$ such that (5.2) holds for $\ell = 1$ and $R + c_1 r_1 = c_0 r_0 + c_1 r_1$ is a (σ_y,σ_z)-remainder for all $c_0, c_1 \in \mathbb{F}$. Applying σ_x to both sides of (5.1), together with step 4.1, one sees that step 4.2 gives (5.3) for $\ell = 1$. The correctness of step 4.2 for each iteration of the loop in step 4 then follows by induction on ℓ.

If f does not have a telescoper then the algorithm halts after step 3. Otherwise, telescopers for f exist by Theorem 5.1. Let $L = \sum_{\ell=0}^{\rho} c_\ell \,\mathrm{S}_x^\ell \in \mathbb{F}[\mathrm{S}_x]$ be a telescoper for f of minimal order. Then $c_\rho \neq 0$ and by (5.3), applying L to f gives

$$L(f) = \sum_{\ell=0}^{\rho} c_\ell \sigma_x^\ell(f) = (\mathrm{S}_y - 1)\left(\sum_{\ell=0}^{\rho} c_\ell g_\ell\right) + (\mathrm{S}_z - 1)\left(\sum_{\ell=0}^{\rho} c_\ell h_\ell\right) + \sum_{\ell=0}^{\rho} c_\ell r_\ell.$$

Notice that $\sum_{\ell=0}^{\rho} c_\ell r_\ell$ is a (σ_y, σ_z)-remainder by step 4.1. It follows from Theorem 3.10 that $L(f)$ is (σ_y, σ_z)-summable, namely L is a telescoper for f, if and only if $\sum_{\ell=0}^{\rho} c_\ell r_\ell = 0$. This implies that the linear system over $\mathbb{F}$ obtained by equating $\sum_{\ell=0}^{\rho} c_\ell r_\ell$ to zero has a nontrivial solution, which yields a telescoper of minimal order. The algorithm thus terminates. □

Recall that an operator $L \in \mathbb{F}[\mathrm{S}_x]$ is a *common left multiple* of operators $L_1, \ldots, L_m \in \mathbb{F}[\mathrm{S}_x]$ if there exist operators $L_1', \ldots, L_m' \in \mathbb{F}[\mathrm{S}_x]$ such that $L = L_1' L_1 = \cdots = L_m' L_m$. Amongst all such common left multiples, the monic one of lowest possible degree with respect to S_x is called the *least common left multiple*. In view of the computation, many efficient algorithms are available, see [7] and the references therein.

In analogy to [30, Theorem 2], we have the following lemma.

Lemma 5.3. *Let $r = r_1 + \cdots + r_m$ with $r_i \in \mathbb{F}(y, z)$ and let $L_1, \ldots, L_m \in \mathbb{F}[\mathrm{S}_x]$ be the respective minimal telescopers for $r_1, \ldots, r_m$. Then the least common left multiple L of the L_i is a telescoper for r. Moreover, if any telescoper for r is also a telescoper for each r_i with $1 \le i \le m$, then L is a minimal telescoper for r.*

Then one sees that the least common multiple is a minimal telescoper for the given sum provided that the denominators of distinct summands comprise distinct (x, y, z)-shift equivalence classes.

Proposition 5.4. *Let $r \in \mathbb{F}(y, z)$ be a rational function of the form*

$$r = r_1 + r_2 + \cdots + r_m,$$

where $r_i = a_i/d_i$ with $a_i, d_i \in \mathbb{F}[y, z]$ satisfying the conditions

(i) $\deg_z(a_i) < \deg_z(d_i)$;

(ii) any monic irreducible factor of d_i of positive degree in z is (x, y, z)-shift inequivalent to all factors of $d_{i'}$ whenever $1 \le i, i' \le m$ and $i \neq i'$.

Let $L_1, \ldots, L_m \in \mathbb{F}[\mathrm{S}_x]$ be respective minimal telescopers for $r_1, \ldots, r_m$. Then the least common left multiple L of the L_i is a minimal telescoper for r. Moreover, for each $1 \le i \le m$, let (g_i, h_i) be a corresponding certificate for L_i and let $L_i' \in \mathbb{F}[\mathrm{S}_x]$ be the cofactor of L_i so that $L = L_i' L_i$. Then

$$\left(\sum_{i=1}^{m} L_i'(g_i), \sum_{i=1}^{m} L_i'(h_i)\right)$$

is a corresponding certificate for L.

Proof. Let $\tilde{L} \in \mathbb{F}[S_x]$ be a telescoper for r. In order to show the first assertion, by Lemma 5.3, it suffices to verify that $\tilde{L}$ is also a telescoper for each r_i with $1 \leq i \leq m$. Notice that the application of a nonzero operator from $\mathbb{F}[S_x]$ does not change the (x, y, z)-shift equivalence classes, with representatives being monic irreducible polynomials of positive degrees in z, that appear in a given polynomial in $\mathbb{F}[y, z]$. Hence condition (ii) remains valid when d_i and $d_{i'}$ are replaced by $\tilde{L}(d_i)$ and $\tilde{L}(d_{i'})$, respectively. It then follows that any two monic irreducible factors of positive degrees in z from distinct d_i are (y, z)-shift inequivalent to each other. By the definition of telescopers, $\tilde{L}(r)$ is (σ_y, σ_z)-summable, and then so is each $\tilde{L}(r_i)$ according to [26, Lemma 3.2]. This implies that $\tilde{L}$ is indeed a telescoper for each r_i. The second assertion follows by observing that $(S_y - 1)$ and $(S_z - 1)$ both commute with operators from $\mathbb{F}[S_x]$. □

The above proposition provides an alternative way to construct a minimal telescoper for a given rational function.

5.1 Examples

Example 5.5. *Consider the rational function f_1 given in Example 3.11. Note that f_1 is a remainder fraction and satisfies conditions (i)-(ii) in Theorem 5.1. So telescopers for f_1 exist. Applying the algorithm* **ReductionCT** *to f_1, we obtain in step 4 that*

$$\sigma_x^\ell(f_1) = (S_y - 1)(g_\ell) + (S_z - 1)(h_\ell) + r_\ell \quad \textit{for } \ell = 0, 1, 2,$$

where

$$r_0 = f_1, \quad r_1 = \frac{(x+1)^2(z-1)+1}{(x+y)(x+z)^2+1}, \quad r_2 = \frac{(x+2)^2(z-2)+1}{(x+y)(x+z)^2+1}$$

and $g_\ell, h_\ell \in \mathbb{F}(y, z)$ are not displayed here to keep things neat. By finding a $\mathbb{F}$-linear dependency among r_0, r_1, r_2, we see that

$$L_1 = (x^4 + 2x^3 + x^2 + 2x + 1) S_x^2 - 2(x^4 + 4x^3 + 4x^2 + 2x + 2) S_x + (x^4 + 6x^3 + 13x^2 + 14x + 7)$$

is a minimal telescoper for f_1.

Example 5.6. *Consider the rational function f_2 given in Example 3.11. Then it can be decomposed as*

$$f_2 = (S_y - 1)(g_0) + (S_z - 1)(h_0) + \underbrace{\frac{\frac{1}{3}x(2y+3z) + \frac{2}{3}x^2 + 1}{(x+y)((x+2y+3z)^2+1)}}_{r_0} \quad \textit{for some } g_0, h_0 \in \mathbb{F}(y, z).$$

Note that r_0 is a remainder fraction and satisfies conditions (i)-(ii) in Theorem 5.1. Thus telescopers for f_2 exist. Applying the algorithm **ReductionCT** *to f_2, we obtain in step 4 that*

$$\sigma_x^\ell(f_2) = (S_y - 1)(g_\ell) + (S_z - 1)(h_\ell) + r_\ell \quad \textit{for } \ell = 0, 1, \ldots, 6,$$

where $g_\ell, h_\ell \in \mathbb{F}(y, z)$ are again not displayed due to the large sizes, and

$$r_1 = \frac{(\frac{1}{3}x+1)(2y+3z)+\frac{2}{3}x^2+x+\frac{4}{3}}{(x+y+2)((x+2y+3z)^2+1)}, \quad r_2 = \frac{(\frac{1}{3}x+\frac{2}{3})(2y+3z)+\frac{2}{3}x^2+2x+\frac{7}{3}}{(x+y+4)((x+2y+3z)^2+1)}, \quad r_3 = \frac{(\frac{1}{3}x+1)(2y+3z)+\frac{2}{3}x^2+4}{(x+y)((x+2y+3z)^2+1)},$$

$$r_4 = \frac{(\frac{1}{3}x+\frac{4}{3})(2y+3z)+\frac{2}{3}x^2+4x+\frac{19}{3}}{(x+y+2)((x+2y+3z)^2+1)}, \quad r_5 = \frac{(\frac{1}{3}x+\frac{5}{3})(2y+3z)+\frac{2}{3}x^2+5x+\frac{28}{3}}{(x+y+4)((x+2y+3z)^2+1)}, \quad r_6 = \frac{(\frac{1}{3}x+2)(2y+3z)+\frac{2}{3}x^2+6x+13}{(x+y)((x+2y+3z)^2+1)}.$$

Then one finds a $\mathbb{F}$-linear dependency among r_0, r_3, r_6 which yields a minimal telescoper

$$L_2 = (x^2+3x-3)\,\mathrm{S}_x^6 - 2(x^2+6x-3)\,\mathrm{S}_x^3 + x^2+9x+15.$$

The following illustrates the result of Proposition 5.4.

Example 5.7. *Consider the same rational function f as in Example 3.11. Then we know that f_3 is (σ_y,σ_z)-summable. Thus $L_3 = 1$ is a minimal telescoper for f_3. Let $L_1, L_2 \in \mathbb{F}[\mathrm{S}_x]$ be the operators computed in Examples 5.5-5.6. It then follows that the least common left multiple L of $\{L_1, L_2, L_3\}$, given by*

$$\begin{aligned} L \;=\; \mathrm{S}_x^8 \;& - \tfrac{2(x^2+5x+1)(3x^2+24x+31)}{(x^2+7x+7)(3x^2+21x+19)}\,\mathrm{S}_x^7 + \tfrac{(x^2+3x-3)(3x^2+27x+43)}{(x^2+7x+7)(3x^2+21x+19)}\,\mathrm{S}_x^6 - \tfrac{2(x^2+10x+13)}{x^2+7x+7}\,\mathrm{S}_x^5 \\ & + \tfrac{4(3x^2+24x+31)(x^2+8x+4)}{(x^2+7x+7)(3x^2+21x+19)}\,\mathrm{S}_x^4 - \tfrac{2(x^2+6x-3)(3x^2+27x+43)}{(x^2+7x+7)(3x^2+21x+19)}\,\mathrm{S}_x^3 + \tfrac{x^2+13x+37}{x^2+7x+7}\,\mathrm{S}_x^2 \\ & - \tfrac{2(x^2+11x+25)(3x^2+24x+31)}{(x^2+7x+7)(3x^2+21x+19)}\,\mathrm{S}_x + \tfrac{(x^2+9x+15)(3x^2+27x+43)}{(x^2+7x+7)(3x^2+21x+19)}, \end{aligned}$$

is a telescoper for f. On the other hand, by directly applying the algorithm **ReductionCT** *to f, one sees that L is in fact a minimal telescoper for f.*

5.2 Efficiency considerations

The efficiency of Algorithm **ReductionCT** can be enhanced by incorporating two modifications in the algorithm.

Simplification of remainder step 4.1

For each iteration of the loop in step 4, rather than using the overall (σ_y,σ_z)-remainder $R = \sum_{k=0}^{\ell-1} c_k r_k$ in step 4.1, we can apply the remainder linearization to the shift value $\sigma_x(r_{\ell-1})$ with respect to the initial (σ_y,σ_z)-remainder r_0 only. This is sufficient as, for any (σ_y,σ_z)-remainder r_ℓ of $\sigma_x(r_{\ell-1})$ with $\ell \geq 1$, if $r_0 + r_\ell$ is a (σ_y,σ_z)-remainder then so is $R + c_\ell r_\ell$, provided that the algorithm proceeds in the described iterative fashion.

The intuition for this simplification is as follows. Notice that if the algorithm continues after passing through step 3 then $r_0 \neq 0$. Since distinct (y,z)-shift equivalence classes can be tackled separately, we restrict ourselves to the case where the denominator of r_0 is of the form

$$d\,\sigma_x^{i_1}(d)\cdots\sigma_x^{i_m}(d)$$

with $d \in \mathbb{F}[y,z]$ being monic, irreducible and of positive degree in z, $i_1,\ldots,i_m$ being distinct positive integers such that $d, \sigma_x^{i_1}(d),\ldots,\sigma_x^{i_m}(d)$ are (y,z)-shift inequivalent to each other. For simplicity, we call $(0,i_1,\ldots,i_m)$ the x-shift exponent sequence of d in r_0. By Theorem 5.1, there exists a positive integer ξ such that $\sigma_x^{\xi}(d) \sim_{y,z} d$ and so we let ξ be the smallest one with such a property. Then there are only ξ many (y,z)-shift equivalence classes produced by shifting d with respect to x, with $d, \sigma_x(d),\ldots,\sigma_x^{\xi-1}(d)$ as respective representatives. Without loss of generality, we further assume that $0 < i_1 < \cdots < i_m < \xi$. For $\ell \geq 1$, let r_ℓ be the output of the remainder linearization when applied to $\sigma_x(r_{\ell-1})$ with respect to r_0. By induction on ℓ, one sees that the x-shift exponent sequence of d in r_ℓ is given by

$$(\ell, i_1+\ell,\ldots,i_m+\ell) \mod \xi,$$

whose entries form an $(m+1)$-subset of $\{0, 1, \ldots, \xi - 1\}$. It thus follows from Definition 3.7 that $R + c_\ell r_\ell$ is also a (σ_y, σ_z)-remainder.

Simplification of remainder step 4.3

Our second modification is in step 4.3, where we first individual derive from $R = 0$ the equation for each remainder fraction $a/(b\, d^j)$ appearing in the remainder form of R, and then build a linear system over $\mathbb{F}$ from the coefficients of the numerator of the equation with respect to y and $Z = \alpha y + \beta z$, instead of y and z, in the case where d is (y, z)-integer linear of (α, β)-type. Notice that $R = c_0 r_0 + c_1 r_1 + \cdots + c_\ell r_\ell$ at the stage of step 4.3. Let $d_1, \ldots, d_m$ be all monic irreducible polynomials of positive degrees in z that appear in the denominator of R, with multiplicities $n_1, \ldots, n_m$, respectively. For $1 \leq i \leq m$, $1 \leq j \leq n_i$ and $0 \leq k \leq \ell$, let $a_{ij}^{(k)} \in \mathbb{F}[y, z]$ and $b_{ij}^{(k)} \in \mathbb{F}[y]$ be such that $a_{ij}^{(k)}/(b_{ij}^{(k)} d_i^j)$ is a remainder fraction appearing in the remainder form of r_k. By coprimeness among the d_i, one gets that

$$R = 0 \quad \Longleftrightarrow \quad \sum_{k=0}^{\ell} c_k \cdot \frac{a_{ij}^{(k)}}{b_{ij}^{(k)}} = 0 \quad \text{for all } i = 1, \ldots, m \text{ and } j = 1, \ldots, n_i.$$

If d_i is (y, z)-integer linear of (α_i, β_i)-type, then by Definition 3.6, every $a_{ij}^{(k)}$ can be viewed as a polynomial in $Z_i = \alpha_i y + \beta_i z$ with coefficients all having degrees in y less than $\deg_y(b_{ij}^{(k)})$. In this case, rather than naively considering the coefficients with respect to y and z, we instead force all the coefficients with respect to y and Z_i of the numerator of $\sum_{k=0}^{\ell} c_k \cdot (a_{ij}^{(k)}/b_{ij}^{(k)})$ to zero. This way ensures that the resulting linear system over $\mathbb{F}$ typically has smaller size than the naive one.

6 Implementation and timings

We have implemented our new algorithm **ReductionCT** in the computer algebra system Maple 2018. Our implementation includes the two enhancements to step 4 discussed in the previous subsection. In order to get an idea about the efficiency of our algorithm, we applied our implementation to certain examples and tabulated their runtime in this section. All timings were measured in seconds on a Linux computer with 128GB RAM and fifteen 1.2GHz Dual core processors. The computations for the experiments did not use any parallelism.

We considered trivariate rational functions of the form

$$f(x, y, z) = \frac{a(x, y, z)}{d_1(x, y, z) \cdot d_2(x, y, z)}, \tag{6.1}$$

where

- $a \in \mathbb{Z}[x, y, z]$ of total degree $m \geq 0$ and max-norm $||a||_\infty \leq 5$, in other words, the maximal absolute value of the coefficients of a with respect to x, y, z are no more than 5;
- $d_i = p_i \cdot \sigma_x^\xi(p_i)$ with $p_1 = P_1(\xi y - \zeta x, \xi z + \zeta x)$ and $p_2 = P_2(\zeta x + \xi y + 2\xi z)$ for two nonzero integers ξ, ζ and two integer polynomials $P_1(y, z) \in \mathbb{Z}[y, z]$, $P_2(z) \in \mathbb{Z}[z]$, both of which have total degree $n > 0$ and max-norm no more than 5.

For a selection of random rational functions of this type for different choices of (m, n, ξ, ζ), Table 1 collects the timings of four variants of the algorithm **ReductionCT** from Section 5: for the columns RCT_1 and RCT_2, we both compute the telescoper as well as the certificate, but the difference lies in that the former brings the certificate to a common denominator while the latter leaves the certificate as an unnormalized linear combination of rational functions; for RCT_3 we only compute the telescoper and neglect almost everything related to the certificate; for RCTLM_1, RCTLM_2 and RCTLM_3, we perform the same functionality as RCT_1, RCT_2 and RCT_3 but based on the idea from Proposition 5.4. As indicated by the table, the timings for RCT_2 (resp. RCTLM_2) are virtually the same as for RCT_3 (resp. RCTLM_3).

(m, n, ξ, ζ)	RCT_1	RCT_2	RCT_3	RCTLM_1	RCTLM_2	RCTLM_3	order
(1, 1, 1, 1)	0.196	0.098	0.979	0.220	0.109	0.110	1
(1, 1, 1, 5)	7.319	0.112	0.123	9.483	0.131	0.123	1
(1, 1, 1, 9)	105.548	0.123	0.121	104.514	0.128	0.125	1
(1, 1, 1, 13)	2586.295	0.114	0.136	3078.043	0.133	0.126	1
(1, 1, 1, 3)	0.574	0.098	0.097	0.712	0.107	0.104	1
(1, 2, 1, 3)	17.812	0.258	0.256	17.299	0.268	0.263	1
(1, 3, 1, 3)	266.206	2.008	1.999	220.209	2.027	1.997	1
(1, 4, 1, 3)	2838.827	37.052	37.358	3039.199	33.599	30.547	1
(1, 5, 1, 3)	19403.916	1085.659	1074.295	18309.000	1111.333	1119.393	1
(2, 3, 1, 3)	31678.706	2.257	2.540	15825.876	2.295	2.224	3
(3, 3, 1, 3)	44243.254	5.106	5.378	16869.097	4.512	4.295	3
(3, 2, 1, 3)	710.810	0.480	0.492	670.501	0.522	0.487	3
(3, 2, 2, 3)	1314.809	0.751	0.701	941.009	0.792	0.756	6
(3, 2, 4, 3)	1558.440	1.528	1.525	1121.624	1.598	1.550	12
(3, 2, 8, 3)	1878.424	4.567	4.215	986.017	4.133	4.245	24
(3, 2, 16, 3)	2800.050	25.027	21.136	1317.603	38.399	38.504	48

Table 1: Timings for sixth variants of the algorithm **ReductionCT**.

7 Conclusion and future work

In this paper, we have studied the class of trivariate rational functions and presented a creative telescoping algorithm for this class. The procedure is based on a bivariate extension of Abramov's reduction method initiated in [1]. Our algorithm finds a minimal telescoper for a given trivariate rational function without also needing to compute an associated certificate. This in turn provides a more efficient way to deal with rational double summations in practice.

We are interested in the more important problem of computing hypergeometric multiple summations or proving identities which involve such summations. A function $f(x, y_1, \ldots, y_n)$ is called a multivariate *hypergeometric term* if the quotients

$$\frac{f(x+1, y_1, \ldots, y_n)}{f(x, y_1, \ldots, y_n)}, \frac{f(x, y_1+1, \ldots, y_n)}{f(x, y_1, \ldots, y_n)}, \ldots, \frac{f(x, y_1, \ldots, y_n+1)}{f(x, y_1, \ldots, y_n)}$$

are all rational functions in $x, y_1, \ldots, y_n$. The problem of hypergeometric multiple summations tends to appear more often than the rational case, particularly in combinatorics [10, 15], and it is also more challenging.

Since a large percent of hypergeometric terms falls into the class of holonomic functions, the problem of hypergeometric multiple summations can also be considered in a more general framework of multivariate holonomic functions. In this context, several creative telescoping approaches have already been developed in [37, 35, 23, 22, 29]. The algorithms in the first three papers are based on elimination and suffer from the disadvantage of inefficiency in practice. The algorithm in [22], also known as Chyzak's algorithm, deals with single sums (and single integrals) and can only be used to solve multiple ones in a recursive manner. A fast but heuristic approach was given in [29] in order to eliminate the bottleneck in Chyzak's algorithm of solving a coupled first-order system. This approach generalizes to multiple sums (and multiple integrals). We refer to [28] for a detailed and excellent exposition of these approaches. We remark that all these approaches find the telescoper and the certificate simultaneously, with the exception of Takayama's algorithm in [35] where natural boundaries have to be assured a priori. Note also that holonomicity is a sufficient but not necessary condition for the applicability of creative telescoping applied to hypergeometric terms (cf. [5, 16]).

Restricted to the hypergeometric setting, partial solutions for the problem of multiple summations were proposed in [21] and [15]. In the former paper, the authors presented a heuristic method to find telescopers for trivariate hypergeometric terms, through which they also managed to prove certain famous hypergeomeric double summation identities. In the second paper, the authors mainly focused on a subclass of proper hypergeometric summations – multiple binomial sums. They first showed that the generating function of a given multiple binomial sum is always the diagonal of a rational function and vice versa. They then constructed a differential equation for the diagonal by a reduction-based telescoping approach. Finally the differential equation is translated back into a recurrence relation satisfied by the given binomial sum. In the future, we hope to explore this topic further and aim at developing a complete reduction-based telescoping algorithm for hypergeometric terms in three or more variables.

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