Non-minimality of Minimal Telescopers Explained by Residues *

Shaoshi Chen[a,b], Manuel Kauers[c], Christoph Koutschan[d]
Xiuyun Li[a,b,c], Rong-Hua Wang[e], and Yisen Wang[a,b,d]

[a]KLMM, Academy of Mathematics and Systems Science,
Chinese Academy of Sciences, Beijing, 100190, (China)
[b]School of Mathematical Sciences,
University of Chinese Academy of Sciences,
Beijing 100049, (China)
[c]Institute for Algebra, Johannes Kepler University
Linz A-4040, (Austria)
[d]RICAM, Austrian Academy of Sciences
Linz A-4040, (Austria)
[e]School of Mathematical Sciences, Tiangong University
Tianjin, 300387, (China)
schen@amss.ac.cn, manuel.kauers@jku.at
lixiuyun@amss.ac.cn, christoph.koutschan@oeaw.ac.at
wangronghua@tiangong.edu.cn, wangyisen@amss.ac.cn

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Abstract

Elaborating on an approach recently proposed by Mark van Hoeij, we continue to investigate why creative telescoping occasionally fails to find the minimal-order annihilating operator of a given definite sum or integral. We offer an explanation based on the consideration of residues.

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1 Introduction

Creative telescoping is the standard approach to definite summation and integration in computer algebra. Its purpose is to find an annihilating operator for a given definite sum $\sum_k f(n,k)$ or a given definite integral $\int_\Omega f(x,y)dy$.

Such operators are obtained from annihilating operators of the summand or integrand that have a particular form. In the case of summation, suppose that we have

$$(L - (S_k - 1)Q) \cdot f(n,k) = 0 \tag{1.1}$$

for some operator L that only involves n and the shift operator S_n but neither k nor the shift operator S_k, and another operator Q that may involve any of n, k, S_n, S_k. Summing the equation over all k yields

$$L \cdot \sum_k f(n,k) = \big[Q \cdot f(n,k)\big]_{k=-\infty}^{\infty}.$$

If the right-hand side happens to be zero, we find that L is an annihilating operator for the sum.

In the case of integration, having

$$(L - D_y Q) \cdot f(x,y) = 0 \tag{1.2}$$

for some operator L that only involves x and the derivation D_x but neither y nor the derivation D_y, and some other operator Q that may involve any of x, y, D_x, D_y, implies the equation

$$L \cdot \int_\Omega f(x,y)\,dy = \big[Q \cdot f(x,y)\big]_\Omega.$$

If the right-hand side happens to be zero, we find that L is an annihilating operator for the integral.

An operator L as in equations (1.1) and (1.2) is called a *telescoper* for f, and Q is called a *certificate* for L. The degree of S_n or D_x in L is called the order of L. If L is such that there is no telescoper of lower order, then L is called a *minimal telescoper*. The minimal telescoper is unique up to multiplication by rational functions (from the left).

Algorithms for computing telescopers meanwhile have a long history in computer algebra, see [26, 27, 23, 16] for classical results and recent developments on the matter. In his recent paper [25], van Hoeij proposed a fresh view on creative telescoping. He explains why a telescoper can often be written as a least common left multiple of smaller operators, and why the minimal telescoper is sometimes not the minimal-order annihilating operator for the sum or integral under consideration.

Let C be a field of characteristic zero and $C(n,k)$ be the field of rational functions in n, k over C. Let $A_{n,k} = C(n,k)\langle S_n, S_k\rangle$ be the ring of all linear recurrence operators in S_n, S_k with rational function coefficients, and $A_n =$

$C(n)\langle S_n\rangle$ be the subalgebra consisting of all operators that do not involve k or S_k. For a given summand $f(n,k)$, consider the A_n-module $\Omega := A_{n,k}\cdot f(n,k)$ and the quotient module $M := \Omega/((S_k-1)\Omega)$. An operator $L\in A_n$ is then a telescoper for $H=f(n,k)$ if and only if it is an annihilating operator of the image $\overline{H}$ of H in M.

In this setting, van Hoeij makes the following observations:

- If M can be written as a direct sum of submodules, say $M = M_1 \oplus M_2$, then the minimal telescoper of H is the least common left multiple of the minimal annihilating operators of the projections $\pi_1(H)$ and $\pi_2(H)$ of H in M_1 and M_2, respectively.
- If, moreover, the definite sum whose summand corresponds to $\pi_1(H)$ happens to be zero identically, then every annihilating operator of $\pi_2(H)$ is already an annihilating operator of the definite sum over H, even though it may not be a telescoper for H.

In order to take advantage of the second observation, it is necessary to understand under which circumstances a definite sum can be zero. Such "vanishing sums" are themselves examples when a minimal telescoper fails to be a minimal annihilator. For example, we have $\sum_k(-1)^k\binom{2n+1}{k}^2 = 0$, so the minimal annihilator is 1. However, the minimal telescoper of $(-1)^k\binom{2n+1}{k}^2$ is $L=(2n+3)S_n+(8n+8)$. Note that since L is irreducible, the module M, which is isomorphic to A_n/LA_n, has no nontrivial submodules.

We propose an explanation of why certain sums are identically zero which is based on the investigation of residues. Also based on residues, we will explain why telescopers tend to be least common left multiples. We are not the first to use residues in the context of creative telescoping. For rational functions and algebraic functions in the differential case, it was observed by Chen, Kauers, and Singer [11] that telescopers and residues are closely related. Chen and Singer also used residues in the context of summation problems [12]. Residues are also tied to creative telescoping through the equivalence of extracting residues with taking diagonals and positive parts and the computation of Hadamard products [7].

2 Residues and Telescopers for Rational Functions

Residues have played an important role in rational integration and summation [8, 20, 12, 5, 4]. In this section, we will first use residues in the continuous setting to explain why minimal telescopers may not always lead to minimal annihilators for integrals and then use residues in the discrete setting to explain some vanishing sums.

2.1 The integration case

Let $F = C(x)$, so that the bivariate rational function field $C(x, y)$ can be viewed as a univariate rational function field $F(y)$. An element f of $F(y)$ is said to be *integrable in* $F(y)$ if $f = D_y(g)$ for some $g \in F(y)$.

Any rational function $f = a/b \in F(y)$ with $a, b \in F[y]$ and $\gcd(a, b) = 1$ can be uniquely written as

$$f = p + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{\alpha_{i,j}}{(y - \beta_i)^j},$$

where $p \in F[y], n, m_i \in \mathbb{N}, \alpha_{i,j}, \beta_i \in \overline{F}$, and the β_i's are distinct roots of b. Note that all the $\alpha_{i,j}$'s are in the field $F(\beta_1, \ldots, \beta_n)$. The value $\alpha_{i,1} \in \overline{F}$ is called the *residue* (in y) of f at β_i, denoted by $\operatorname{res}_y(f, \beta_i)$. Let $P, Q \in F[y]$ be such that $\gcd(P, Q) = 1$ and Q is squarefree and let $\beta \in \overline{F}$ be a zero of Q. Then we have Lagrange's residue formula

$$\operatorname{res}_y\left(\frac{P}{Q}, \beta\right) = \frac{P(\beta)}{D_y(Q)(\beta)}.$$

It is well-known that a rational function is integrable in $F(y)$ if and only if all its residues in y are zero (see [12, Proposition 2.2]). So residues are the obstruction to the integrability in $F(y)$. From this fact and the commutativity between the derivation in x and taking the residue in y, we have that the minimal telescoper of a rational function in $C(x, y)$ is the least common left multiple of the minimal annihilating operators of its residues in y which are algebraic functions in $\overline{C(x)}$ (see [11, Theorem 6]).

Now consider the integral

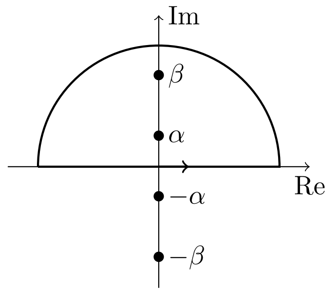
$$I(x) := \int_{-\infty}^{\infty} f(x, y)\, dy \quad \text{with } f := \frac{1}{y^4 + xy^2 + 1} \text{ and } x > 2.$$

We have $I(x) = \pi/\sqrt{x + 2}$, so the integral has the minimal annihilator $(2x + 4)D_x + 1$. The minimal telescoper for f however is $L = (4x^2 - 16)D_x^2 + 12xD_x + 3$. Let us see why the minimal telescoper overshoots in this example.

Let $\alpha, \beta \in \overline{\mathbb{Q}(x)}$ be such that $\alpha, -\alpha, \beta, -\beta$ are the poles of f and $\beta = \alpha(\alpha^2 + x)$. Then we have the residues

$$\operatorname{res}_y(f, \pm\alpha) = \pm\frac{\alpha(2 - x^2 - \alpha^2 x)}{2(x^2 - 4)} \text{ and } \operatorname{res}_y(f, \pm\beta) = \pm\frac{\alpha(2\alpha^2 + x)}{2(x^2 - 4)}.$$

Note that each of the four residues has the telescoper L as its minimal annihilator. This does not explain yet why the telescoper factors and overshoots. To explain this, we need to observe that the sum $\operatorname{res}_y(f, \alpha) + \operatorname{res}_y(f, \beta)$ is annihilated by $(2x + 4)D_x + 1$. By the residue theorem, the sum of these residues is equal (up to a multiplicative constant) to the following contour integral:



By increasing the contour indefinitely, we see that it is also the value of the real integral $I(x) = \pi/\sqrt{x+2}$. As creative telescoping does not know the contour but only the integrand, it must return a telescoper that works for every contour, in particular one that encircles only one of the poles. For such a contour, the minimal telescoper is indeed the minimal annihilator.

In van Hoeij's language of submodules, translated to the differential case, consider $\Omega = C(x, y)$, $M = \Omega/D_y\Omega$, and $A_x = C(x)\langle D_x\rangle$. The submodule generated by f in M is $N = \operatorname{span}_{C(x)}(f + D_y\Omega, y^2 f + D_y\Omega)$. Note that $\dim_{C(x)} N = \operatorname{ord}(L) = 2$. The module N admits a decomposition $N = N_+ \oplus N_-$ where $N_\pm = \operatorname{span}_{C(x)}((1 \pm y^2)f + D_y\Omega)$, which suggests writing

$$f = \frac{1+y^2}{2} f + \frac{1-y^2}{2} f.$$

Indeed, the minimal telescoper of $\frac{1+y^2}{2} f$ is $(2x+4)D_x + 1$, the minimal telescoper of $\frac{1-y^2}{2} f$ is $(2x-4)D_x + 1$, and L is the least common left multiple of these operators. Because of

$$\operatorname{res}_y(f, \alpha) = \operatorname{res}_y(y^2 f, \beta) \quad \text{and} \quad \operatorname{res}_y(f, \beta) = \operatorname{res}_y(y^2 f, \alpha),$$

the residues of $\frac{1-y^2}{2} f$ at α and β cancel each other, so

$$\int_{-\infty}^{\infty} \frac{1-y^2}{2} f \, dy = 0,$$

and that's why the factor $(2x-4)D_x + 1$ of L is not needed for $I(x)$.

2.2 The summation case

As a discrete analogue of residues for rational integration, discrete residues are introduced to study the summability problem and the existence problem of telescopers for rational functions in [12]. Efficient algorithms for computing discrete residues and their variants are given in [5, 4, 6].

Let S_x and S_y denote the usual shift operators of $C(x, y)$ with respect to x and y, respectively. Let Δ_y denote the difference operator defined by $\Delta_y(r) = S_y(r) - r$ for any $r \in F(y)$. A rational function $f \in F(y)$ is said to be *summable* *in* $F(y)$ if $f = \Delta_y(g)$ for some $g \in F(y)$. For any elements $\beta \in \overline{F}$, we call the set $\{\beta + i \,|\, i \in \mathbb{Z}\}$ a $\mathbb{Z}$-*orbit* of β in $\overline{F}$, denoted by $[\beta]_{\mathbb{Z}}$. Any rational function $f \in F(y)$ can be decomposed into the form

$$f = p + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \sum_{\ell=0}^{d_{i,j}} \frac{\alpha_{i,j,\ell}}{(y - (\beta_i + \ell))^j},$$

where $p \in F[y]$, $m, n_i, d_{i,j} \in \mathbb{N}$, $\alpha_{i,j,\ell}, \beta_i \in \overline{F}$, and the β_i's are in distinct $\mathbb{Z}$-orbits. The sum $\sum_{\ell=0}^{d_{i,j}} \alpha_{i,j,\ell}$ is called the *discrete residue in* y of f at the $\mathbb{Z}$-orbit $[\beta_i]_{\mathbb{Z}}$ of multiplicity j, denoted by $\mathrm{dres}_y(f, [\beta_i]_{\mathbb{Z}}, j)$. By Proposition 2.5 in [12], discrete residues are the precise obstruction for rational functions to be summable, i.e., $f \in F(y)$ is summable in $F(y)$ if and only if all of the discrete residues of f are zero.

We recall a very old result due to Nicole [24] that describes a family of summable rational functions and then use this result to explain some vanishing sums.

Lemma 2.1 (Nicole, 1717)**.** *Let* $n \geq 2$ *be an integer and* $P \in F[y]$ *be such that* $\deg_y(P) \leq n - 2$. *Then the rational function*

$$f = \frac{P(y)}{(y + \beta_1) \cdots (y + \beta_n)}$$

is summable in $F(y)$ *for all* $\beta_i \in \overline{F}$ *with* $\beta_i - \beta_j \in \mathbb{Z} \setminus \{0\}$ *for* $i \neq j$.

Proof. By partial fraction decomposition, we get

$$f = \sum_{i=1}^{n} \frac{\alpha_i}{y + \beta_i}, \quad \text{where } \alpha_i \in \overline{F}. \tag{2.1}$$

Note that the β_i's are in the same $\mathbb{Z}$-orbit. By Proposition 2.5 in [12], f is summable in $F(y)$ if and only if the sum $\sum_{i=1}^{n} \alpha_i$ is zero. By normalizing f in (2.1), we get

$$P = (\alpha_1 + \cdots + \alpha_n) y^{n-1} + \text{terms with degree lower than } n - 1.$$

Since $\deg_y(P) \leq n - 2$, it holds that $\sum_{i=1}^{n} \alpha_i = 0$. ∎

When F is the field of complex numbers, the identity $\sum_{i=1}^{n} \alpha_i = 0$ also follows from Cauchy's residue theorem since the residue of f at infinity is zero.

As a corollary of Nicole's lemma, we obtain a class of vanishing sums. For any polynomial $P \in F[y]$ with $\deg_y(P) \leq n-1$, we consider the rational function

$$f = \frac{P(y)}{y(y+1)\cdots(y+n)} = \sum_{k=0}^{n} \frac{\alpha_k}{y + k},$$

which is summable in $F(y)$ by Nicole's lemma. Since the denominator of f is squarefree, Lagrange's residue formula implies that

$$\alpha_k = \frac{(-1)^k P(-k)}{k!(n-k)!}.$$

Then we have the vanishing sum

$$\sum_{k=0}^{n} \frac{(-1)^k P(-k)}{k!(n-k)!} = 0.$$

Example 2.2. *To show the combinatorial identity*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} k^j = 0, \quad \text{where } n \geq 2 \text{ and } 0 \leq j < n,$$

we consider the rational function

$$f = \frac{P}{Q} = \frac{n!(-y)^j}{y(y+1)\cdots(y+n)} = \sum_{k=0}^{n} \frac{\alpha_k}{y+k}.$$

By Lagrange's residue formula, we have

$$\alpha_k = (-1)^k \binom{n}{k} k^j.$$

Since $0 \leq j < n$, *we have* $\deg_y(P) \leq \deg_y(Q) - 2$. *Then the identity* $\sum_{k=0}^{n} \alpha_k = 0$ *holds.*

Example 2.3. *To show the combinatorial identity*

$$\sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{n-k} \frac{1}{2k-1} = 0, \quad \text{where } n \geq 1,$$

we consider the rational function

$$f = \frac{P}{Q} = -\frac{2^n \prod_{i=1}^{n-1} (2(y+i)+1)}{y(y+1)\cdots(y+n)} = \sum_{k=0}^{n} \frac{\alpha_k}{y+k}.$$

By Lagrange's residue formula, we get

$$\alpha_k = \binom{2k}{k} \binom{2n-2k}{n-k} \frac{1}{2k-1}.$$

Since $\deg_y(P) = n-1$ *and* $\deg_y(Q) = n+1$, *Nicole's lemma implies the identity* $\sum_{k=0}^{n} \alpha_k = 0$.

We will see more applications of Nicole's lemma in Section 3.3.

3 Residual Forms and Prescopers for Hypergeometric Terms

We now focus on creative telescoping for hypergeometric terms. We will use residual forms introduced in [10] to construct submodules in order to find right factors of minimal telescopers and then investigate the automorphisms and the non-minimality phenomenon of minimal telescopers for hypergeometric sums. These studies continue the development of the submodule approach initialized by van Hoeij [25].

To be more compatible with the customary usage, we will now use n and k instead of x and y, respectively. A sequence $H(n,k)$ is called a *hypergeometric term* over $C(n,k)$ with respect to n and k if the two shift quotients $S_n(H)/H$ and $S_k(H)/H$ are rational functions in $C(n,k)$. A hypergeometric term H is said to be *hypergeometric summable* in k if $H = \Delta_k(G)$ for some hypergeometric term G. A nonzero linear operator $L \in C(n)\langle S_n \rangle$ is called a *telescoper* for H if there exists another hypergeometric term $G(n,k)$ such that

$$L(H(n,k)) = \Delta_k(G(n,k)). \tag{3.1}$$

Recall that $p \in C(n)[k]$ is *shift-free* in k if $\gcd(p, S_k^i(p)) = 1$ for all $i \in \mathbb{Z} \setminus \{0\}$. A rational function $f = a/b \in C(n,k)$ is *shift-reduced* in k if $\gcd(a, S_k^i(b)) = 1$ for all $i \in \mathbb{Z}$. A nonzero polynomial $p \in C(n)[k]$ is *strongly prime* with a rational function $f = a/b$ if $\gcd(p, S_k^{-i}(a)) = \gcd(p, S_k^i(b)) = 1$ for all $i \in \mathbb{N}$. By computing rational normal forms as in [2], one can write $f \in C(n,k)$ as

$$f = \frac{S_k(S)}{S} \cdot K, \tag{3.2}$$

where $S, K \in C(n,k)$ such that K is shift-reduced in k. The rational functions K and S are called *kernel* and *shell* of f, respectively. Let $f = S_k(H)/H$. Then $H = S \cdot H_0$ with $S_k(H_0)/H_0 = K$. Write $K = u/v$ with $u, v \in C(n)[k]$ and $\gcd(u,v) = 1$. Let $\phi_K \colon C(n)[k] \to C(n)[k]$ be a $C(n)$-linear map defined by

$$\phi_K(p) = uS_k(p) - vp \quad \text{for all } p \in C(n)[k].$$

Let W_K be the standard complement of the image $\operatorname{im}(\phi_K)$ in $C(n)[k]$ such that $C(n)[k] = \operatorname{im}(\phi_K) \oplus W_K$. By the modified Abramov–Petkovšek reduction [10] we can decompose H into

$$H = \Delta_k(r \cdot H_0) + \left(\frac{a}{b} + \frac{p}{v}\right) H_0 \tag{3.3}$$

where $r \in C(n,k), p \in W_K$, and $a, b \in C(n)[k]$ such that $\deg_k(a) < \deg_k(b)$, $\gcd(a,b) = 1$, and b is shift-free in k and strongly prime with K. By Proposition 4.7 and Theorem 4.8 in [10], we have W_K is finite-dimensional over $C(n)$ and H is hypergeometric summable in k if and only if $a = 0$ and $p = 0$. So the form $(a/b + p/v)H_0$ is the obstruction to the hypergeometric summability. For this reason, we call $(a/b + p/v)H_0$ a *residual form* of H with respect to Δ_k.

Let Ω be the A_n-module $C(n,k)\cdot H$. Note that $\Delta_k(\Omega)$ is an A_n-submodule of Ω. Let M denote the quotient module $\Omega/\Delta_k(\Omega)$. An operator $L \in A_n$ is a telescoper for H if and only if L is an annihilator of the image $\overline{H}$ of H in M.

Lemma 3.1. *Let H_0 and v be defined as in* (3.3) *and let*

$$N := \left\{ \frac{p}{v}H_0 + \Delta_k(\Omega) \;\middle|\; p \in W_K \right\}.$$

Then N is an A_n-submodule of M.

Proof. By [15, Proposition 5.2] with $b_0 = 1$, for any $i \in \mathbb{N}$,

$$S_n^i\left(\frac{p}{v}H_0\right) \equiv \frac{p_i}{v}H_0 \bmod \Delta_k(\Omega)$$

for some $p_i \in W_K$. The lemma follows. ∎

Note that N is independent of the choice of S and K in the rational normal form (3.2). We will call N a *kernel submodule* of M which is an A_n-submodule and a finite-dimensional vector space over $C(n)$. Recall that an operator L is a telescoper for H if it annihilates $\overline{H}$ in M. Therefore, if N is any submodule of M, then for an operator L to be a telescoper, it is *necessary* that L maps $\overline{H}$ into N, although this condition is in general *not sufficient* for being a telescoper. This observation motivates the following definition of prescopers for hypergeometric terms. An analogous definition was introduced in [14, Section 6.2] for hyperexponential functions.

Definition 3.2. *A nonzero operator $R \in C(n)\langle S_n\rangle$ is called a* prescoper *for H with respect to k if $R(H) + \Delta_k(\Omega) \in N$, i.e., there exists $p \in W_K$ such that*

$$R(H) \equiv \frac{p}{v}H_0 \bmod \Delta_k(\Omega).$$

A prescoper is said to be minimal if it has minimal degree in S_n.

By definition, it is clear that telescopers are prescopers. The next lemma shows that the minimal prescoper for H is a right factor of the minimal telescoper for H if they exist.

Lemma 3.3. *Let $N \subseteq M$ be A_n-modules and $m \in M$. Suppose that $R \in A_n$ is the minimal annihilator for $m + N \in M/N$ and T is the minimal annihilator for $R(m)$, then $T \cdot R$ is the minimal annihilator for $m \in M$.*

Proof. We firstly observe that $T\cdot R$ is an annihilator for $m \in M$. Let L be any annihilator for m. Then L must be an annihilator for $m + N \in M/N$, which implies that L is right divisible by R. Let $L = \widetilde{L}\cdot R$, then $\widetilde{L}$ is an annihilator for $R(m)$. By the minimality of T, we have that $\widetilde{L}$ is right divisible by T and then L is right divisible by $T \cdot R$. Hence $T\cdot R$ is the minimal annihilator for m. ∎

The following lemma will be used in the next sections to explore the LCLM structure of annihilators of elements in A_n-modules.

Lemma 3.4. *Let M be an A_n-module and $M = \bigoplus_{i=1}^{n} M_i$ be a direct-sum decomposition of M. For any element $m = m_1 + \cdots + m_n \in M$, the minimal annihilator for m is the least common left multiple of the minimal annihilators for the m_i's.*

Proof. Let L_i be the minimal annihilator for $m_i \in M_i$. Suppose L is an annihilator for m, then

$$L(m) = L(m_1) + \cdots + L(m_n) = 0.$$

Since $L(m_i) \in M_i$, we have $L(m_i) = 0$, which implies that L is right-divisible by L_i. Thus L is right-divisible by $\operatorname{lclm}(L_1, \ldots, L_n)$. Note that $\operatorname{lclm}(L_1, \ldots, L_n)$ is an annihilator for m. The lemma follows. ∎

3.1 Constructing minimal prescopers

We now present a method to construct minimal prescopers for hypergeometric terms. We first recall some terminologies from [3, Section 4] and [15, Section 3] about properties of polynomials under shifts. Let F be a field of characteristic zero. Two polynomials $q_1, q_2 \in F[z]$ are *σ-equivalent* with respect to the F-automorphism σ of $F[z]$ if $q_1 = \sigma^j(q_2)$ for some $j \in \mathbb{Z} \setminus \{0\}$, denoted as $q_1 \sim_\sigma q_2$. Two shift-free polynomials $b_1, b_2 \in C(n)[k]$ are *shift-related* (with respect to k) if for any nontrivial monic irreducible factor q_1 of b_1, there exists a unique monic irreducible factor q_2 of b_2 with the same multiplicity as q_1 in b_1 such that q_1 and q_2 are S_k-equivalent and vice versa. An irreducible polynomial $p \in C[n,k]$ is *integer-linear* over C if there exist a univariate polynomial $P \in C[z]$ and a nonzero vector $(m, \ell) \in \mathbb{Z}^2$ such that $p(n,k) = P(mn + \ell k)$. A polynomial $p \in C[n,k]$ is *integer-linear* if all of its irreducible factors are integer-linear.

By the existence criterion on telescopers [1], a hypergeometric term H as in (3.3) has a nonzero telescoper in A_n if and only if b is an integer-linear polynomial. From now on, we always assume that the given hypergeometric term H has a nonzero telescoper. Since b is integer-linear, shift-free in k, and strongly prime with K, we can decompose b as

$$b = \prod_{i=1}^{I} \prod_{j=0}^{\ell_i - 1} S_k^{\mu_{i,j}} \left(P_i(m_i n + \ell_i k + j) \right)^{\lambda_{i,j}},$$

where each $P_i \in C[z]$ is irreducible, $\lambda_{i,j} \in \mathbb{N}$ and $m_i, \ell_i, \mu_{i,j} \in \mathbb{Z}$ satisfying $\ell_i > 0$, $\gcd(m_i, \ell_i) = 1$, and $S_k^{\mu_{i,j}} \left(P_i(m_i n + \ell_i k + j) \right)$ is strongly prime with K. Moreover, one can ensure that for all $i, i' \in \{1, \ldots, I\}$ with $i \neq i'$, at least one of the following three relations is not satisfied:

$$m_i = m_{i'}, \ \ell_i = \ell_{i'}, \text{ and } P_i \sim_{S_z} P_{i'}. \tag{3.4}$$

Let $\lambda_i := \max\{\lambda_{i,0}, \ldots, \lambda_{i,\ell_i - 1}\}$ and set

$$B_{i,j} := S_k^{\mu_{i,j}} \left(P_i(m_i n + \ell_i k + j) \right)^{\lambda_i}.$$

Then we can write a/b in the residual form of H as

$$\frac{a}{b} = \sum_{i=1}^{I} \sum_{j=0}^{\ell_i - 1} \frac{q_{i,j}}{B_{i,j}}, \tag{3.5}$$

where $q_{i,j} \in C(n)[k]$ such that $\deg_k(q_{i,j}) < \deg_k(B_{i,j})$. Let $\hat{H} = a/b \cdot H_0$. By Definition 3.2, the minimal prescoper for H is equal to the minimal prescoper for $\hat{H}$. From the above decomposition we obtain

$$\hat{H} = \sum_{i=1}^{I} \hat{H}_i \quad \text{with} \quad \hat{H}_i := \sum_{j=0}^{\ell_i - 1} \frac{q_{i,j}}{B_{i,j}} \cdot H_0.$$

Lemma 3.5. *The minimal prescoper for $\hat{H}$ is the least common left multiple of the minimal prescopers for the $\hat{H}_i$'s.*

Proof. Let $V_i \subseteq M/N$ be the set that consists of the elements

$$\sum_{j=0}^{\ell_i - 1} \frac{a_{i,j}}{B_{i,j}} H_0 + N$$

with $a_{i,j} \in C(n)[k]$ and $\deg_k(a_{i,j}) < \deg_k(B_{i,j})$. By [15, Proposition 5.4], for any $d \in \mathbb{N}$, there exist $\widetilde{a}_{i,j} \in C(n)[k]$ with $\deg_k(\widetilde{a}_{i,j}) < \deg_k(B_{i,j})$ and $p_d \in W_K$ such that

$$S_n^d \left(\sum_{j=0}^{\ell - 1} \frac{a_{i,j}}{B_{i,j}} H_0 \right) \equiv \left(\sum_{j=0}^{\ell - 1} \frac{\widetilde{a}_{i,j}}{B_{i,j}} + \frac{p_d}{v} \right) H_0 \bmod \Delta_k(\Omega).$$

This implies that V_i is an A_n-submodule of M/N. Let $V = \sum_{i=1}^{I} V_i$. Then $\hat{H} + N$ is an element of V. By Lemma 3.4, it remains to show that $V = \bigoplus_{i=1}^{I} V_i$. By [15, Proposition 3.2] the following holds: if there exist $p_1, p_2 \in W_K$ such that

$$\left(\frac{a_1}{b_1} + \frac{p_1}{v} \right) H_0 \equiv \left(\frac{a_2}{b_2} + \frac{p_2}{v} \right) H_0 \bmod \Delta_k(\Omega),$$

where b_1, b_2 satisfy the conditions as in Equation (3.3), then b_1 and b_2 are shift-related to each other. Therefore, we have $V_i \cap V_j = \{0\}$ for any $i \neq j$. ∎

We next deal with the question how to compute the minimal prescoper for each $\hat{H}_i$. For each $d \in \mathbb{N}$, the modified Abramov–Petkovšek reduction [10] decomposes

$$S_n^d(\hat{H}_i) \equiv \left(r_{i,d} + \frac{p_{i,d}}{v} \right) H_0 \bmod \Delta_k(\Omega),$$

where $p_{i,d} \in W_K$ and $r_{i,d} \in C(n,k)$, which are also contained in a finite-dimensional $C(n)$-vector space. Take the minimal $\rho_i \in \mathbb{N}$ such that we have $\sum_{d=0}^{\rho_i} e_{i,d} r_{i,d} = 0$ with $e_{i,d} \in C(n)$ and $e_{i,\rho_i} = 1$. Then we have

$$R_i := \sum_{d=0}^{\rho_i} e_{i,d} S_n^d$$

is the minimal prescoper for $\hat{H}_i$.

For a rational function $f \in C(n,k)$ of the form

$$f = \frac{1}{(mn+\ell k)^s},$$

where s is a positive integer and $m, \ell \in \mathbb{Z}$ with $\ell \neq 0$ and $\gcd(m,\ell)=1$, one can observe that $S_n^\ell - 1$ is the minimal telescoper for f. Based on this observation, Le [19] gave a direct method for computing minimal telescopers for rational functions which avoids the process of item-by-item examination of the order of the ansatz operators in Zeilberger's algorithm. Motivated by van Hoeij's example in [25, Section 3], we partially extend Le's direct method to special hypergeometric terms of the form

$$H = \frac{q(n,k)}{(mn+\ell k+\alpha)^\lambda} \cdot H_0, \tag{3.6}$$

where $\alpha \in C$, $\deg_k(q) < \lambda$, $\gcd(q, mn+\ell k+\alpha) = 1$, and $(mn+\ell k+\alpha)$ is strongly prime with K. For a nonzero operator $R \in C(n)\langle S_n\rangle$ and a positive integer $\ell \in \mathbb{N}$, we can write R as

$$R = R_0 + \cdots + R_{\ell-1}, \tag{3.7}$$

with $R_i \in S_n^i \cdot C(n)\langle S_n^\ell\rangle$. This decomposition is called the *ℓ-exponent separation* of R, see [9, Section 4].

Lemma 3.6. *Let H be as in* (3.6) *and let R have the ℓ-exponent separation as in* (3.7)*. If R is the minimal prescoper for H, then $R = R_0$.*

Proof. Note that any two polynomials in $\{S_n^i(mn+\ell k+\alpha)^\lambda\}_{i=0}^{\ell-1}$ are not S_k-equivalent, but for all $j \in \mathbb{N}$ we have that $S_n^{j\ell}(mn+\ell k+\alpha)^\lambda$ and $(mn+\ell k+\alpha)^\lambda$ are S_k-equivalent. Then $\{R_i(H)+\Delta_k(\Omega)\}_{i=0}^{\ell-1}$ is linearly independent over $C(n)$ modulo N. If R is the minimal prescoper for H, then

$$R(H) + \Delta_k(\Omega) = \sum_{i=0}^{\ell-1}\big(R_i(H)+\Delta_k(\Omega)\big) \in N,$$

which implies that $R_i(H)+\Delta_k(\Omega) \in N$, i.e., each R_i is a prescoper for H. Since N is also closed under S_n^{-1}, the trailing coefficient of R is nonzero, which leads to $R_0 \neq 0$. For $i \neq j$, we have $\operatorname{ord}(R_i) \neq \operatorname{ord}(R_j)$, unless both are zero. We deduce that actually $R_i = 0$ for each $i = 1, \ldots, \ell-1$, because otherwise we could find some prescoper R_j with order less than $\operatorname{ord}(R)$. ∎

Using Lemma 3.6, we now present a recursive algorithm according to the value λ for computing the minimal prescoper for H as in (3.6). Since $S_n^\ell S_k^{-m}$ fixes the linear form $(mn+\ell k+\alpha)$, we have

$$h = \frac{S_n^\ell S_k^{-m}(H)}{H} = \frac{S_n^\ell S_k^{-m}(qH_0)}{qH_0} \in C(n,k). \tag{3.8}$$

Since $\gcd(q, mn+\ell k+\alpha) = 1$ and $(mn+\ell k+\alpha)$ is strongly prime with K, the evaluation of h at $k = -mn/\ell - \alpha/\ell$, assigned to $r \in C(n)$, is well-defined.

For $\lambda = 1$, we have $q \in C(n)$. It can be decomposed into

$$S_n^\ell(H) \equiv S_n^\ell S_k^{-m}(H) \equiv \left(\frac{r \cdot q}{mn+\ell k+\alpha} + \frac{p'}{v}\right) H_0 \bmod \Delta_k(\Omega),$$

for some $p' \in W_K$. Then $(S_n^\ell - r)\cdot H + \Delta_k(\Omega) \in N$. By Lemma 3.6, we have that $R := S_n^\ell - r$ is the minimal prescoper for H.

For $\lambda > 1$, we let $\widetilde{A}_n := C(n)\langle S_n^\ell\rangle$ which is a subring of A_n and let M_i be the set consisting of the elements

$$\left(\frac{a}{(mn+\ell k+\alpha)^i} + \frac{p}{v}\right) H_0 + \Delta_k(\Omega)$$

where $a \in C(n)[k]$ with $\deg_k(a) < i$ and $p \in W_K$. We claim that M_i is a $\widetilde{A}_n$-submodule of M. Indeed, for any $H_i + \Delta_k(\Omega) \in M_i$ and $j \in \mathbb{N}$,

$$S_n^{j\ell}(H_i) \equiv S_n^{j\ell} S_k^{-jm}(H_i) \equiv \left(\frac{a'}{(mn+\ell k+\alpha)^i} + \frac{p'}{v}\right) H_0 \bmod \Delta_k(\Omega),$$

for some $a' \in C(n)[k]$ with $\deg_k(a') < i$ and $p' \in W_K$. By definition, we have $N \subseteq M_i$ and M_{i-1} is an $\widetilde{A}_n$-submodule of M_i. By the modified Abramov–Petkovšek reduction, we can decompose H into

$$S_n^\ell(H) \equiv S_n^\ell S_k^{-m}(H) \equiv \left(\frac{r \cdot q}{(mn+\ell k+\alpha)^\lambda}\right) H_0 + \widetilde{H} \bmod \Delta_k(\Omega),$$

where $\widetilde{H} + \Delta_k(\Omega) \in M_{\lambda-1}$. Then $(S_n^\ell - r)\cdot H + \Delta_k(\Omega) \in M_{\lambda-1}$. Since $R := S_n^\ell - r$ is of order 1 in $\widetilde{A}_n$ and $H + \Delta_k(\Omega) \notin M_{\lambda-1}$, it is the minimal annihilator for $H + \Delta_k(\Omega) + M_{\lambda-1} \in M_\lambda/M_{\lambda-1}$. We can recursively compute the minimal prescoper $\widetilde{L}$ for $\widetilde{H}$. By Lemma 3.3, we have $\widetilde{L}\cdot R$ is the minimal prescoper for H.

The following example, sent to us by Hui Huang, indicates that the above method outperforms the existing codes for Zeilberger's algorithm in Maple and the reduction-based method in [10].

Example 3.7. *Consider the hypergeometric term*

$$H = \frac{1}{2n+k} H_0 \quad \text{with } H_0 = \frac{\binom{5n}{3k}^2}{\binom{n}{k}}.$$

Then the shift-quotient with respect to k is

$$K = \frac{S_k(H_0)}{H_0} = \frac{(3k-5n)^2(3k-5n+1)^2(3k-5n+2)^2}{9(n-k)(k+1)(3k+1)^2(3k+2)^2},$$

which is already shift-reduced in k. Let v be the denominator of K and

$$N = \left\{ \frac{p}{v} H_0 + \Delta_k(\Omega) \;\middle|\; p \in W_K \subset \mathbb{Q}(n)[k] \right\}.$$

Observe that $H \notin N$. Evaluating $S_n S_k^{-2}(H)/H$ at $k = -2n$ yields

$$r = \frac{3(3n+1)(3n+2)\prod_{i=1}^{5}(5n+i)^2 \prod_{i=0}^{5}(6n+i)^2}{2n(2n+1)\prod_{i=1}^{11}(11n+i)^2}.$$

Then $R = S_n - r$ is the minimal prescoper for H. It remains to compute the minimal telescoper for

$$\widetilde{H} := (S_n - r) \cdot H,$$

which is of order 6. It takes 13 seconds on a Dell Optiplex 7090 (CPU 3.70GHz, RAM 128G) with the reduction-based method in [10], compared with 31 seconds with the Maple codes for Zeilberger's algorithm.

3.2 Automorphisms of the kernel submodule

In his paper [25], van Hoeij presents examples in which a symmetry of a summation problem translates into an automorphism of the submodule N. The eigenspaces of the automorphism give rise to a decomposition of N into submodules, and this decomposition explains why the minimal telescoper is not a minimal annihilating operator of the sum.

Automorphisms of N can be found algorithmically. By Lemma 3.1, the A_n-module N has a finite dimension as $C(n)$-vector space. Let $\{v_1, \ldots, v_d\}$ be a vector space basis. Any A_n-automorphism $\phi\colon N \to N$ is in particular a $C(n)$-linear map. As such, it can be written in the form

$$\begin{pmatrix} \phi(v_1) \\ \vdots \\ \phi(v_d) \end{pmatrix} = \Phi \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix}, \tag{3.9}$$

for a certain matrix $\Phi \in C(n)^{d\times d}$. The requirement for a linear map to be an A_n-module automorphism is that it is invertible and compatible with the shift. If $\Sigma \in C(n)^{d\times d}$ is defined by

$$\begin{pmatrix} S_n(v_1) \\ \vdots \\ S_n(v_d) \end{pmatrix} = \Sigma \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix}, \tag{3.10}$$

then the latter requirement means that the commutation rule $\Sigma\Phi = S_n(\Phi)\Sigma$ must hold.

In order to find automorphisms, we can therefore make an ansatz with undetermined coefficients for the entries of Φ. The requirement $\Sigma\Phi = S_n(\Phi)\Sigma$ leads to a coupled system of linear recurrence equations for the undetermined coefficients. This system can be solved by the command `SolveCoupledSystem` of Koutschan's Mathematica package `HolonomicFunctions` [17]. The result is a C-linear subspace of $C(n)^{d\times d}$. Automorphisms correspond to all the matrices in this space whose determinant is nonzero.

Example 3.8. *For the hypergeometric term $H_0 := \binom{n}{2k}^2$, the kernel module N computed by the modified Abramov–Petkovšek reduction is a $C(n)$-vector space of dimension 3, given by the following basis:*

$$\left\{ \frac{k^i}{4(2k+1)^2(k+1)^2} H_0 + \Delta_k(\Omega) \;\middle|\; i = 0, 1, 2 \right\}.$$

The matrix $\Sigma \in C(n)^{3\times 3}$ is determined by N, but it is too large to display it here. We make an ansatz for $\Phi := (\phi_{i,j})_{1\le i,j\le 3}$ with undetermined entries $\phi_{i,j}$. Then the condition $\Sigma\Phi = S_n(\Phi)\Sigma$ yields a 9×9 coupled first-order linear system of difference equations, whose rational solutions are computed with the command `SolveCoupledSystem`*. It returns a two-dimensional solution space over the constant field C, which is spanned by the identity matrix I and by the matrix*

$$\Psi = \frac{\begin{pmatrix} 12n^3+16n+64 & -64n^2+32n+192 & 128n+192 \\ 4n^4-2n^3+4n^2-8n-48 & -20n^3+16n^2-16n-128 & 32n^2-16n-96 \\ n^5-2n^4+4n^3+32 & -4n^4+16n^3-16n^2-32n+64 & 4n^3-40n^2-48n+32 \end{pmatrix}}{4(n+2)^3}.$$

The matrix Ψ corresponds to the automorphism $(n, k) \to (n, k + 1/2)$, and it satisfies $\Psi^2 = I$, as expected. By inspecting the symmetry of H_0, one could anticipate the existence of another automorphism, namely $(n, k) \to (n, n/2-k)$. However, it turns out that this map is not compatible with S_n and hence is not an A_n-module automorphism.

3.3 Zero-sum submodules

The submodule approach introduced by van Hoeij [25] can not only speed-up the computation of minimal telescopers, but also explain (by examples) why the minimal telescoper for a hypergeometric sum may not be its minimal recurrence. The explanation of the non-minimality phenomenon by anti-symmetry has been given in [13, 22, 21] that leads to the method of creative symmetrizing [18]. A concrete example is the identity

$$\sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k}^2 = 0.$$

The summand $H := (-1)^k \binom{2n+1}{k}^2$ satisfies the anti-symmetry relation

$$H(n, k) = -H(n, 2n + 1 - k).$$

So summing H for k from 0 to $2n+1$ leads to zero. The minimal telescoper for H is the first-order operator $S_n + 8(n+1)/(2n+3)$, but the minimal recurrence for the above vanishing sum is any nonzero element of $C(n)$.

As a research question, van Hoeij [25, Section 7] proposed to study the zero-sum submodules, especially how to detect and find such submodules. We call $Z \subseteq N$ a zero-sum submodule if it only contains terms whose summation with respect to k gives 0. Note that every operator T with $T(\overline{H}) \in Z$ is then an annihilating operator of $\sum_k H$, but not necessarily a telescoper.

The following two examples show how the techniques from the previous sections, especially Nicole's lemma, can be used to construct zero-sum submodules and explain the non-minimality phenomenon. In the first example, we find that the minimal prescoper R maps $\overline{H}$ not only into N but even into Z. It is therefore an annihilator of the sum. However, since $R(\overline{H}) \neq 0 \in M$, it is not a telescoper. In the second example, the minimal prescoper is $R = 1$. Nevertheless, the minimal telescoper is not the minimal annihilator of the sum because it turns out that there is an operator T with $T(\overline{H}) \in Z$ but $T(\overline{H}) \neq 0$.

Example 3.9. *The minimal telescoper for the hypergeometric term*

$$H := (-1)^k \binom{3n+1}{k} \binom{3n-k}{n}^3$$

is of order 2*, which is not the minimal recurrence satisfied by the sum*

$$\sum_{k=-\infty}^{+\infty} H(n,k) = 1.$$

To explain this non-minimality, we let $H_0 = (k-3n-1)H$ *and let*

$$K := \frac{S_k(H_0)}{H_0} = \frac{(k-2n)^3}{(k+1)(k-3n)^2} =: \frac{u}{v}.$$

Then the algorithm in Section 3.1 can compute the minimal prescoper $R = S_n - 1$ *for* H *so that* $R(H) + \Delta_k(\Omega)$ *is in the submodule*

$$N := \left\{ \frac{p}{v} \cdot H_0 + \Delta_k(\Omega) \;\middle|\; p \in W_K \right\},$$

where W_k *has a* $\mathbb{Q}(n)$*-basis* $\{1, k^3\}$*. We now use Nicole's lemma in Section 2.2 to show that for all* $p \in \mathbb{Q}(n)[k]$ *with* $\deg_k(p) \leq 2$*, we have the vanishing-sum identity*

$$\sum_{k=-\infty}^{+\infty} \frac{p}{v} \cdot H_0 = 0, \quad \text{where } n \geq 1.$$

Similar to Examples 2.2 and 2.3, we consider the rational function

$$\begin{aligned} f = \frac{P}{Q} &= \frac{p(n,-x)(3n+1)!(x+3n-1)^2 \cdots (x+2n+1)^2}{(n!)^3(x-1)x(x+1)\cdots(x+2n)} \\ &= \sum_{k=-1}^{2n} \frac{\alpha_k}{x+k}. \end{aligned}$$

Since Q *is squarefree, Lagrange's residue formula implies that*

$$\alpha_k = \frac{p(n,k)(3n+1)!(3n-k-1)^2 \cdots (2n-k+1)^2}{(n!)^3(-k-1)(-k)(-k+1)\cdots(-k+2n)} = \frac{p}{v} \cdot H_0.$$

By Lemma 2.1, f is summable in $\mathbb{C}(x)$ since $\deg_k(p) \leq 2$. Then the above vanishing-sum identity holds. By this identity, we have a zero-sum submodule

$$Z := \left\{ \frac{p}{v} \cdot H_0 + \Delta_k(\Omega) \;\middle|\; p \in W_K \textit{ with } \deg_k(p) = 0 \right\}.$$

Applying the prescoper $R = S_n - 1$ to H yields

$$S_n(H) - H \equiv \frac{37n^7 + 96n^6 + 81n^5 + 22n^4}{8(n+1)^3(9n^2+10n+3)} \frac{H_0}{v} \bmod \Delta_k(\Omega).$$

So $S_n(H) - H + \Delta_k(\Omega) \in Z$ which contributes zero to the sum. Then $S_n - 1$ is the minimal annihilator for the sum $\sum_{k=-\infty}^{+\infty} H(n,k)$.

Example 3.10. *We now explain why minimal telescopers overshoot in the following combinatorial identity*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{3k}{n} = (-3)^n.$$

This is a special case of the identity in [22, Section 4.3] which was originally used to show the non-minimality phenomenon. The minimal telescoper for the summand $H := (-1)^k \binom{n}{k}\binom{3k}{n}$ is

$$S_n^2 + \frac{3(5n+7)}{2(2n+3)} S_n + \frac{9(n+1)}{2(2n+3)},$$

but this is not the minimal recurrence $S_n + 3$ satisfied by the sum. In this example, we let $H_0 = H$ and

$$K := \frac{S_k(H_0)}{H_0} = \frac{3(k-n)(3k+1)(3k+2)}{(3k-n+1)(3k-n+2)(3k-n+3)} =: \frac{u}{v}.$$

The corresponding kernel submodule is

$$N := \left\{ \frac{p}{v} \cdot H_0 + \Delta_k(\Omega) \;\middle|\; p \in W_K \right\},$$

where W_k has a $\mathbb{Q}(n)$-basis $\{1, k^3\}$. Since $H + \Delta_k(\Omega) \in N$, the minimal prescoper of H is $R = 1$. Similar to the previous example, considering the rational function

$$f = \frac{p(n,-x)(-3x)(-3x-1)\cdots(-3x-n+4)}{x(x+1)\cdots(x+n)}$$

yields the vanishing-sum identity

$$\sum_{k=0}^{n} \frac{p}{v} \cdot H_0 = 0, \quad \textit{where } n \geq 3,$$

for all $p \in \mathbb{Q}(n)[k]$ *with* $\deg_k(p) \leq 2$*. So we obtain the zero-sum submodule*

$$Z := \left\{ \frac{p}{v} \cdot H_0 + \Delta_k(\Omega) \;\middle|\; p \in W_K \text{ with } \deg_k(p) = 0 \right\}.$$

We can verify that Z *is closed under any operator in* A_n*. In fact,*

$$S_n\left(\frac{H_0}{v}\right) \equiv \frac{-9n^3 - 21n^2 + 36n + 84}{2(n+2)(2n+5)(3n+4)} \frac{H_0}{v} \bmod \Delta_k(\Omega).$$

The remaining task is to find an operator $T \in \mathbb{Q}(n)\langle S_n \rangle$ *such that* $T(H) + \Delta_k(\Omega) \in Z$*. The modified Abramov–Petkovšek reduction decomposes* H_0 *and* $S_n(H_0)$ *as*

$$H_0 \equiv \frac{81k^3n - n^4 + 108k^3 + 4n^3 - 12n^2 + 12n + 18}{(3n+4) \cdot v} H_0 \bmod \Delta_k(\Omega).$$

$$S_n(H_0) \equiv \frac{-243k^3n + n^4 - 324k^3 - 9n^3 + 41n^2 - 42n - 54}{(3n+4) \cdot v} H_0 \bmod \Delta_k(\Omega).$$

Note that $T = S_n + 3$ *brings* H_0 *into the zero-sum submodule* Z*. Therefore,* T *annihilates the sum, and since the sum evaluates to* $(-3)^n$*, we find that* T *is actually its minimal annihilator.*

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