How to generate all possible rational Wilf–Zeilberger forms?

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Abstract

Wilf–Zeilberger pairs are fundamental in the algorithmic theory of Wilf and Zeilberger for computer-generated proofs of combinatorial identities. Wilf–Zeilberger forms are their high-dimensional generalizations, which can be used for proving and discovering convergence acceleration formulas. This paper presents a structural description of all possible rational such forms, which can be viewed as an additive analog of the classical Ore–Sato theorem. Based on this analog, we show a structural decomposition of so-called multivariate hyperarithmetic expressions, which extend multivariate hypergeometric terms to the additive setting.

Keywords: Wilf–Zeilberger form, additive Ore–Sato theorem, hyperarithmetic expression, orbital decomposition

1. Introduction

In the 1990s, Wilf and Zeilberger developed an algorithmic theory for proving combinatorial identities [37, 32]. The notion of WZ-pairs is one of the core concepts in their theory, which was originally introduced in [38] with a recent brief description in [36]. It is an elegant and powerful tool for proving identities involving sums of hypergeometric terms in an algorithmic fashion, and there are

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implementations in several computer algebra systems. A WZ-pair is a pair of functions $\big(F(n,k), G(n,k)\big)$ satisfying the relation

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k),$$

where both F and G are hypergeometric terms, i.e., their shift quotients with respect to n and k are rational functions in n and k. Assume that $F(n,k)$ vanishes except for k in some finite interval for each n. Therefore, one can sum both sides of the above equation w.r.t. k from 0 to ∞ to get, by telescoping,

$$\sum_{k=0}^{\infty} F(n+1,k) - \sum_{k=0}^{\infty} F(n,k) = \lim_{k\to\infty} G(n,k+1) - G(n,0).$$

If the boundary terms on the right-hand side vanish, we obtain that

$$\sum_{k=0}^{\infty} F(n+1,k) = \sum_{k=0}^{\infty} F(n,k),$$

which implies that the definite sum $\sum_{k=0}^{\infty} F(n,k)$ is independent of n. Thus, we get the identity $\sum_{k=0}^{\infty} F(n,k) = c$, where the constant c can be determined by evaluating the sum for one specific value of n. We can also get a companion identity by summing w.r.t. n. For example, the WZ-pair (F,G) with

$$F = \frac{\binom{n}{k}^2}{\binom{2n}{n}} \quad \text{and} \quad G = \frac{(2k-3n-3)k^2}{2(2n+1)(-n-1+k)^2} \cdot \frac{\binom{n}{k}^2}{\binom{2n}{n}}$$

leads to two identities

$$\sum_{k=0}^{\infty} \binom{n}{k}^2 = \binom{2n}{n} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{(3n-2k+1)}{(2n+1)\binom{2n}{n}} \binom{n}{k}^2 = 2.$$

WZ-pairs have been employed by Guillera to prove Ramanujan-type series [21, 22, 23, 24] and new congruences involving harmonic numbers and Apéry numbers conjectured by Sun [35, 39, 26]. This shows that WZ-pairs are not only beneficial in combinatorics but also in the fields of mathematical analysis and number theory. Wilf–Zeilberger forms, in short WZ-forms, are a direct generalization of WZ-pairs to tuples with more than two entries. The naming "WZ-form" is reminiscent of the classical concepts of differential forms [7] and difference forms, to which it is indeed related. To be more precise, we first recall some basic terminologies and properties about difference forms from [40, 29]. Let $\mathcal{M}$ be a well-chosen module of discrete functions defined on some region of $\mathbb{Z}^n$ so that one can define the usual shift operators $\sigma_1, \ldots, \sigma_n$ that commute. Let $\delta x_1, \ldots, \delta x_n$ be "indeterminates" satisfying the relations

$$\delta x_i \, \delta x_j = -\delta x_j \, \delta x_i \quad \text{for all } i, j \text{ with } 1 \le i \le j \le n.$$

An (exterior) *difference k-form* is a linear combination of "words" in the alphabet $\{\delta x_1, \ldots, \delta x_n\}$ with coefficients from the module $\mathcal{M}$, which can be written as

$$\omega = \sum_{i_1,\ldots,i_k} f_{i_1,\ldots,i_k}\, \delta x_{i_1} \cdots \delta x_{i_k}, \quad \text{where } f_{i_1,\ldots,i_k} \in \mathcal{M}.$$

Any element of $\mathcal{M}$ can be seen as a 0-*form*. The *exterior difference* δ of ω is the $(k+1)$-form defined by

$$\delta\omega = \sum_{i_1,\ldots,i_k} \left(\sum_{i=1}^{n} \Big(\sigma_i(f_{i_1,\ldots,i_k}) - f_{i_1,\ldots,i_k} \Big) \delta_{x_i} \right) \delta x_{i_1} \cdots \delta x_{i_k}$$

A difference form ω is called *closed* if $\delta\omega = 0$ and it is called *exact* if $\omega = \delta\theta$ for some θ. Analogous to the de Rham complex for differential forms, the property $\delta^2 = 0$ holds for difference forms since the shift operators commute. That is to say, every exact form is always closed, but the converse is not true in general. Closed difference 1-forms with hypergeometric coefficients are called *WZ-forms* in [40] and *WZ-cohomology* is the quotient of the modules of closed forms modulo that of exact forms. Similar to WZ-pairs, WZ-forms as well can be used to prove combinatorial identities and to derive convergence acceleration formulas [41]. For example, Dixon's identity

$$\sum_k (-1)^k \frac{(a+b)!(a+c)!(b+c)!a!b!c!}{(a+k)!(a-k)!(b+k)!(b-k)!(c+k)!(c-k)!(a+b+c)!} = 1 \qquad (1)$$

can be derived from the closed difference 1-form

$$\omega = F\,\delta k + G\,\delta a + H\,\delta b + I\,\delta c,$$

where F denotes the summand in (1), and

$$G := -\frac{(b+k)(c+k)}{2(a-k+1)(a+b+c+1)} \cdot F,$$
$$H := -\frac{(a+k)(c+k)}{2(b-k+1)(a+b+c+1)} \cdot F,$$
$$I := -\frac{(a+k)(b+k)}{2(c-k+1)(a+b+c+1)} \cdot F.$$

The idea is similar to that used in deriving identities from WZ-pairs. For any given closed difference 1-form $\omega = f_1\,\delta x_1 + \cdots + f_n\,\delta x_n$, Zeilberger applied the discrete Stokes theorem [40, p. 583] under some finitely supported conditions on the f_i's to prove that $g(\widehat{\mathbf{x}}_i) = \sum f_i\,\delta x_i$ is identically constant in $\widehat{\mathbf{x}}_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ and this constant can be determined by checking the sum for one special value of $\widehat{\mathbf{x}}_i$. To get convergence acceleration formulas, the idea is to apply the discrete Stokes theorem to a closed difference 1-form $\omega = F(n,k)\,\delta k + G(n,k)\,\delta n$ to obtain

$$\sum_{n=0}^{\infty} G(n,0) = \sum_{n=1}^{\infty} \Big(F(n, n-1) + G(n-1, n-1) \Big)$$

whenever both sums converge. For example, the closed form $\omega = F\,\delta k + G\,\delta n$ with

$$F = \frac{(-1)^{n+k} k!^2 (n-k-1)!}{(n+k+1)!} \quad \text{and} \quad G = \frac{2\,(-1)^{n+k} k!^2 (n-k)!}{(n+1)(n+k+1)!}.$$

leads to the convergence acceleration formula [33] for $\zeta(2)$

$$\zeta(2) = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^2} = 3\sum_{n=1}^{\infty} \frac{1}{\binom{2n}{n} n^2}. \tag{2}$$

Then the natural questions arise how one can construct such WZ-forms? And how can we decide whether a WZ-form is exact or not? Note that the idea of deriving WZ-forms from known identities and then employing them to generate new identities has been shown in [19, 41, 30, 5]. In this paper, we restrict our attention to WZ-forms with rational functions instead of hypergeometric entries and use n-tuples $(f_1, \ldots, f_n)$ to denote WZ-forms instead of difference forms. We shall describe the structure of rational WZ-forms which is an additive analog of the Ore–Sato theorem [31, 34, 17, 4, 1] in Theorem 4. Before proving the main theorem, we first recall the structure theorem on WZ-pairs from [8] in Section 3 which will be used as the base case in our induction proof and then overview some basic properties about orbital decomposition and orbital residues in Section 4. The proof of Theorem 4 splits into two steps: the first step is to show that any WZ-form can be decomposed into one exact WZ-form plus several uniform WZ-forms in Section 5 and the second step is to describe the explicit integer-linear structure of uniform WZ-forms in Section 6. In the last section, we present an algorithm for computing the additive structure of WZ-forms that minimizes the uniform part. With the help of this minimal decomposition, we can detect the exactness of WZ-forms by just checking whether the uniform part vanishes.

2. Preliminaries

Throughout this paper, let $\mathbb{N}$ denote the set of nonnegative integers. Let K be an algebraically closed field of characteristic zero and $K(x_1, \ldots, x_n)$ be the field of rational functions in the variables $x_1, \ldots, x_n$ over K, which is also written as $K(\mathbf{x})$. For a multivariate function f, the shift maps σ_i are defined as

$$\sigma_i f(x_1, \ldots, x_n) = f(x_1, \ldots, x_i + 1, \ldots, x_n), \quad \forall i \in \{1, \ldots, n\}.$$

The action of operators on functions is also denoted by $\bullet$, e.g., $\sigma_i \bullet f = \sigma_i(f)$. Analogously, the forward difference operators are defined as

$$\Delta_i(f) := \sigma_i(f) - f, \quad \forall i \in \{1, \ldots, n\}.$$

Definition 1 (Hypergeometric, hyperarithmetic)**.** *A nonzero expression H is said to be* hypergeometric *over $K(\mathbf{x})$ if there exist rational functions $f_1, \ldots, f_n \in K(\mathbf{x})$ such that*

$$\frac{\sigma_i(H)}{H} = f_i, \quad \forall i \in \{1, \ldots, n\}.$$

Analogously, H is said to be hyperarithmetic *over $K(\mathbf{x})$ if there exist rational functions $f_1, \ldots, f_n \in K(\mathbf{x})$ such that*

$$\sigma_i(H) - H = f_i, \quad \forall i \in \{1, \ldots, n\}.$$

In both cases, the rational functions $f_1, \ldots, f_n$ are called the certificates *of H. Two hypergeometric (resp. hyperarithmetic) expressions H_1 and H_2 are called* conjugate*, denoted by $H_1 \simeq H_2$, if they have the same certificates.*

Since σ_i and σ_j commute, the certificates $f_1, \ldots, f_n$ of a hypergeometric term H satisfy the following compatibility conditions:

$$\frac{\sigma_i(f_j)}{f_j} = \frac{\sigma_j(f_i)}{f_i}, \quad \forall i, j \in \{1, \ldots, n\}. \tag{3}$$

Similarly, the certificates $f_1, \ldots, f_n$ of a hyperarithmetic expression H satisfy the following compatibility conditions:

$$\sigma_i(f_j) - f_j = \sigma_j(f_i) - f_i, \quad \forall i, j \in \{1, \ldots, n\}. \tag{4}$$

Definition 2. *An n-tuple $(f_1, \ldots, f_n) \in K(\mathbf{x})^n$ is called a* WZ-form *with respect to $(\Delta_1, \ldots, \Delta_n)$ if $\Delta_i(f_j) = \Delta_j(f_i)$ for all $i, j \in \{1, \ldots, n\}$. Note that $(f_1, \ldots, f_n)$ is a WZ-form with respect to $(\Delta_1, \ldots, \Delta_n)$ if and only if $\omega = f_1\delta_1 + \cdots + f_n\delta_n$ is a closed difference 1-form.*

The classical Ore–Sato theorem plays an important role in the theory of multivariate hypergeometric terms [17, 4, 1], because it describes the multiplicative structure of nonzero rational functions $f_1, \ldots, f_n \in K(\mathbf{x})$ that satisfy the compatibility conditions (3). The bivariate case was proven by Ore [31] and the multivariate case by Sato [34]. According to this theorem, any multivariate hypergeometric term can be decomposed into a product of a rational function and several factorial terms (which are basically products of Gamma functions).

Theorem 3 (Ore–Sato theorem)**.** *Let $f_1, \ldots, f_n \in K(\mathbf{x})$ be nonzero rational functions satisfying the compatibility conditions* (3)*. Then there exist a rational function $a \in K(\mathbf{x})$, constants $\mu_1, \ldots, \mu_n \in K$, a finite set $V \subset \mathbb{Z}^n$, and for each $\mathbf{v} \in V$ a univariate monic rational function $r_{\mathbf{v}} \in K(z)$ such that*

$$f_j = \frac{\sigma_j(a)}{a}\mu_j \prod_{\mathbf{v}\in V} \prod_{\ell}{}_0^{v_j} r_{\mathbf{v}}(\mathbf{v} \cdot \mathbf{x} + \ell),$$

where $\mathbf{v} \cdot \mathbf{x} := v_1 x_1 + \cdots + v_n x_n$ *and where the product notation is defined as follows: for* $s, t \in \mathbb{Z}$,

$$\prod_{\ell}{}_{s}^{t} \alpha_\ell := \begin{cases} \alpha_s \alpha_{s+1} \cdots \alpha_{t-1}, & \text{if } t \geq s; \\ \dfrac{1}{\alpha_t \alpha_{t+1} \cdots \alpha_{s-1}}, & \text{if } t < s. \end{cases}$$

Christopher's theorem [12, 42] is an analog of the Ore–Sato theorem in the continuous case. Other analogs concern the q-discrete case [15] and the continuous-discrete case [9]. In this paper, we want to explore the additive structure of nonzero rational functions $f_1, \ldots, f_n \in K(\mathbf{x})$ satisfying the compatibility conditions (4), i.e., $(f_1, \ldots, f_n)$ is a WZ-form. Our main result, which is stated in the following theorem, reveals this additive structure and therefore implies an additive decomposition of hyperarithmetic expressions.

Theorem 4 (Additive Ore–Sato theorem)**.** *Let* $f_1, \ldots, f_n \in K(\mathbf{x})$ *be nonzero rational functions satisfying the compatibility conditions* (4)*. Then there exist a rational function* $a \in K(\mathbf{x})$*, constants* $\mu_1, \ldots, \mu_n \in K$*, a finite set* $V \subset \mathbb{Z}^n$*, and for each* $\mathbf{v} \in V$ *a univariate monic rational function* $r_{\mathbf{v}} \in K(z)$ *such that*

$$f_j = \sigma_j(a) - a + \mu_j + \sum_{\mathbf{v} \in V} \sum_{\ell}{}_{0}^{v_j} r_{\mathbf{v}}(\mathbf{v} \cdot \mathbf{x} + \ell),$$

where $\mathbf{v} \cdot \mathbf{x} := v_1 x_1 + \cdots + v_n x_n$ *and where we use the sum notation (for* $s, t \in \mathbb{Z}$*)*

$$\sum_{\ell}{}_{s}^{t} \alpha_\ell := \begin{cases} \alpha_s + \alpha_{s+1} + \cdots + \alpha_{t-1}, & \text{if } t \geq s; \\ -(\alpha_t + \alpha_{t+1} + \cdots + \alpha_{s-1}), & \text{if } t < s. \end{cases}$$

In the proof of the classical Ore–Sato theorem, the complete irreducible factorization was used as a key ingredient. When it comes to the additive case, we need another auxiliary tool, the so-called orbital decomposition, which compensates the missing of partial fraction decompositions of multivariate rational functions. Hence, our additive Ore–Sato theorem is not just a straightforward analog of its multiplicative predecessor, but is based on a significantly different proof strategy.

3. WZ-forms and structure of WZ-pairs

The goal of this section is to introduce some notions that will help us to describe the proofs in the later sections more concisely.

Definition 5 ((Pairwise) shift-invariant)**.** *A rational function* $f \in K(\mathbf{x})$ *is called* shift-invariant *if there exists a nonzero integer vector* $\mathbf{v} \in \mathbb{Z}^n$ *such that* $f(\mathbf{x} + \mathbf{v}) = f(\mathbf{x})$*. It is called* pairwise shift-invariant *if for all* $i, j \in \{1, \ldots, n\}$*, there exist* $s, t \in \mathbb{Z}$*, not both zero, such that* $\sigma_i^s(f) = \sigma_j^t(f)$*.*

Definition 6 (Integer-linearity)**.** *An irreducible polynomial $p \in K[\mathbf{x}]$ is called* integer-linear *over K if there exist a univariate polynomial $P \in K[z]$ and a nonzero integer vector $\mathbf{v} \in \mathbb{Z}^n$ such that*

$$p(\mathbf{x}) = P(\mathbf{v} \cdot \mathbf{x}).$$

Without loss of generality, we can assume that $\gcd(v_1, \ldots, v_n) = 1$, because a common factor can be extracted and absorbed by P. Such a vector $\mathbf{v}$ is called the integer-linear type *of p. We say that $f \in K(\mathbf{x})$ is integer-linear of type $\mathbf{v}$ if all the irreducible factors of its numerator and its denominator are of the common integer-linear type $\mathbf{v}$.*

There is an efficient algorithm for the computation of the integer-linear decomposition of multivariate polynomials [20], which will be used for computing additive decompositions in Section 7. The next lemma reveals the equivalence between the pairwise shift-invariance and the integer-linearity of a rational function.

Lemma 7 ([4, Proposition 7])**.** *A rational function $f \in K(\mathbf{x})$ is pairwise shift-invariant if and only if there exist a nonzero integer vector $\mathbf{v} \in \mathbb{Z}^n$ and a univariate rational function $r \in K(z)$ such that*

$$f(\mathbf{x}) = r(\mathbf{v} \cdot \mathbf{x}),$$

i.e., f is integer-linear of type $\mathbf{v}$.

Given the integer-linear type of f, one can easily see that f is pairwise shift-invariant. In contrast, the opposite direction of Lemma 7 is not that obvious. However, it follows, by using an inductive argument, from the bivariate case that is illustrated in the following remark.

Remark 8. *Let $f \in K(x, y)$ satisfy $\sigma_x^s \sigma_y^t(f) = f$ with $s, t \in \mathbb{Z}$ not both zero. If $s = 0$, then f is free of y, which implies that f is integer-linear of type $(1, 0)$. Similarly, if $t = 0$, then f is integer-linear of type $(0, 1)$. If both of them are nonzero, then f is integer-linear of type $(\bar{t}, \bar{s})$, where $\bar{t} = t/\gcd(s, t)$ and $\bar{s} = s/\gcd(s, t)$.*

According to Definition 6, an element in K can be viewed as having any integer-linear type. But for a non-constant rational function whose factors are of the same integer-linear type, its type is unique. Such a type remains unchanged under addition and under application of shift operators.

We now introduce two special kinds of WZ-forms, namely exact WZ-forms and uniform WZ-forms, which will play an important role in describing the structure of general WZ-forms (see Theorem 4).

Definition 9 (Exact WZ-form)**.** *A WZ-form $(f_1, \ldots, f_n)$ with respect to $(\Delta_1, \ldots, \Delta_n)$ is said to be* exact *if there exists $g \in K(\mathbf{x})$ such that $f_i = \Delta_i(g)$ for all $i \in \{1, \ldots, n\}$.*

Definition 10 (Uniform WZ-form)**.** *A WZ-form* $(f_1, \ldots, f_n)$ *with respect to* $(\Delta_1, \ldots, \Delta_n)$ *is called a* uniform WZ-form *if there exists an integer vector* $\mathbf{v}$ *such that each* f_i *is integer-linear of type* $\mathbf{v}$*.*

Remark 11. *A WZ-form can be both exact and uniform: for example,* $\big(\Delta_x(\frac{1}{x+y}), \Delta_y(\frac{1}{x+y})\big)$ *is an exact WZ-pair where each component is integer-linear of type* $(1, 1)$*.*

In the remaining part of this section, we recall the structure theorem [8] on WZ-pairs that is described in terms of exact and cyclic pairs, see Theorem 14.

Definition 12 (Cyclic operator)**.** *Let* $G = \langle \sigma_1, \ldots, \sigma_n \rangle$ *be the free abelian group generated by the shift operators* $\sigma_1, \ldots, \sigma_n$*. For any* $m \in \mathbb{Z}$ *and* $\tau \in G$*, define*

$$\frac{\tau^m - 1}{\tau - 1} := \begin{cases} 1 + \tau + \cdots + \tau^{m-1}, & \text{if } m > 0; \\ 0, & \text{if } m = 0; \\ -(\tau^m + \cdots + \tau^{-1}), & \text{if } m < 0. \end{cases}$$

Definition 13 (Cyclic pair)**.** *A WZ-pair* (f, g) *w.r.t.* (Δ_x, Δ_y) *is called a* cyclic pair *if there exists* $h \in K(x, y)$ *that satisfies* $\sigma_x^s(h) = \sigma_y^t(h)$ *for some* $s, t \in \mathbb{Z}$*, not both zero, such that*

$$f = \frac{\sigma_y^t - 1}{\sigma_y - 1} \bullet h \quad \text{and} \quad g = \frac{\sigma_x^s - 1}{\sigma_x - 1} \bullet h.$$

Note that any cyclic pair is a uniform WZ-pair by Remark 8. The following theorem shows that each WZ-pair can be decomposed into one exact WZ-pair plus several cyclic pairs.

Theorem 14 (Structure of WZ-pairs, [8, Theorem 3])**.** *Any WZ-pair can be decomposed into one exact WZ-pair plus several cyclic WZ-pairs.*

When it comes to a multivariate generalization of Theorem 14, cyclic pairs will be replaced by uniform WZ-forms, see Theorem 21. For this purpose, we define orbital decompositions and orbital residues of rational functions in the next section.

4. Orbital decompositions and orbital residues

In this section, we recall the notion of orbital decomposition of a rational function, which was first used for studying the existence problem of telescopers [10], and propose a modified definition of discrete residues, which were originally introduced in [11] with polynomial and elliptic analogs in [27, 14].

Definition 15 (Shift-equivalence)**.** *Let* F *be a subgroup of* $\langle \sigma_1, \ldots, \sigma_n \rangle$*. For* $a, b \in K(\mathbf{x})$*, we say that* a *and* b *are* F-equivalent*, denoted by* $a \sim_F b$*, if there exists* $\tau \in F$ *with* $\tau(a) = b$*. We call the set*

$$[a]_F := \{\tau(a) \mid \tau \in F\}$$

the F-orbit *of* a*. Note that* $a \sim_F b$ *implies* $[a]_F = [b]_F$*.*

Example 16. *Let $b = 4x + 6y + 5z$ and let F be the subgroup $\langle \sigma_x, \sigma_y \rangle$ of $G = \langle \sigma_x, \sigma_y, \sigma_z \rangle$. Then b and $b+1$ are G-equivalent because $\tau(b) = b + 1$ for $\tau = \sigma_x^{-1}\sigma_z \in G$. In contrast, b and $b+1$ are not F-equivalent.*

The orbital decomposition of a rational function $f = P/Q \in K(\mathbf{x})$ depends on the variable x_1 and a subgroup F. In order to define it, we first focus on its denominator as a polynomial in x_1, that is, $Q \in K(\widehat{\mathbf{x}})[x_1]$ with $\widehat{\mathbf{x}} := x_2, \ldots, x_n$. The first step consists in factoring the polynomial Q completely over $K(\widehat{\mathbf{x}})$. We sort all of its irreducible factors into distinct F-orbits as follows:

$$Q = c \cdot \prod_{i=1}^{I} \prod_{j=1}^{J} \prod_{\tau \in \Lambda_{i,j}} \tau(b_i^j),$$

where $c \in K(\widehat{\mathbf{x}})$, $\Lambda_{i,j}$ are finite subsets of F, and the $b_i \in K(\widehat{\mathbf{x}})[x_1]$ are monic irreducible polynomials in distinct F-orbits. Note that this factorization is unique up to the choice of the representative b_i in each F-orbit. Moreover, we impose on the sets $\Lambda_{i,j}$ the condition that $\tau(b_i) \neq \tau'(b_i)$ for $\tau, \tau' \in \Lambda_{i,j}$ with $\tau \neq \tau'$. In the second step, we compute the unique irreducible partial fraction decomposition of f with respect to the above factorization:

$$f = p + \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{\tau \in \Lambda_{i,j}} \frac{a_{i,j,\tau}}{\tau(b_i^j)}, \tag{5}$$

where $p, a_{i,j,\tau} \in K(\widehat{\mathbf{x}})[x_1]$ with $\deg_{x_1}(a_{i,j,\tau}) < \deg_{x_1}(b_i)$ for all i, j, τ. For a polynomial $b \in K(\widehat{\mathbf{x}})[x_1]$, a subgroup $F \leq G$, and $j > 0$, we define the following linear $K(\widehat{\mathbf{x}})$-subspace:

$$U_{b,j}^F := \operatorname{Span}_{K(\widehat{\mathbf{x}})} \left\{ \frac{a}{\tau(b^j)} \;\middle|\; \tau \in F,\ a \in K(\widehat{\mathbf{x}})[x_1],\ \deg_{x_1}(a) < \deg_{x_1}(b) \right\}. \tag{6}$$

Note that in Equation (5), each sum $\sum_\tau \frac{a_{i,j,\tau}}{\tau(b_i^j)}$ lies in the corresponding subspace $U_{b_i,j}^F$. Since the decomposition (5) exists for any $f \in K(\mathbf{x})$, and since the orbits $[b]_F$ do not overlap, we obtain the following direct sum decomposition:

$$K(\mathbf{x}) = K(\widehat{\mathbf{x}})[x_1] \oplus \left(\bigoplus_{j>0} \bigoplus_{[b]_F} U_{b,j}^F \right), \tag{7}$$

where $[b]_F$ runs over all orbits in $K(\widehat{\mathbf{x}})[x_1]/\sim_F$. Such a direct sum decomposition is called [10] *the orbital decomposition of $K(\mathbf{x})$ with respect to the variable x_1 and the group F.*

According to the definition of $U_{b,j}^F$, it is easy to check that this linear subspace is closed under the application of any operator in $K(\widehat{\mathbf{x}})[F]$, that is, any operator of the form $\sum_{\tau \in F} c_\tau \tau$ with $c_\tau \in K(\widehat{\mathbf{x}})$. The following lemma is a direct generalization of [10, Lemma 5.1].

Lemma 17. *If $f \in U_{b,j}^F$ and $\theta \in K(\widehat{\mathbf{x}})[F]$, then $\theta(f) \in U_{b,j}^F$.*

Theorem 18. *Let $f = p + \sum_{i=1}^{I} \sum_{j=1}^{J} f_{i,j}$ with $p \in K(\widehat{\mathbf{x}})[x_1]$ and $f_{i,j} \in U_{b_i,j}^F$ be an orbital decomposition of f with respect to x_1 and F, and let $\theta_1, \theta_2 \in K(\widehat{\mathbf{x}})[F]$. For $g \in K(\mathbf{x})$, we have $\theta_1(f) = \theta_2(g)$ if and only if there exist $q \in K(\widehat{\mathbf{x}})[x_1]$ and $g_{i,j} \in U_{b_i,j}^F$ such that $\theta_1(p) = \theta_2(q)$ and $\theta_1(f_{i,j}) = \theta_2(g_{i,j})$ for all i, j.*

Proof. The sufficiency is due to the linearity of the operators $\theta_1, \theta_2 \in K(\widehat{\mathbf{x}})[F]$. For the necessity, suppose $g = q + \sum_{i=1}^{I} \sum_{j=1}^{J} g_{i,j}$, where $q \in K(\widehat{\mathbf{x}})[x_1]$ and $g_{i,j} \in U_{b_i,j}^F$ for each i, j. By Lemma 17, the orbital decomposition of $\theta_1(f)$ with respect to x_1 and F is

$$\theta_1(f) = \theta_1(p) + \sum_{i=1}^{I} \sum_{j=1}^{J} \theta_1(f_{i,j}).$$

Similarly, we get

$$\theta_2(g) = \theta_2(q) + \sum_{i=1}^{I} \sum_{j=1}^{J} \theta_2(g_{i,j}).$$

By the uniqueness of the direct sum decomposition (7), we have $\theta_1(p) = \theta_2(q)$ and $\theta_1(f_{i,j}) = \theta_2(g_{i,j})$ for each i, j. □

For $f \in K(\mathbf{x})$, we say that f is σ_i-*summable* if there exists $g \in K(\mathbf{x})$ such that $f = \Delta_i(g)$. Let $(f_1, \ldots, f_n)$ be a WZ-form w.r.t. $(\Delta_1, \ldots, \Delta_n)$. Then $\Delta_i(f_1)$ is σ_1-summable, because we have $\Delta_i(f_1) = \Delta_1(f_i)$. The first step in our proof of Theorem 4 is to decompose f_1 and to find the shift-invariance of each part.

Next, for the definition of orbital residues, let us look at the orbital decomposition of $f \in K(\mathbf{x})$ with respect to x_1 and the subgroup $F = \langle \sigma_1 \rangle$. In this case, the decomposition (5) can be written as

$$f = p + \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{\ell=0}^{L} \frac{a_{i,j,\ell}}{\sigma_1^{\ell}(d_i^j)}, \tag{8}$$

where the d_i are irreducible polynomials in distinct $\langle \sigma_1 \rangle$-orbits.

Definition 19 (Orbital residue)**.** *Let f be given in the form (8), let $d \in K(\widehat{\mathbf{x}})[x_1]$ be irreducible, and let $j \in \{1, \ldots, J\}$. If there exists $i \in \{1, \ldots, I\}$ such that $d_i \in [d]_{\langle \sigma_1 \rangle}$ (by the properties of the orbital decomposition, such i is uniquely determined), then the* orbital residue *of f at d of multiplicity j, denoted by $\operatorname{res}_{\sigma_1}(f, d, j)$, is defined to be the $\langle \sigma_1 \rangle$-orbit $[r]_{\langle \sigma_1 \rangle}$ with*

$$r := \sum_{\ell=0}^{L} \sigma_1^{-\ell}(a_{i,j,\ell}).$$

If no such i exists, we define $\operatorname{res}_{\sigma_1}(f, d, j) = 0$. If it is clear from the context, we will abbreviate $[r]_{\langle \sigma_1 \rangle}$ by $[r]$.

Note that the definition of orbital residue does not depend on the representation (5) of f: if instead of d_i some other representative of $[d_i]_{\langle\sigma_1\rangle}$ is used, at the cost of changing the range of ℓ, then also the polynomial r in Definition 19 changes, but it will stay in the same $\langle\sigma_1\rangle$-orbit. This is the reason why the residue is defined to be an orbit, instead of a single polynomial. Similarly, we have $\operatorname{res}_{\sigma_1}(f, d, j) = \operatorname{res}_{\sigma_1}(f, d', j)$ whenever $d \sim_{\langle\sigma_1\rangle} d'$.

Example 20. *Let $b = 4x + 6y + 5z$ as in Example 16 and let*

$$f = \frac{x}{b^2} + \frac{x+y}{(b+1)^2} + \frac{2x}{(b-3)^2} + \frac{2x+3}{(b+3)^2}.$$

We observe that $b+1 = \sigma_x(b-3)$ and that $b, b-3, b+3$ are in distinct $\langle\sigma_x\rangle$-orbits. By Definition 19, we have

$$\operatorname{res}_{\sigma_x}(f, b, 2) = [x], \quad \operatorname{res}_{\sigma_x}(f, b-3, 2) = [3x+y-1], \quad \operatorname{res}_{\sigma_x}(f, b+3, 2) = [2x+3].$$

5. Additive decompositions of WZ-forms

Exact and uniform WZ-forms are special kinds of WZ-forms. Conversely, the following theorem shows that these two forms are the only basic building blocks of all possible WZ-forms. This section is dedicated to proving the following theorem, which is a multivariate generalization of Theorem 14.

Theorem 21. *Any WZ-form can be decomposed into one exact WZ-form plus several uniform WZ-forms.*

First we recall the notion of isotropy group, which was introduced by Sato [34] in order to prove the classical Ore–Sato theorem.

Definition 22 (Isotropy group)**.** *Let $p \in K[\mathbf{x}]$. The set*

$$G_p = \{\tau \in G \mid \tau(p) = p\}$$

is a subgroup of G, called the isotropy group *of p in G.*

Example 23. *Let $b = 4x + 6y + 5z$ and $G = \langle\sigma_x, \sigma_y, \sigma_z\rangle$ be as in Example 16, and let $c = 3y + 2z$. The isotropy group of b consists of all monomials $\sigma_x^i\sigma_y^j\sigma_z^k$ that satisfy $4i + 6j + 5k = 0$, i.e., $G_b = \langle\sigma_x\sigma_y\sigma_z^{-2}, \sigma_y^5\sigma_z^{-6}\rangle$. Similarly, one finds $G_c = \langle\sigma_x, \sigma_y^2\sigma_z^{-3}\rangle$ and $G_{b\cdot c} = G_b \cap G_c = \langle\sigma_x^3\sigma_y^8\sigma_z^{-12}\rangle$.*

Definition 22 can directly be extended to rational functions. The next lemma shows that shift-equivalent elements have the same isotropy group.

Lemma 24. *Let $f, g \in K(\mathbf{x})$. If $f \sim_G g$, then $G_f = G_g$.*

Proof. Let $\sigma \in G$ such that $f = \sigma(g)$. For $\tau \in G_g$ we have $\tau(g) = g$. Applying σ to both sides of the equation yields $\sigma(\tau(g)) = \sigma(g)$. Since σ and τ commute, we have $\tau(\sigma(g)) = \sigma(g)$, i.e., $\tau(f) = f$. Thus $\tau \in G_f$, which implies that $G_g \subseteq G_f$. Since $\sigma^{-1} \in G$ such that $g = \sigma^{-1}(f)$, we similarly have $G_f \subseteq G_g$. Hence $G_f = G_g$. □

We recall a crucial lemma that led to the structure theorem of WZ-pairs. Here it will be used to conduct the induction step in the proof of Theorem 21.

Lemma 25 ([8, Lemma 6]). *Let $f \in K(x,y)$ be a rational function of the form*

$$f = \frac{a_0}{b^m} + \frac{a_1}{\sigma_y(b^m)} + \cdots + \frac{a_n}{\sigma_y^n(b^m)},$$

where $m, n \in \mathbb{N}$ with $m > 0$, $a_0, \ldots, a_n, b \in K(y)[x]$ with $a_n \neq 0$. Moreover, we assume that $\deg(a_i) < \deg(b)$, b is irreducible and monic, and that $\sigma_y^i(b) \not\sim_{\langle \sigma_x \rangle} \sigma_y^j(b)$ for all $i, j \in \{0, \ldots, n\}$ with $i \neq j$. If for some $g \in K(x,y)$ we have $\Delta_y(f) = \Delta_x(g)$, then there exists $t \in \mathbb{Z}$ such that $\sigma_y^{n+1}(a_0) = \sigma_x^t(a_0)$, $\sigma_y^{n+1}(b) = \sigma_x^t(b)$, and $a_\ell = \sigma_y^\ell(a_0)$ for all $\ell \in \{0, \ldots, n\}$. Furthermore, for some $g_0 \in K(y)$ we get

$$f = \frac{\sigma_y^{n+1} - 1}{\sigma_y - 1} \bullet \frac{a_0}{b^m} \quad \text{and} \quad g = \frac{\sigma_x^t - 1}{\sigma_x - 1} \bullet \frac{a_0}{b^m} + g_0.$$

According to Remark 8, the bivariate function f in Lemma 25 is of a certain integer-linear type. We will use this lemma to reduce the problem from the multivariate case to the bivariate case, see the proof of Lemma 29 below.

Recall that $G = \langle \sigma_1, \ldots, \sigma_n \rangle$ and $\widehat{\mathbf{x}} = x_2, \ldots, x_n$. Let $\omega = (f_1, \ldots, f_n) \in K(\mathbf{x})^n$ be a WZ-form w.r.t. $(\Delta_1, \ldots, \Delta_n)$. Applying the orbital decomposition (5) with respect to x_1 and G to f_1 yields

$$f_1 = p + \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{\tau \in \Lambda_{i,j}} \frac{a_{i,j,\tau}}{\tau(b_i^j)}, \tag{9}$$

where for all i, j, τ we have $p, a_{i,j,\tau} \in K(\widehat{\mathbf{x}})[x_1]$ with $\deg_{x_1}(a_{i,j,\tau}) < \deg_{x_1}(b_i)$ and $\Lambda_{i,j} \subset G$. The following reduction formula is crucial in Abramov's algorithm for rational summation [2, 3].

Fact 26. *For all $a, u \in K[\mathbf{x}]$ with $u \neq 0$ and automorphism ϕ of $K(\mathbf{x})$, we have*

$$\frac{a}{\phi^m(u)} = \phi(g) - g + \frac{\phi^{-m}(a)}{u}, \tag{10}$$

where

$$g = \begin{cases} \displaystyle\sum_{i=0}^{m-1} \frac{\phi^{i-m}(a)}{\phi^i(u)}, & \text{if } m \geq 0; \\ \displaystyle -\sum_{i=m}^{-1} \frac{\phi^{i-m}(a)}{\phi^i(u)}, & \text{if } m < 0. \end{cases} \tag{11}$$

Let $E := \langle \sigma_2, \ldots, \sigma_n \rangle$. Then each $\tau \in G$ can be written as $\sigma_1^m \lambda$ for some $m \in \mathbb{Z}$ and $\lambda \in E$. By taking $\phi = \sigma_1$ and $u = \lambda(b)$ in Formula (10), we get

$$\frac{a}{\tau(b)} = \frac{a}{\sigma_1^m(u)} = \Delta_1(g) + \frac{\sigma_1^{-m}(a)}{u} = \Delta_1(g) + \frac{\sigma_1^{-m}(a)}{\lambda(b)}, \tag{12}$$

for some $g \in K(\mathbf{x})$ of the form (11). Applying the above reduction (12) to each summand $a_{i,j,\tau}/\tau(b_i^j)$ in Equation (9) yields

$$f_1 = \Delta_1(g_0) + \sum_{i=1}^{I}\sum_{j=1}^{J} \widetilde{f}_{1,i,j} \quad \text{with} \quad \widetilde{f}_{1,i,j} = \sum_{\lambda \in \widetilde{\Lambda}_{i,j}} \frac{\widetilde{a}_{i,j,\lambda}}{\lambda(b_i^j)}, \tag{13}$$

where $g_0 \in K(\mathbf{x})$, $\widetilde{\Lambda}_{i,j} \subseteq E$, and $\lambda(b_i) \not\sim_{\langle \sigma_1 \rangle} \lambda'(b_i)$ whenever λ, λ' are two distinct elements from $\widetilde{\Lambda}_{i,j}$. Since the shift operators σ_1^{-m} preserve the degrees of the polynomials $a_{i,j,\lambda}$, we have for all i,j that $\widetilde{f}_{1,i,j} \in U_{b_i,j}^G$. In fact,

$$[\widetilde{a}_{i,j,\lambda}] = \operatorname{res}_{\sigma_1}\big(f_1, \lambda(b_i), j\big).$$

We give an illustrative example to show how we can immediately obtain the orbital residue via the reduction (13). Note that the result is the same as specified in Definition 19.

Example 27 (Continuing Example 20)**.** *Rewrite f as*

$$f = \frac{x}{b^2} + \frac{x+y}{\sigma_x^{-1}\sigma_z(b^2)} + \frac{2x}{\sigma_x\sigma_y^{-2}\sigma_z(b^2)} + \frac{2x+3}{\sigma_x^{-3}\sigma_z^3(b^2)}.$$

First we get rid of the operator σ_x among all the denominators,

$$\begin{aligned} f = \Delta_x\bigg(&-\frac{x+y}{\sigma_x^{-1}\sigma_z(b^2)} + \frac{2x-2}{\sigma_y^{-2}\sigma_z(b^2)} - \frac{2x+3}{\sigma_x^{-3}\sigma_z^3(b^2)} - \frac{2x+5}{\sigma_x^{-2}\sigma_z^3(b^2)} - \frac{2x+7}{\sigma_x^{-1}\sigma_z^3(b^2)}\bigg) \\ &+ \frac{x}{b^2} + \frac{x+y+1}{\sigma_z(b^2)} + \frac{2x-2}{\sigma_y^{-2}\sigma_z(b^2)} + \frac{2x+9}{\sigma_z^3(b^2)}. \end{aligned}$$

Note that $\sigma_y^{-2}\sigma_z(b^2) = \sigma_x^{-3}\sigma_z(b^2)$, so we continue the reduction as follows:

$$\frac{2x-2}{\sigma_y^{-2}\sigma_z(b^2)} = \Delta_x\bigg(-\frac{2x-2}{\sigma_x^{-3}\sigma_z(b^2)} - \frac{2x}{\sigma_x^{-2}\sigma_z(b^2)} - \frac{2x+2}{\sigma_x^{-1}\sigma_z(b^2)}\bigg) + \frac{2x+4}{\sigma_z(b^2)}.$$

Hence

$$f = \Delta_x(g) + \frac{x}{b^2} + \frac{3x+y+5}{\sigma_z(b^2)} + \frac{2x+9}{\sigma_z^3(b^2)},$$

for some $g \in K(\mathbf{x})$. We observe that $\{b^2, \sigma_z(b^2), \sigma_z^3(b^2)\} = \{b^2, (b+5)^2, (b+15)^2\}$ are pairwise $\langle \sigma_x \rangle$-inequivalent, hence the reduction is done. We have

$$\operatorname{res}_{\sigma_x}(f, b, 2) = [x],\ \operatorname{res}_{\sigma_x}\big(f, \sigma_z(b), 2\big) = [3x+y+5],\ \operatorname{res}_{\sigma_x}\big(f, \sigma_z^3(b), 2\big) = [2x+9].$$

Using the g_0 that was obtained by Abramov's reduction (10), we define an exact WZ-form $\omega_0 := \big(\Delta_1(g_0), \ldots, \Delta_n(g_0)\big)$, which we remove from the given WZ-form ω. To this end, we let $\widetilde{f}_i := f_i - \Delta_i(g_0)$ and observe that $\big(\widetilde{f}_1, \ldots, \widetilde{f}_n\big)$ is still a WZ-form, which implies that for each $k \in \{2, \ldots, n\}$, $\Delta_k(\widetilde{f}_1)$ is σ_1-summable. Note that $\sum_{i=1}^I \sum_{j=1}^J \widetilde{f}_{1,i,j}$ is the orbital decomposition of $\widetilde{f}_1$ with respect to x_1 and G. By Theorem 18, for each i,j, we have $\Delta_k\big(\widetilde{f}_{1,i,j}\big)$ is σ_1-summable. Then we can focus on each orbital component of $\widetilde{f}_1$ in a linear $K(\widehat{\mathbf{x}})$-subspace $U_{b,m}^G$.

Remark 28. *We claim that* $a \in K(\mathbf{x}) \setminus K(\widehat{\mathbf{x}})$ *is pairwise shift-invariant if and only if for each* $k \in \{2, \ldots, n\}$*, there exist* $L_k, N_k \in \mathbb{Z}$ *with* $L_k \neq 0$*, such that* $\sigma_k^{L_k}(a) = \sigma_1^{N_k}(a)$*. The necessity follows from Definition 5. For the sufficiency, we combine for any* $k, s \in \{2, \ldots, n\}$ *the* N_s*-fold application of* $\sigma_k^{L_k}(a) = \sigma_1^{N_k}(a)$ *with the* N_k*-fold application of* $\sigma_s^{L_s}(a) = \sigma_1^{N_s}(a)$ *to obtain*

$$\sigma_k^{L_k N_s}(a) = \sigma_1^{N_k N_s}(a) = \sigma_s^{L_s N_k}(a).$$

If $N_k = N_s = 0$*, then* a *is free of* x_k *and* x_s *which implies that* $\sigma_k^1(a) = \sigma_s^1(a)$*.*

Lemma 29. *Let* $f_1 = \sum_{\lambda \in \Lambda} a_\lambda / \lambda(b^m) \in U_{b,m}^G$ *with* $\Lambda \subset E$ *and the* $\lambda(b)$ *being in distinct* $\langle \sigma_1 \rangle$*-orbits. If* $\Delta_k(f_1)$ *is* σ_1*-summable for each* $k \in \{2, \ldots, n\}$*, then all of the* a_λ *and* b *are integer-linear of the same type.*

Proof. By Remark 28 and Lemma 7, it is sufficient to show that for each $k \in \{2, \ldots, n\}$, there exist $L_k, N_k \in \mathbb{Z}$ with L_k nonzero such that $\sigma_k^{L_k}(b) = \sigma_1^{N_k}(b)$ and $\sigma_k^{L_k}(a_\lambda) = \sigma_1^{N_k}(a_\lambda)$ for all $\lambda \in \Lambda$. Let $E_k := \langle \sigma_2, \ldots, \sigma_{k-1}, \sigma_{k+1}, \ldots, \sigma_n \rangle$. For each $\lambda \in \Lambda \subset E$, there exist $t_\lambda \in \mathbb{Z}$, $\eta_\lambda \in E_k$ such that $\lambda = \sigma_k^{t_\lambda} \eta_\lambda$, and therefore

$$f_1 = \sum_{\lambda \in \Lambda} \frac{a_\lambda}{\sigma_k^{t_\lambda} \eta_\lambda(b^m)}.$$

By applying the reduction formula (10) once again, we can rewrite f_1 in the form

$$f_1 = \Delta_1(f_{1,k}) + \sum_{\eta \in \Lambda_k} \sum_{\ell=0}^{T_\eta} \frac{\widetilde{a}_{\eta,\ell}}{\sigma_k^\ell \eta(b^m)}, \tag{14}$$

where $\Lambda_k \subset E_k$, $\eta(b) \not\sim_{\langle \sigma_1, \sigma_k \rangle} \eta'(b)$ if $\eta \neq \eta'$, $\sigma_k^\ell(b) \not\sim_{\langle \sigma_1 \rangle} \sigma_k^{\ell'}(b)$ if $\ell \neq \ell'$, and $\widetilde{a}_{\eta, T_\eta} \neq 0$ for each η. Furthermore, we assume that this representation is such that $T_\eta \geq 0$ is as small as possible. Note that $\sum_{\ell=0}^{T_\eta} \widetilde{a}_{\eta,\ell} / \sigma_k^\ell \eta(b^m) \in U_{\eta(b),m}^{\langle \sigma_1, \sigma_k \rangle}$. Recall that by our assumption $\Delta_k(f_1)$ is σ_1-summable. Then by Theorem 18, we have that $\Delta_k\left(\sum_{\ell=0}^{T_\eta} \widetilde{a}_{\eta,\ell} / \sigma_k^\ell \eta(b^m)\right)$ is σ_1-summable for each η. Now Lemma 25 implies that there exist integers S_η such that

$$\sigma_k^{T_\eta+1}\big(\eta(b)\big) = \sigma_1^{S_\eta}\big(\eta(b)\big), \tag{15}$$

$$\sigma_k^{T_\eta+1}(\widetilde{a}_{\eta,0}) = \sigma_1^{S_\eta}(\widetilde{a}_{\eta,0}), \tag{16}$$

$$\widetilde{a}_{\eta,\ell} = \sigma_k^\ell(\widetilde{a}_{\eta,0}), \quad \forall \ell \in \{0, \ldots, T_\eta\}. \tag{17}$$

Applying η^{-1} to both sides of Equation (15) yields $\sigma_k^{T_\eta+1}(b) = \sigma_1^{S_\eta}(b)$ since G is commutative. Since the $\sigma_k^\ell(b)$ are in distinct $\langle \sigma_1 \rangle$-orbits, we have $T_\eta = T_{\eta'}$ and $S_\eta = S_{\eta'}$ for any two $\eta, \eta' \in \Lambda_k$. Let $L_k := T_\eta + 1$ and $N_k := S_\eta$, then L_k is the minimal positive integer such that $\sigma_k^{L_k}(b) \sim_{\langle \sigma_1 \rangle} b$ and $\sigma_k^{L_k}(b) = \sigma_1^{N_k}(b)$. According to Equations (16) and (17), we have $\sigma_k^{L_k}(\widetilde{a}_{\eta,\ell}) = \sigma_1^{N_k}(\widetilde{a}_{\eta,\ell})$ for each η and ℓ. We observe that

$$\operatorname{res}_{\sigma_1}\big(f_1, \lambda(b), m\big) = [a_\lambda] \quad \text{and} \quad \operatorname{res}_{\sigma_1}\big(f_1, \sigma_k^\ell \eta(b), m\big) = [\widetilde{a}_{\eta,\ell}].$$

For each $\lambda \in \Lambda$, there exists a unique pair (η, ℓ) where $\eta \in \Lambda_k, \ell \in \{0, \ldots, T_\eta\}$ such that $\lambda(b) \sim_{\langle\sigma_1\rangle} \sigma_k^\ell \eta(b)$. By Definition 19 we have $a_\lambda \sim_{\langle\sigma_1\rangle} \widetilde{a}_{\eta,\ell}$. Now Lemma 24 implies that $\sigma_k^{L_k}(a_\lambda) = \sigma_1^{N_k}(a_\lambda)$. ☐

We are now ready to give the proof of Theorem 21.

Proof. We proceed by induction on n. For the base case $n = 1$, the theorem follows from the fact that any univariate rational function is a uniform WZ-form. For $n \geq 2$ suppose that the theorem holds for any WZ-forms in $n-1$ variables. As in Lemma 29, we focus on each component of the orbital decomposition of f_1 and rewrite it as in (14). Next we use the cyclic operator to describe f_1 in a more precise way as

$$f_1 = \Delta_1(f_{1,k}) + \frac{\sigma_k^{L_k} - 1}{\sigma_k - 1} \bullet \sum_{\eta \in \Lambda_k} \frac{\widetilde{a}_{\eta,0}}{\eta(b^m)}.$$

Suppose that $L_k, N_k \in \mathbb{Z}$ with $L_k \neq 0$ such that

$$\sigma_k^{L_k}\left(\frac{\widetilde{a}_{\eta,0}}{\eta(b^m)}\right) = \sigma_1^{N_k}\left(\frac{\widetilde{a}_{\eta,0}}{\eta(b^m)}\right).$$

For each $k \in \{2, \ldots, n\}$, let

$$f_k' = \Delta_k(f_{1,k}) + \frac{\sigma_1^{N_k} - 1}{\sigma_1 - 1} \bullet \sum_{\eta \in \Lambda_k} \frac{\widetilde{a}_{\eta,0}}{\eta(b^m)}.$$

Then one can easily check that $\Delta_k(f_1) = \Delta_1(f_k')$ with f_k' and f_1 being integer-linear of the same type. For $k, \ell \in \{2, \ldots, n\}$ with $k \neq \ell$, we have $\Delta_k(f_1) = \Delta_1(f_k')$ and $\Delta_\ell(f_1) = \Delta_1(f_\ell')$, from which it follows that

$$\Delta_\ell \Delta_1(f_k') = \Delta_\ell \Delta_k(f_1) = \Delta_k \Delta_1(f_\ell').$$

Thus $\Delta_1\big(\Delta_\ell(f_k') - \Delta_k(f_\ell')\big) = 0$, i.e., $\Delta_\ell(f_k') - \Delta_k(f_\ell') \in K(\widehat{\mathbf{x}})$. By construction, we have $f_{1,k} \in U_{b,m}^G$ and $f_2', \ldots, f_n' \in U_{b,m}^G$. By Lemma 17, also $\Delta_\ell(f_k') - \Delta_k(f_\ell')$ is an element of $U_{b,m}^G$. According to the definition of $U_{b,m}^G$ in (6), one has

$$U_{b,m}^G \cap K(\widehat{\mathbf{x}}) = \{0\}.$$

Thus $\Delta_\ell(f_k') - \Delta_k(f_\ell') = 0$. By Definition 10, $(f_1, f_2', \ldots, f_n')$ is a uniform WZ-form in $U_{b,m}^G$, denoted by $\omega_{i,j}$ for some i, j.

In conclusion, from the orbital decomposition of f_1, we can obtain a WZ-form $(f_1, f_2', \ldots, f_n')$ which is one exact WZ-form ω_0 plus several uniform WZ-forms $\omega_{i,j}$. Note that there may remain a WZ-form of the form $(0, f_2 - f_2', \ldots, f_n - f_n')$. From the compatibility conditions (4), we have for each $k \in \{2, \ldots, n\}$ that $\Delta_1(f_k - f_k') = \Delta_k(0) = 0$, so $f_k - f_k' \in K(\widehat{\mathbf{x}})$. Hence the remaining form can be viewed as an $(n-1)$-variable WZ-form w.r.t. $(\Delta_2, \ldots, \Delta_n)$. By the induction hypothesis, the proof is completed. ☐

Note that this decomposition is not unique, because of two aspects. When a WZ-form is both exact and uniform, see Remark 11, we choose to put it into the exact part, which minimizes the uniform part. Second, the final result depends on the operators in G that are chosen for the orbital decomposition. Next we give an example to illustrate how the decomposition works.

Example 30. *Let $\omega = (f, g, h) \in K(x, y, z)^3$ be a WZ-form with*

$$f = \sum_{\ell=0}^{3} \frac{1}{4x + 6y + 5z + \ell},$$
$$g = \sum_{\ell=0}^{5} \frac{1}{4x + 6y + 5z + \ell} + \sum_{\ell=0}^{2} \frac{1}{3y + 2z + \ell},$$
$$h = \sum_{\ell=0}^{4} \frac{1}{4x + 6y + 5z + \ell} + \sum_{\ell=0}^{1} \frac{1}{3y + 2z + \ell}.$$

It is easy to check that (f, g, h) satisfy the following compatibility conditions:

$$\{\Delta_y(f) = \Delta_x(g),\ \Delta_z(f) = \Delta_x(h),\ \Delta_z(g) = \Delta_y(h)\}.$$

Let $b = 4x+6y+5z$ be the same as in Example 20, while the rational function f here is different. In terms of b it can be written as

$$f = \frac{1}{b} + \frac{1}{\sigma_x^{-1}\sigma_z(b)} + \frac{1}{\sigma_x^{-1}\sigma_y(b)} + \frac{1}{\sigma_x^{-3}\sigma_z^3(b)},$$

but note that this representation is not unique. Similarly, let $c = 3y + 2z$ and rewrite

$$\sum_{\ell=0}^{2} \frac{1}{3y + 2z + \ell} = \frac{1}{c} + \frac{1}{\sigma_y^{-1}\sigma_z^2(c)} + \frac{1}{\sigma_z(c)}.$$

Then we can decompose ω into an exact WZ-form plus two uniform WZ-forms:

$$\begin{aligned}
f &= \Delta_x(a + \bar{a}) + \left(\Delta_x(a_2) + \frac{\sigma_y^2 - 1}{\sigma_y - 1} \cdot \frac{\sigma_z^2 - 1}{\sigma_z - 1} \bullet \frac{1}{b}\right) + \frac{\sigma_y^0 - 1}{\sigma_y - 1} \cdot \frac{\sigma_z^3 - 1}{\sigma_z - 1} \bullet \frac{1}{c} \\
&= \Delta_x(a + \bar{a}) + \left(\Delta_x(a_3) + \frac{\sigma_z^4 - 1}{\sigma_z - 1} \cdot \frac{\sigma_y - 1}{\sigma_y - 1} \bullet \frac{1}{b}\right) + \frac{\sigma_z^3 - 1}{\sigma_z - 1} \cdot \frac{\sigma_y^0 - 1}{\sigma_y - 1} \bullet \frac{1}{c}, \\
g &= \Delta_y(a + \bar{a}) + \left(\Delta_y(a_2) + \frac{\sigma_x^3 - 1}{\sigma_x - 1} \cdot \frac{\sigma_z^2 - 1}{\sigma_z - 1} \bullet \frac{1}{b}\right) + \frac{\sigma_x - 1}{\sigma_x - 1} \cdot \frac{\sigma_z^3 - 1}{\sigma_z - 1} \bullet \frac{1}{c}, \\
h &= \Delta_z(a + \bar{a}) + \left(\Delta_z(a_3) + \frac{\sigma_x^5 - 1}{\sigma_x - 1} \cdot \frac{\sigma_y - 1}{\sigma_y - 1} \bullet \frac{1}{b}\right) + \frac{\sigma_x - 1}{\sigma_x - 1} \cdot \frac{\sigma_y^2 - 1}{\sigma_y - 1} \bullet \frac{1}{c}.
\end{aligned}$$

where

$$a = -\frac{1}{\sigma_x^{-1}\sigma_z(b)} - \frac{1}{\sigma_x^{-1}\sigma_y(b)} - \frac{1}{\sigma_x^{-3}\sigma_z^3(b)} - \frac{1}{\sigma_x^{-2}\sigma_z^3(b)} - \frac{1}{\sigma_x^{-1}\sigma_z^3(b)},$$
$$a_2 = \frac{1}{\sigma_y\sigma_z(b)}, \quad a_3 = -\frac{1}{\sigma_x^{-1}\sigma_z^2(b)}, \quad \bar{a} = -\frac{1}{\sigma_y^{-1}\sigma_z^2(c)}.$$

As we can see, the first uniform WZ-form has type $(4, 6, 5)$*, while the second one has type* $(0, 3, 2)$*.*

6. Structure of uniform WZ-forms

Theorem 21 tells us how every WZ-form can be decomposed into exact and uniform WZ-forms. While exact WZ-forms are easy to describe and to construct, Definition 10 only allows us to check whether a given tuple is a uniform WZ-form, but this characterization is not explicit enough to construct such forms. In this section, we use a difference homomorphism in order to write a uniform WZ-form in terms of its integer-linear type and a single univariate rational function. Then we finish our proof of the additive Ore–Sato theorem.

Let $(A, \boldsymbol{\sigma})$ and $(A, \boldsymbol{\tau})$ be two difference rings, where $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$ and $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n)$. A homomorphism (resp. isomorphism) $\phi \colon A \to A$ is called a difference homomorphism (resp. isomorphism) from $(A, \boldsymbol{\sigma})$ to $(A, \boldsymbol{\tau})$ if $\phi \circ \sigma_i = \tau_i \circ \phi$ for each $i \in \{1, \dots, n\}$. In other words, for each i there is a commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\sigma_i} & A \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} \\ A & \xrightarrow{\tau_i} & A \end{array}$$

Lemma 31. *Given a unimodular matrix* $\mathbf{D} \in \mathbb{Z}^{n\times n}$*, i.e.,* $\mathbf{D}^{-1} \in \mathbb{Z}^{n\times n}$*, we define a ring isomorphism* $\phi \colon K(\mathbf{x}) \to K(\mathbf{x})$ *by* $\phi(\mathbf{x}) = \mathbf{D} \cdot \mathbf{x}$*. Furthermore, we let the* σ_i *act on vectors as* $\sigma_i(\mathbf{x}) = \mathbf{x} + \mathbf{e}_i$*, where* $\mathbf{e}_i$ *denotes the* i*-th unit vector. If we define* $\tau_i(\mathbf{x}) = \mathbf{x} + \mathbf{D}^{-1} \cdot \mathbf{e}_i$ *for all* $i \in \{1, \dots, n\}$*, then* ϕ *is a difference isomorphism from* $\big(K(\mathbf{x}), \boldsymbol{\sigma}\big)$ *to* $\big(K(\mathbf{x}), \boldsymbol{\tau}\big)$*.*

Proof. We have to check that $\phi \circ \sigma_i = \tau_i \circ \phi$. For the left-hand side we get

$$\phi(\sigma_i(f(\mathbf{x}))) = \phi(f(\mathbf{x} + \mathbf{e}_i)) = f(\mathbf{D} \cdot \mathbf{x} + \mathbf{e}_i),$$

and the right-hand side gives

$$\tau_i(\phi(f(\mathbf{x}))) = \tau_i\big(f(\mathbf{D} \cdot \mathbf{x})\big) = f\big(\mathbf{D} \cdot (\mathbf{x} + \mathbf{D}^{-1} \cdot \mathbf{e}_i)\big) = f\big(\mathbf{D} \cdot \mathbf{x} + \mathbf{e}_i\big).$$

This completes the proof. □

Given $f_1, \dots, f_n \in K(\mathbf{x})$ satisfying the compatibility conditions (4), then [6, Theorem 2] shows that there exists a difference ring extension $\big(K(\mathbf{x})[H], \boldsymbol{\sigma}\big)$ of $\big(K(\mathbf{x}), \boldsymbol{\sigma}\big)$, where H is a hyperarithmetic expression with certificates $f_1, \dots, f_n$. A difference homomorphism from $\big(K(\mathbf{x}), \boldsymbol{\sigma}\big)$ to $\big(K(\mathbf{x}), \boldsymbol{\tau}\big)$ can naturally be extended to the corresponding difference ring extensions.

Lemma 32 ([4, Proposition 9])**.** *For every integer vector* $\mathbf{v} = (v_1, \dots, v_n)$ *there is an integer matrix* $\mathbf{D} \in \mathbb{Z}^{n\times n}$ *with the first row* $\mathbf{v}$ *and* $\det(\mathbf{D}) = \gcd(v_1, \dots, v_n)$*.*

Next we use such a matrix $\mathbf{D}$ to construct the difference homomorphism.

Theorem 33. *Let* $\big(f_1(\mathbf{v}\cdot\mathbf{x}),\ldots,f_n(\mathbf{v}\cdot\mathbf{x})\big)$ *be a uniform WZ-form of type* $\mathbf{v}$*, then there exist constants* $\mu_1,\ldots,\mu_n \in K$ *and a univariate rational function* $r \in K(z)$ *such that for each* $i \in \{1,\ldots,n\}$*,*

$$f_i(\mathbf{v}\cdot\mathbf{x}) = \mu_i + \sum\nolimits_{\ell}{}_0^{v_i} r(\mathbf{v}\cdot\mathbf{x}+\ell).$$

Proof. Let $H(\mathbf{x})$ be a hyperarithmetic expression, and let $f_1(\mathbf{v}\cdot\mathbf{x}),\ldots,f_n(\mathbf{v}\cdot\mathbf{x})$ be its certificates. That is to say for each i,

$$\sigma_i\big(H(\mathbf{x})\big) = H(\mathbf{x}) + f_i(\mathbf{v}\cdot\mathbf{x}). \tag{18}$$

Without loss of generality, we can assume that $\gcd(v_1,\ldots,v_n)=1$. By Lemma 32, there exists an integer matrix $\mathbf{D}=(d_{ij})\in\mathbb{Z}^{n\times n}$ with $\det(\mathbf{D})=1$ whose first row is $\mathbf{v}$. Let $\phi\colon K(\mathbf{x})\to K(\mathbf{x})$ such that

$$\phi\big(f(\mathbf{x})\big) = f\big(\mathbf{D}^{-1}\cdot\mathbf{x}\big), \text{ for all } f(\mathbf{x})\in K(\mathbf{x}).$$

By Lemma 31, ϕ is a difference isomorphism from $\big(K(\mathbf{x})[H],\boldsymbol{\sigma}\big)$ to $\big(K(\mathbf{x})[H],\boldsymbol{\tau}\big)$, where $\tau_i(\mathbf{x}) = \mathbf{x}+\mathbf{D}\cdot\mathbf{e}_i$ for all i in $\{1,\ldots,n\}$. Applying the operator ϕ to Equation (18) yields

$$\begin{aligned}\phi\big(\sigma_i\big(H(\mathbf{x})\big)\big) &= \phi\big(H(\mathbf{x})\big)+\phi\big(f_i(\mathbf{v}\cdot\mathbf{x})\big),\\ \tau_i\big(\phi\big(H(\mathbf{x})\big) &= \phi\big(H(\mathbf{x})\big)+f_i(x_1).\end{aligned}$$

Let $H'(\mathbf{x}) = \phi\big(H(\mathbf{x})\big)$, then it follows that $\tau_i\big(H'(\mathbf{x})\big) = H'(\mathbf{x})+f_i(x_1)$. For any integer $m>0$ and $i\in\{1,\ldots,n\}$ we have

$$\tau_i^m\big(H'(\mathbf{x})\big) = H'(\mathbf{x}) + \sum_{j=0}^{m-1} f_i\big(x_1+jd_{1i}\big) =: H'(\mathbf{x})+f_{i,m}(x_1), \tag{19}$$

$$\tau_i^{-m}\big(H'(\mathbf{x})\big) = H'(\mathbf{x}) - \sum_{j=1}^{m} f_i\big(x_1-jd_{1i}\big) =: H'(\mathbf{x})+f_{i,-m}(x_1). \tag{20}$$

Let $\mathbf{D}^{-1} =: \big(\widetilde{d}_{ij}\big)_{n\times n}$, then for all $i\in\{1,\ldots,n\}$ we can rewrite σ_i in terms of $\tau_1,\ldots,\tau_n$:

$$\sigma_i = \prod_{j=1}^{n}\tau_j^{\widetilde{d}_{ji}}.$$

By applying (19) and (20) repeatedly, we obtain $\sigma_i\big(H'(\mathbf{x})\big) = H'(\mathbf{x})+f_i'(x_1)$ for some univariate rational function f_i'. That is to say, $\Delta_i\big(H'(\mathbf{x})\big) = f_i'(x_1)$. By the compatibility conditions (4) we have that $\Delta_1\big(f_k'(x_1)\big) = \Delta_k\big(f_1'(x_1)\big) = 0$, and thus $f_k'\in K$ for all $k\in\{2,\ldots,n\}$. Then an easy induction shows that

$$H'(\mathbf{x}) \simeq F'(x_1) + \sum_{k=2}^{n} f_k' x_k,$$

where $F'(x_1)$ is a solution of the difference equation $y(x_1+1)-y(x_1)=f'_1(x_1)$. Next, we can recover $H(\mathbf{x})$ as follows,

$$\begin{aligned} H(\mathbf{x}) &= \phi^{-1}\big(H'(\mathbf{x})\big) \\ &= H'\big(\mathbf{D}\cdot\mathbf{x}\big) \\ &\simeq F'(\mathbf{v}\cdot\mathbf{x}) + \sum_{k=2}^{n} f'_k\Bigg(\sum_{i=1}^{n} d_{ki}x_i\Bigg) \\ &= F'(\mathbf{v}\cdot\mathbf{x}) + \sum_{i=1}^{n}\Bigg(\sum_{k=2}^{n} f'_k d_{ki}\Bigg)x_i, \end{aligned}$$

where $F'(\mathbf{v}\cdot\mathbf{x}+1)-F'(\mathbf{v})=f'_1(\mathbf{v}\cdot\mathbf{x})$. Write that $\mu_i := \sum_{k=2}^{n} f'_k d_{ki}$. Then for each $i\in\{1,\dots,n\}$,

$$f_i(\mathbf{v}\cdot\mathbf{x}) = \Delta_i\big(H(\mathbf{x})\big) = \begin{cases} \mu_i + \displaystyle\sum_{\ell=0}^{v_i-1} f'_1(\mathbf{v}\cdot\mathbf{x}+\ell), & \text{if } v_i>0; \\ \mu_i, & \text{if } v_i = 0; \\ \mu_i - \displaystyle\sum_{\ell=v_i}^{-1} f'_1(\mathbf{v}\cdot\mathbf{x}+\ell), & \text{if } v_i<0. \end{cases}$$

Finally we let the univariate rational function r be defined as f'_1. □

Eventually we obtain Theorem 4 by combining Theorems 21 and 33. Note that we can disregard the μ_i since the constant tuple $(\mu_1,\dots,\mu_n)$ itself can be viewed as an exact WZ-form. We show that any hyperarithmetic expression can be described, up to conjugation, as a rational function plus a K-linear combination of polygamma functions. First we employ the partial fraction decomposition on the univariate function r over K:

$$r(z) = \sum_s \sum_t \frac{\beta_{s,t}}{(z+\alpha_s)^t},$$

where $\alpha_s, \beta_{s,t}\in K$ and $s,t\in\mathbb{N}$, both with finite support.

According to the recurrence formula of polygamma functions [13, (5.15)]

$$\psi^{(t)}(z+1)-\psi^{(t)}(z) = \frac{(-1)^t t!}{z^{t+1}}, \quad t=0,1,\dots$$

we have

$$\psi^{(t)}(z+\alpha_s+1)-\psi^{(t)}(z+\alpha_s) = \frac{(-1)^t t!}{(z+\alpha_s)^{t+1}}.$$

Then the hyperarithmetic expression H' with certificates

$$\Big(\sum\nolimits_{\ell}{}_{0}^{v_1} r(\mathbf{v}\cdot\mathbf{x}+\ell),\dots,\sum\nolimits_{\ell}{}_{0}^{v_n} r(\mathbf{v}\cdot\mathbf{x}+\ell)\Big)$$

is conjugate to

$$\sum_s \sum_t \frac{\beta_{s,t+1}}{(-1)^t t!} \psi^{(t)}(\mathbf{v} \cdot \mathbf{x} + \alpha_s).$$

Corollary 34. *Any hyperarithmetic expression is conjugate to*

$$a + \sum_{\mathbf{v} \in V} \sum_s \sum_t \beta_{\mathbf{v},s,t} \psi^{(t)}(\mathbf{v} \cdot \mathbf{x} + \alpha_{\mathbf{v},s}),$$

where $a \in K(\mathbf{x})$*,* $V \subset \mathbb{Z}^n$*,* $s, t \in \mathbb{N}$*, and for each* $\mathbf{v}$*, we have* $\beta_{\mathbf{v},s,t}, \alpha_{\mathbf{v},s} \in K$*.*

Example 35. *Let* H *be a hyperarithmetic expression with certificates* (f, g, h) *as in Example 30. Then* H *is conjugate to* $\psi^{(0)}(4x + 6y + 5z) + \psi^{(0)}(3y + 2z)$*.*

7. An algorithm for the minimal decomposition of WZ-forms

Now we will present an algorithm for computing additive representations of WZ-forms based on the recursive idea in the proof of Theorem 4. Furthermore, we require that in such a representation the uniform WZ-forms are minimal, which we call "the minimal decomposition". Therefore with this algorithm we can decide the exactness of WZ-forms by checking whether the uniform part is zero or not.

Definition 36 (Additive representation)**.** *Given a WZ-form* $\omega = (f_1, \ldots, f_n)$*, there is a decomposition of the form*

$$\omega = \big(\Delta_1(a), \ldots, \Delta_n(a)\big) + \sum_{\mathbf{v} \in V} \Big(\sum_{\ell}{}_0^{v_1} r_{\mathbf{v}}(\mathbf{v} \cdot \mathbf{x} + \ell), \ldots, \sum_{\ell}{}_0^{v_n} r_{\mathbf{v}}(\mathbf{v} \cdot \mathbf{x} + \ell) \Big).$$

We call the list $\big(a, V, \{r_{\mathbf{v}}\}_{\mathbf{v} \in V}\big)$ *an* additive representation *of* ω*. This decomposition is called "minimal "if it has minimal degree in the denominator of each* $r_{\mathbf{v}}(z)$*.*

Let $\omega = (f_1, \ldots, f_n) \in K(\mathbf{x})^n$ be a WZ-form. First, we apply Abramov's reduction [2] with respect to the variable x_1 to decompose f_1 into

$$f_1 = \Delta_1(g_0) + \sum_{i=1}^{I} \sum_{j=1}^{J} \frac{a_{i,j}}{b_i^j},$$

where $g_0 \in K(\widehat{\mathbf{x}})[x_1]$, $a_{i,j}, b_i \in K[\widehat{\mathbf{x}}][x_1]$ with $\deg_{x_1}(a_{i,j}) < \deg_{x_1}(b_i)$, and the b_i are in distinct $\langle \sigma_1 \rangle$-orbits.

By Lemma 29, each $a_{i,j}/b_i^j$ is integer-linear of some type $\mathbf{v}_i$. In order to compute the type of each simple fraction in the above decomposition, we are reduced to the following problem.

Problem 37 (Integer-linear testing)**.** *Given a polynomial* $p \in K[\mathbf{x}]$*, decide whether there exist* $u \in K[z]$ *and* $\mathbf{v} \in \mathbb{Z}^n$ *such that* $p = u(\mathbf{v} \cdot \mathbf{x})$*.*

This problem can be solved by the algorithm `IntegerLinearDecomp` [20]. Applying it to the numerator and the denominator of each simple fraction $a_{i,j}/b_i^j$ yields

$$\frac{a_{i,j}}{b_i^j} = u_{i,j}(\mathbf{v}_i \cdot \mathbf{x}),$$

where $u_{i,j} \in K(z)$ and $\mathbf{v}_i \in \mathbb{Z}^n$ with the first entry $v_{i,1}$ being nonzero. By collecting the simple fractions of the same type, we obtain

$$f_1 = \Delta_1(g_0) + \sum_{\mathbf{v} \in V} u_{\mathbf{v}}(\mathbf{v} \cdot \mathbf{x}),$$

where $V \subset \mathbb{Z}^n$ is a finite set and $u_{\mathbf{v}} \in K(z)$ for each $\mathbf{v} \in V$. The next step is to write the rational function $u_{\mathbf{v}}$ into the form

$$u_{\mathbf{v}}(z) = \sum\nolimits_{\ell}{}_0^{v_1} r_{\mathbf{v}}(z+\ell),$$

where $r_{\mathbf{v}} \in K(z)$. Note that $r_{\mathbf{v}}$ must be a rational solution of the difference equation

$$y(z+v_1) - y(z) = u_{\mathbf{v}}(z+1) - u_{\mathbf{v}}(z),$$

which can also be solved by Abramov's reduction.

Let $\omega_0 := \big(\Delta_1(g_0), \ldots, \Delta_n(g_0)\big)$ and $\omega_{\mathbf{v}} := \big(f_{1,\mathbf{v}}, \ldots, f_{n,\mathbf{v}}\big)$, where for each $k \in \{1, \ldots, n\}$,

$$f_{k,\mathbf{v}} := \sum\nolimits_{\ell}{}_0^{v_k} r_{\mathbf{v}}(\mathbf{v} \cdot \mathbf{x} + \ell).$$

Then ω can be written as a summation of one exact WZ-form, several uniform WZ-forms and a "degenerate" WZ-form:

$$\omega = \omega_0 + \sum_{\mathbf{v} \in V} \omega_{\mathbf{v}} + \widetilde{\omega}.$$

In order to get the minimal uniform WZ-forms in the sense that the denominator of each $r_{\mathbf{v}}$ has the lowest possible degree in z, we require the application of Abramov's reduction on $r_{\mathbf{v}}$ again. The following example illustrates that this step is needed to minimize $r_{\mathbf{v}}$.

Example 38. *Let $\omega = (f, g, h) \in K(x, y, z)^3$ be a WZ-form with*

$$f = \Delta_x\left(\frac{1}{(4x+6y+5z+1)^2}\right) + \sum_{\ell=0}^{3} \frac{1}{4x+6y+5z+\ell},$$

$$g = \Delta_y\left(\frac{1}{(4x+6y+5z+1)^2}\right) + \sum_{\ell=0}^{5} \frac{1}{4x+6y+5z+\ell},$$

$$h = \Delta_z\left(\frac{1}{(4x+6y+5z+1)^2}\right) + \sum_{\ell=0}^{4} \frac{1}{4x+6y+5z+\ell}.$$

Since the residue of Abramov's reduction is not unique, we may get the following decomposition of f,

$$f = \Delta_x\left(\frac{1}{(4x+6y+5z+1)^2} - \frac{1}{4x+6y+5z+1}\right)$$
$$+\frac{1}{4x+6y+5z} + \frac{1}{4x+6y+5z+5} + \frac{1}{4x+6y+5z+2} + \frac{1}{4x+6y+5z+3}.$$

The univariate function in Z corresponding to the uniform WZ-form is

$$r_{\mathbf{v}}(Z) = \frac{1}{Z} - \frac{1}{Z+1} + \frac{1}{Z+2}.$$

Obviously we would anticipate it to be $1/Z$. This accident can happen because $4x+6y+5z+1$ and $4x+6y+5z+5$ are in the same σ_x-orbit, and Abramov's reduction on f cannot see which one will lead to the minimal $r_{\mathbf{v}}$. However, we observe that the difference is the summable part of $r_{\mathbf{v}}$ with respect to Z, which finally can be removed by modifying the exact part of f. In this case,

$$\sum_{\ell=0}^{3}\left(\frac{1}{Z+2+\ell} - \frac{1}{Z+1+\ell}\right) = \frac{1}{Z+5} - \frac{1}{Z+1}.$$

After substituting Z with $4x+6y+5z$ on the right-hand side of the above equation, we obtain $\Delta_x\big(\frac{1}{4x+6y+5z+1}\big)$. The modification is done by absorbing it into the previous exact part of f.

If $\widetilde{\omega}$ is nonzero, then we proceed with the induction step by repeating the above process for $\widetilde{\omega}$ which only involves $(n-1)$-variables. The above process for computing additive representations of WZ-forms is summarized in Algorithm 1 and is illustrated in Example 39. Note that ω is exact if and only if the output is $(a, \varnothing, \varnothing)$, which is equivalent to that the uniform part is zero. Our Maple code for implementing Algorithm 1 is available at

`http://www.mmrc.iss.ac.cn/~schen/AddOreSato.html`

We provide a worst-case complexity analysis of Algorithm 1 in terms of arithmetic operations in the base field. Let $K[\mathbf{x}]_d$ denote the set of polynomials in $K[\mathbf{x}]$ whose degree in x_i is no more than d for each $i = 1, 2, \ldots, n$. Let $K(\mathbf{x})_d$ denote the set of rational functions in $K(\mathbf{x})$ with numerators and denominators in $K[\mathbf{x}]_d$. The "big Oh" notation O is referred to the cost of an algorithm "up to a constant factor" and the "soft Oh" notation $\widetilde{O}$ may further neglect some logarithmic factors. We define the max-norm of the multivariate polynomial $f = \sum_{i_1,\ldots,i_n} f_{i_1,\ldots,i_n} x_1^{i_1}\cdots x_n^{i_n} \in \mathbb{Q}[\mathbf{x}]$ as

$$\|f\|_\infty = \max_{i_1,\ldots,i_n} |f_{i_1,\ldots,i_n}|.$$

We first recall some known facts on complexity estimates. The irreducible factorization of a univariate polynomial $f \in \mathbb{Z}[x]$ of degree $d \geq 1$ with max-norm

$\|f\|_\infty = A$ takes $\widetilde{O}(d^6 \cdot (d + \log A))$ operations in $\mathbb{Z}$ (see [16, Theorem 16.23]). The partial fraction decomposition of an element in $K(x)_d$ with a given factorization of its denominator takes no more than $\widetilde{O}(d)$ operations in K (see [16, Section 5.11]). Multi-point evaluation and interpolation in $K[\mathbf{x}]_d$ from the given values on $O(d^m)$ points which form an m-dimensional tensor product grid can be done with $\widetilde{O}(md^m)$ operations in K (see [25, Theorem 1]). Let $f(\mathbf{x}) = \prod_{\ell=1}^s \bar{f}_\ell(\mathbf{x})$ be a multivariate factorization of $f(\mathbf{x}) \in \mathbb{Z}[\mathbf{x}]$ over $\mathbb{Q}$. Then Mahler's bound in [28] using Gelfond's inequality leads to

$$\prod_{\ell=1}^{s} \|\bar{f}_\ell\|_\infty \leq 2^{nd}(d+1)^{n/2}\|f\|_\infty.$$

According to [20], the multivariate integer-linear decomposition of $f \in \mathbb{Z}[\mathbf{x}]$ with $\|f\|_\infty = A$ takes $(n + \log A + d)^{O(1)}$ word operations.

Let $f_1, \ldots, f_n \in \mathbb{Z}(\mathbf{x})_d$ with their max-norms bounded by $A \in \mathbb{N}$. We claim that the total cost of Algorithm 1 is $\widetilde{O}\big(n^{O(1)}d^{O(n)} \log A\big)$ operations in $\mathbb{Q}$. The cost of the algorithm before the recursive loop (line 30 of Algorithm 1) is dominated by the irreducible partial fraction decomposition of $f_1 \in \mathbb{Q}(\mathbf{x})_d$ with respect to x_1, where all simple fractions are of degree in x_j, $j = 2, \ldots, n$ bounded by d^2. To obtain partial fraction decompositions with respect to x_1 over $\mathbb{Q}(x_2, \ldots, x_n)$, we take the strategy of multi-point evaluation and interpolation, i.e., first evaluating the rational function for variables $(x_2, \ldots, x_n)$ at many points, performing the univariate irreducible partial fraction decompositions with respect to x_1, and then recovering the desired decompositions over $\mathbb{Q}(x_2, \ldots, x_n)$ by rational interpolation. To be sufficient for recovering the final result, we need to evaluate $x_2, \ldots, x_n$ at $O(d^{2(n-1)})$ many good points which takes $\widetilde{O}\big((n-1)d^{2(n-1)}\big)$ operations in $\mathbb{Q}$. For each point, the univariate irreducible partial fraction decompositions takes $\widetilde{O}(d^6 \cdot (d + \log A))$ operations in $\mathbb{Q}$, which is the dominating cost of the irreducible factorization of the denominator. Finally, we need the rational interpolation to recover $O(d)$ coefficients in $\mathbb{Q}(x_2, \ldots, x_n)$, where each coefficient costs $\widetilde{O}\big((n-1)d^{2(n-1)}\big)$ operations in $\mathbb{Q}$. Thus before line 30, the total cost is

$$\widetilde{O}\big((n-1)d^{2(n-1)} \cdot d + d^{2(n-1)} \cdot d^6 \cdot (d + \log A)\big).$$

From the special structure of WZ-forms, we observe that the degree bound will not be changed in the recursive steps, and the max-norms of denominators appeared in irreducible partial fraction decompositions are uniformly bounded by $\widetilde{A} := 2^{nd}(d+1)^{n/2}A$. The exact parts during the computation will always be expressed in a sparse representation. This idea was used to get a polynomial-time algorithm for univariate rational summation in [18]. Let $T(n, d, A)$ denote the worst-case running cost of Algorithm 1 in terms of operations in $\mathbb{Q}$. According to the previous discussions, we have

$$T(n, d, A) = \widetilde{O}\left((n-1)d^{2n-1} + d^{2n+4} \cdot (d + \log A)\right) + \widetilde{T}(n-1, d, \widetilde{A}),$$

where $\widetilde{T}(n-1,d,\widetilde{A})$ satisfies the recursive formulae

$$\widetilde{T}(m-1,d,\widetilde{A}) = \widetilde{O}\big((m-2)d^{2m-3} + d^{2m+2}\cdot(d+\log\widetilde{A})\big) + \widetilde{T}(m-2,d,\widetilde{A})$$

for all $m = 3,\ldots,n$. By solving the recurrence relation, we conclude that $T(n,d,A)$ is $\widetilde{O}\big(n^{O(1)}d^{O(n)}\log A\big)$.

Example 39. *Let $\omega = (f,g,h) \in K(x,y,z)^3$ be a WZ-form with respect to $(\Delta_x,\Delta_y,\Delta_z)$, specifically,*

$$\begin{aligned} f &= \frac{xyz - y^2z - yz^2 + yz - 1}{x-y-z+1},\\ g &= \frac{x^2z - xyz - xz^2 + xy - y^2 - yz - 1}{x-y-z},\\ h &= \frac{x^2y - xy^2 - xyz + xz - yz - z^2 - 1}{x-y-z}. \end{aligned}$$

Employing Abramov's reduction on f yields

$$f = \Delta_x(xyz) + \frac{1}{-x+y+z-1}.$$

Then we record the following exact WZ-form as a part of ω:

$$\omega_0 := \big(\Delta_x(xyz), \Delta_y(xyz), \Delta_z(xyz)\big).$$

Obviously from the decomposition of f there is only one integer-linear type $\mathbf{v} = (-1,1,1)$ and the corresponding univariate rational function is $r_{\mathbf{v}} = 1/Z$. It is easily checked that there is no summable part in $r_{\mathbf{v}}$. Then a uniform WZ-form shows up as a part of ω:

$$\omega_{\mathbf{v}} = \Big(\frac{1}{-x+y+z-1}, \frac{1}{-x+y+z}, \frac{1}{-x+y+z}\Big).$$

Then we can update ω by subtracting ω_0 and $\omega_{\mathbf{v}}$ and obtain $\widetilde{\omega} = (0,y,z)$, which is equivalent to the WZ-pair (y,z) with respect to (Δ_y,Δ_z). By simple manipulations we can see that it is an exact WZ-pair:

$$\Big(\Delta_y\big(\tfrac{1}{2}y^2 + \tfrac{1}{2}z^2\big), \Delta_z\big(\tfrac{1}{2}y^2 + \tfrac{1}{2}z^2\big)\Big).$$

Combining this exact WZ-form with the previous one we can update ω_0 as:

$$\omega_0 = \Big(\Delta_x\big(xyz + \tfrac{1}{2}y^2 + \tfrac{1}{2}z^2\big), \Delta_y\big(xyz + \tfrac{1}{2}y^2 + \tfrac{1}{2}z^2\big), \Delta_z\big(xyz + \tfrac{1}{2}y^2 + \tfrac{1}{2}z^2\big)\Big).$$

Finally the decomposition works as $\omega = \omega_0 + \omega_{\mathbf{v}}$, i.e., the additive representation of ω is

$$\Big(xyz + \tfrac{1}{2}y^2 + \tfrac{1}{2}z^2, \{(-1,1,1)\}, \{1/Z\}\Big).$$

Algorithm 1 WZ-form decomposition algorithm

Function: `WZFormDecomp`$\big((f_1,\dots,f_n),\mathbf{x},Z\big)$
Input: WZ-form $(f_1,\dots,f_n)\in K(\mathbf{x})^n$, $\mathbf{x}=(x_1,\dots,x_n)$, and a new variable Z
Output: Its additive representation: $\big(a,V,R=\{r_{\mathbf{v}}\}_{\mathbf{v}\in V}\big)$

```
if f_1 = 0 then
    (a, V, R) ← WZFormDecomp((f_2, ..., f_n), (x_2, ..., x_n), Z)
    for v = (v_2, ..., v_n) in V do
        v ← (0, v_2, ..., v_n)
    end for
    return (a, V, R)
end if
Call AbramovReduction: f_1 = Δ_1(g_0) + Σ_{i=1}^{I} Σ_{j=1}^{J} a_{i,j}/b_i^j,  a ← g_0
if n = 1 then
    return (g_0, ((1)), (f_1 − Δ_1(g_0)))
end if
for 1 ≤ i ≤ I do
    Call IntegerLinearDecomp: b_i = q_i(w_i · x) with q_i ∈ K[Z]
end for
V ← (v_1, ..., v_m) with {v_1, ..., v_m} = {w_1, ..., w_I} and v_i ≠ v_j for i ≠ j
for 1 ≤ k ≤ m do
    u_k ← 0
    for 1 ≤ i ≤ I do
        if the integer-linear type of b_i is v_k = (v_{k,1}, ..., v_{k,n}) then
            for 1 ≤ j ≤ J do
                Perform the substitution v_k · x → Z in a_{i,j} so that a_{i,j} ∈ K[Z]
                u_k ← u_k + a_{i,j}/q_i^j
            end for
        end if
    end for
    Call AbramovReduction: σ_z(h_k) − h_k = u_k(v_{k,1}z + 1) − u_k(v_{k,1}z)
    r_k ← h_k(1/v_{k,1}Z),  r_k = Δ_Z(g_k) + r̃_k, a ← a + g_k(v_k · x)
end for
R ← (r̃_1, ..., r̃_m)
for 2 ≤ k ≤ n do
    f'_k ← f_k − Δ_k(a) − Σ_{i=1}^{m} Σ_{ℓ=0}^{v_{i,k}} r̃_i(v_i · x + ℓ)
end for
if f'_k ≠ 0 for some k then
    (a', V', R') ← WZFormDecomp((f'_2, ..., f'_n), (x_2, ..., x_n), Z)
    for v' = (v_2, ..., v_n) in V' do
        v' ← (0, v_2, ..., v_n)
    end for
    a ← a + a',  V ← Join(V, V'),  R ← Join(R, R')
end if
return (a, V, R)
```

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