On the Structure of Compatible Rational Functions*

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ABSTRACT

A finite number of rational functions are compatible if they satisfy the compatibility conditions of a first-order linear functional system involving differential, shift and q-shift operators. We present a theorem that describes the structure of compatible rational functions. The theorem enables us to decompose a solution of such a system as a product of a rational function, several symbolic powers, a hyperexponential function, a hypergeometric term, and a q-hypergeometric term. We outline an algorithm for computing this product, and present an application.

Categories and Subject Descriptors

I.1.2 [**Computing Methodologies**]: Symbolic and Algebraic Manipulation—*Algebraic Algorithms*

General Terms

Algorithms, Theory

Keywords

Compatibility conditions, compatible rational functions, hyperexponential function, (q-)hypergeometric term

1. INTRODUCTION

A linear functional system consists of linear partial differential, shift and q-shift operators. The commutativity of these operators implies that the coefficients of a linear functional system satisfy compatibility conditions.

A nonzero solution of a first-order linear partial differential system in one unknown function is called a hyperexponential function. Christopher and Zoladek [9, 21] use the compatibility (integrability) conditions to show that a hyperexponential function can be written as a product of a rational function, finitely many power functions, and an exponential function. Their results generalize a well-known fact, namely, for a rational function $r(t)$,

$$\exp\left(\int r(t)dt\right) = f(t)r_1(t)^{e_1}\cdots r_m(t)^{e_m}\exp(g(t)),$$

where $e_1, \ldots, e_m$ are constants, and $f, r_1, \ldots, r_m, g$ are rational functions. The generalization is useful to compute Liouvilian first integrals.

A nonzero solution of a first-order linear partial difference system in one unknown term is called a hypergeometric term. The Ore-Sato Theorem [16, 18] states that a hypergeometric term is a product of a rational function, several power functions and factorial terms. A q-analogue of the Ore-Sato theorem is given in [11, 8]. All these results are based on compatibility conditions. The Ore-Sato theorem was rediscovered in one way or another, and is important for the proofs of a conjecture of Wilf and Zeilberger about holonomic hypergeometric terms [2, 4, 17]. This theorem and its q-analogue also play a crucial role in deriving criteria on the existence of telescopers for hypergeometric and q-hypergeometric terms, respectively [1, 8].

Consider a first-order mixed system

$$\left\{\frac{\partial z(t,x)}{\partial t} = u(t,x)z(t,x),\ z(t,x+1) = v(t,x)z(t,x)\right\},$$

where u and v are rational functions with $v \neq 0$. Its compatibility condition is $\partial v(t,x)/\partial t = v(t,x)(u(t,x+1) - u(t,x))$. By Proposition 5 in [10], a nonzero solution of the above system can be written as a product $f(t,x)r(t)^x\mathcal{E}(t)\mathcal{G}(x)$, where f is a bivariate rational function in t and x, r is a univariate rational function in t, $\mathcal{E}$ is a hyperexponential function in t, and $\mathcal{G}$ is a hypergeometric term in x. This proposition is used to compute Liouvillian solutions of difference-differential systems.

In fact, the above proposition is also fundamental for the criteria on the existence of telescopers when both differential and shift operators are involved [7]. This motivates us to generalize the proposition to include differential, difference and q-difference cases. Such a generalization will enable us to establish the existence of telescopers when both differential (shift) and q-shift operators appear. Next, the proof of the Wilf-Zeilberger conjecture for hypergeometric terms is

*This work was supported in part by two grants of NSFC No. 60821002/F02 and No. 10901156. The first author was a PhD student in the Chinese Academy of Sciences and INRIA, Paris-Rocquencourt when the first draft of this paper was written. He is now a post doctoral fellow at RISC-Linz, and acknowledges the financial support by Austrian FWF grant Y464-N18

based on the Ore-Sato theorem. So it is reasonable to expect that a structural theorem on compatible rational functions with respect to differential, shift and q-shift operators helps us study the conjecture in more general cases.

The main result of this paper is Theorem 5.4 which reveals a special structure of compatible rational functions. By the theorem, a hyperexponential-hypergeometric solution, defined in Section 2, is a product of a rational function, several symbolic powers, a hyperexponential function, a hypergeometric term, and a q-hypergeometric term (see Proposition 6.1). This paves the way to decompose such solutions by Christopher-Zoladek's generalization, the Ore-Sato Theorem, and its q-analogue.

This paper is organized as follows. The notion of compatible rational functions is introduced in Section 2. The bivariate case is studied in Section 3. After presenting a few preparation lemmas in Section 4, we prove in Section 5 a theorem that describes the structure of compatible rational functions. Section 6 is about algorithms and applications.

2. COMPATIBLE RATIONAL FUNCTIONS

In the rest of this paper, $\mathbb{F}$ is a field of characteristic zero. Let $\mathbf{t} = (t_1, \ldots, t_l)$, $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{y} = (y_1, \ldots, y_n)$. Assume that $q_1, \ldots, q_n \in \mathbb{F}$ are neither zero nor roots of unity. For an element f of $\mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$, define $\delta_i(f) = \partial f/\partial t_i$ for all i with $1 \le i \le l$,

$$\sigma_j(f(\mathbf{t}, \mathbf{x}, \mathbf{y})) = f(\mathbf{t}, x_1, \ldots, x_{j-1},\, x_j + 1,\, x_{j+1}, \ldots, x_m, \mathbf{y})$$

for all j with $1 \le j \le m$, and

$$\tau_k(f(\mathbf{t}, \mathbf{x}, \mathbf{y})) = f(\mathbf{t}, \mathbf{x}, y_1, \ldots, y_{k-1},\, q_k y_k,\, y_{k+1}, \ldots, y_n)$$

for all k with $1 \le k \le n$. They are called derivations, shift operators, and q-shift operators, respectively.

Let $\Delta = \{\delta_1, \ldots, \delta_l, \sigma_1, \ldots, \sigma_m, \tau_1, \ldots, \tau_n\}$. These operators commute pairwise. The field of constants w.r.t. an operator in Δ consists of all rational functions free of the indeterminate on which the operator acts nontrivially.

By a first-order linear functional system over $\mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$, we mean a system consisting of

$$\delta_i(z) = u_i z,\ \sigma_j(z) = v_j z,\ \tau_k(z) = w_k z \tag{1}$$

for some rational functions $u_i, v_j, w_k \in \mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$ and for all i, j, k with $1 \le i \le l$, $1 \le j \le m$ and $1 \le k \le n$. System (1) is said to be *compatible* if

$$v_1 \cdots v_m w_1 \cdots w_n \neq 0 \tag{2}$$

and the conditions listed in (3)-(8) hold:

$$\delta_i(u_j) = \delta_j(u_i), \quad 1 \le i < j \le l, \tag{3}$$

$$\sigma_i(v_j)/v_j = \sigma_j(v_i)/v_i, \quad 1 \le i < j \le m, \tag{4}$$

$$\tau_i(w_j)/w_j = \tau_j(w_i)/w_i, \quad 1 \le i < j \le n, \tag{5}$$

$$\delta_i(v_j)/v_j = \sigma_j(u_i) - u_i, \quad 1 \le i \le l \text{ and } 1 \le j \le m, \tag{6}$$

$$\delta_i(w_k)/w_k = \tau_k(u_i) - u_i, \quad 1 \le i \le l \text{ and } 1 \le k \le n, \tag{7}$$

$$\sigma_j(w_k)/w_k = \tau_k(v_j)/v_j, \quad 1 \le j \le m \text{ and } 1 \le k \le n. \tag{8}$$

Compatibility conditions (3)-(8) are caused by the commutativity of the maps in Δ. A sequence of rational functions: $u_1, \ldots, u_l, v_1, \ldots, v_m, w_1, \ldots, w_n$ is said to be *compatible* w.r.t. Δ if (2)-(8) hold.

By a Δ-extension of $\mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$, we mean a ring extension R of $\mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$ s.t. every derivation and automorphism in Δ can be extended to a derivation and a monomorphism from R to R, and, moreover, the extended maps are commutative with each other. Given a finite number of first-order compatible systems, one can construct a Picard-Vessiot Δ-extension of $\mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$ that contains "all" solutions of these systems. Moreover, every nonzero solution is invertible. Details on Picard-Vessiot extensions of compatible systems may be found in [5]. More general and powerful extensions are described in [12]. By a *hyperexponential-hypergeometric solution h over* $\mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$, we mean a nonzero solution of the system (1). The coefficients u_i, v_j and w_k in (1) are called *δ_i-, σ_j-, and τ_k-certificates* of h, respectively. For brevity, we abbreviate "hyperexponential-hypergeometric solution" as "H-solution". An H-solution is a hyperexponential function when $m = n = 0$ in (1), it is a hypergeometric term if $l = n = 0$, and a q-hypergeometric term if $l = m = 0$.

Remark 2.1. *We opt for the word "solution" rather than "function", since all the t_i, x_j and y_k are regarded as indeterminates. It is more sophisticated to regard hypergeometric terms as functions of integer variables [17, 4, 3].*

As a matter of notation, for an element $f \in \mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$, the denominator and numerator of f are denoted $\mathrm{den}(f)$ and $\mathrm{num}(f)$, respectively. Note that $\mathrm{den}(f)$ and $\mathrm{num}(f)$ are coprime. For a ring $\mathbb{A}$, $\mathbb{A}^\times$ stands for $\mathbb{A}\backslash\{0\}$, and for a field $\mathbb{E}$, $\overline{\mathbb{E}}$ stands for the algebraic closure of $\mathbb{E}$. For every $\phi \in \Delta$ and $f \in \mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})^\times$, we denote by $\ell\phi(f)$ the fraction $\phi(f)/f$. When ϕ is a derivation δ_i, $\ell\delta_i(f)$ stands for the logarithmic derivative of f with respect to t_i. This notation allows us to avoid stacking fractions and subscripts.

Let $\mathbb{E}$ be a field and t an indeterminate. A nonzero element f of $\mathbb{E}(t)$ can be written uniquely as $f = p + r$, where $p \in \mathbb{E}[t]$ and r is a proper fraction. We say that p is the polynomial part of f w.r.t. t.

Remark 2.2. *Let $z \in \{t_1, \ldots, t_l, x_1, \ldots, x_m, y_1, \ldots, y_n\}$ and $f \in \mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})^\times$. For all i with $1 \le i \le l$, the polynomial part of $\ell\delta_i(f)$ w.r.t. z has degree at most zero in z.*

3. BIVARIATE CASE

In this section, we assume that $l = m = n = 1$. For brevity, set $t = t_1$, $x = x_1$, $y = y_1$, $\delta = \delta_1$, $\sigma = \sigma_1$, $\tau = \tau_1$, and $q = q_1$. By (2), (6), (7) and (8), three rational functions u, v, w in $\mathbb{F}(t, x, y)$ are Δ-compatible if $vw \neq 0$,

$$\ell\delta(v) = \sigma(u) - u, \tag{9}$$

$$\ell\delta(w) = \tau(u) - u, \tag{10}$$

$$\ell\sigma(w) = \ell\tau(v). \tag{11}$$

Other compatibility conditions become trivial in this case.

Example 3.1. *Let $\alpha \in \mathbb{F}(t, y)^\times$. The system consisting of $\delta(z) = \ell\delta(\alpha)\, x\, z$ and $\sigma(z) = \alpha\, z$ is compatible w.r.t. δ and σ. Denote a solution of this system by α^x, which is irrational if $\alpha \neq 1$.*

The next lemma is immediate from [10, Proposition 5].

Lemma 3.1. *Let $u, v \in \mathbb{F}(t, x, y)$ with $v \neq 0$. If (9) holds, then $u = \ell\delta(f) + \ell\delta(\alpha)\, x + \beta$ and $v = \ell\sigma(f)\, \alpha\lambda$ for some f in $\mathbb{F}(t, x, y)$, α, β in $\mathbb{F}(t, y)$, and λ in $\mathbb{F}(x, y)$.*

Assume that an H-solution h has δ-certificate u and σ-certificate v. By Lemma 3.1, $h = cf\alpha^x\mathcal{E}\mathcal{G}$ in some Δ-ring, where c is a constant w.r.t. δ and σ, $\mathcal{E}$ is hyperexponential with certificate β, and $\mathcal{G}$ is hypergeometric with certificate λ.

We shall prove two similar results: one is about differential and q-shift variables; the other about shift and q-shift ones. To this end, we recall some terminologies from [2, 4, 12].

Let $\mathbb{A}=\mathbb{F}(t,y)$ and $p \in \mathbb{A}[x]^\times$. The *$\sigma$-dispersion of p* is defined to be the largest nonnegative integer i s.t. for some r in $\overline{\mathbb{A}}$, r and $r+i$ are roots of p. Let $f \in \mathbb{A}(x)^\times$. We say that f is *σ-reduced* if $\mathrm{den}(f)$ and $\sigma^i(\mathrm{num}(f))$ are coprime for every integer i; and that f is *σ-standard* if zero is the σ-dispersion of $\mathrm{num}(f)\mathrm{den}(f)$. A σ-standard rational function is a σ-reduced one, but the converse is false. By Lemma 6.2 in [12], $f = \ell\sigma(a)\, b$ for some a, b in $\mathbb{A}(x)$ with b being σ-standard or σ-reduced.

Let $\mathbb{B} = \mathbb{F}(t,x)$ and $p \in \mathbb{B}[y]^\times$. The *$\tau$-dispersion of p* is defined to be the largest nonnegative integer i s.t. for some *nonzero* $r \in \overline{\mathbb{B}}$, r and $q^i r$ are roots of p. In addition, the τ-dispersion of p is set to be zero if $p = cy^k$ for some $c \in \mathbb{B}$. Let $f \in \mathbb{B}(y)^\times$. The *polar τ-dispersion* is the τ-dispersion of $\mathrm{den}(f)$. The notion of τ-reduced and τ-standard rational functions are defined likewise. One can write $f = \ell\tau(a)\, b$, where $a, b \in \mathbb{B}(y)^\times$ and b is τ-standard or τ-reduced.

Now, we prove a q-analogue of Lemma 3.1.

LEMMA 3.2. *Let $u, w\in\mathbb{F}(t,x,y)$ with $w\neq 0$. If* (10) *holds, then $u=\ell\delta(f)+a$ and $w=\ell\tau(f)\, b$ for some f in $\mathbb{F}(t,x,y)$, a in $\mathbb{F}(t,x)$, and b in $\mathbb{F}(x,y)$.*

PROOF. Set $w = \ell\tau(f)\, b$ for some f, b in $\mathbb{F}(t,x,y)$ with b being τ-standard. Set $b = y^k P/Q$, where $P, Q \in \mathbb{F}(x)[t,y]$ are coprime, and neither is divisible by y. Since b is τ-standard, so is P/Q. Assume $u = \ell\delta(f) + a$. By (10),

$$\ell\delta(P/Q) = \tau(a) - a. \tag{12}$$

Since P/Q is τ-standard, the τ-dispersion of PQ is zero, and so is the polar τ-dispersion of the left-hand side in (12), which, together with [12, Lemma 6.3], implies that a belongs to $\mathbb{F}(t,x)[y,y^{-1}]$. Moreover, a is free of positive powers of y by Remark 2.2 (setting $z = y$); and a is free of negative powers of y, because neither P nor Q is divisible by y. We conclude that a is in $\mathbb{F}(t,x)$. Consequently, $\tau(a) = a$. It follows from (12) that $\delta(P/Q) = 0$, i.e., b is in $\mathbb{F}(x,y)$. $\square$

By the above lemma, an H-solution h can be written as a product of a constant w.r.t. δ and τ, a rational function, a hyperexponential function, and a q-hypergeometric term.

The last lemma is a q-analogue of [4, Theorem 9]. Our proof is based on an easy consequence of [20, Lemma 2.1].

FACT 3.1. *Let $a, b \in \mathbb{F}(t,x,y)^\times$. If $\sigma(a) = ba$, and P is an irreducible factor of $\mathrm{den}(b)$ with $\deg_x P > 0$, then $\sigma^i(P)$ is a factor of $\mathrm{num}(b)$ for some nonzero integer i.*

The same is true if we swap $\mathrm{den}(b)$ and $\mathrm{num}(b)$ in the above assertion.

LEMMA 3.3. *Let $v, w \in \mathbb{F}(t,x,y)^\times$. If* (11) *holds, then $v = \ell\sigma(f)\, a$ and $w = \ell\tau(f)\, b$ for some f in $\mathbb{F}(t,x,y)$, a in $\mathbb{F}(t,x)$, and b in $\mathbb{F}(t,y)$.*

PROOF. In this proof, $P \mid Q$ means that $P, Q \in \mathbb{F}(t)[x,y]^\times$ and $Q = PR$ for some $R \in \mathbb{F}(t)[x,y]$.

Set $v = \ell\sigma(f)\, a$, where $f, a \in \mathbb{F}(t,x,y)$ and a is σ-reduced. Assume $w = \ell\tau(f)\, b$. By (11), $\ell\sigma(b) = \ell\tau(a)$, that is,

$$\sigma(b) = gb, \quad \text{where } g = \frac{\tau(\mathrm{num}(a))\,\mathrm{den}(a)}{\tau(\mathrm{den}(a))\,\mathrm{num}(a)}. \tag{13}$$

First, we show that a is the product of an element in $\mathbb{F}(t,x)$ and an element in $\mathbb{F}(t,y)$. Suppose the contrary. Then there is an irreducible polynomial $P \in \mathbb{F}(t)[x,y]$ with $\deg_x P{>}0$ and $\deg_y P{>}0$ s.t. P divides $\mathrm{den}(a)\mathrm{num}(a)$ in $\mathbb{F}(t)[x,y]$. Assume that $P \mid \mathrm{num}(a)$. If $P \nmid \mathrm{den}(g)$, then $P \mid \tau(\mathrm{num}(a))$ since $\mathrm{num}(a)$ and $\mathrm{den}(a)$ are coprime. So $\tau^{-1}(P) | \mathrm{num}(a)$. If $P | \mathrm{den}(g)$, then $\sigma^i(P) \mid \mathrm{num}(g)$ for some integer i by (13) and Fact 3.1. Thus, $\sigma^i(P) \mid \tau(\mathrm{num}(a))$, because $\mathrm{num}(g)$ is a factor of $\tau(\mathrm{num}(a))\mathrm{den}(a)$ and a is σ-reduced. This implies $\sigma^i\tau^{-1}(P) \mid \mathrm{num}(a)$. In either case, we have that

$$\sigma^j\tau^{-1}(P) \mid \mathrm{num}(a) \quad \text{for some integer } j.$$

Assume $P \mid \mathrm{den}(a)$. Then the same argument implies

$$\sigma^k\tau^{-1}(P) \mid \mathrm{den}(a) \quad \text{for some integer } k.$$

Hence, there exists an integer m_1 s.t. $P_1 := \sigma^{m_1}\tau^{-1}(P_0)$ is an irreducible factor of $\mathrm{den}(a)\mathrm{num}(a)$, where $P_0 = P$. A repeated use of the above reasoning leads to an infinite sequence of irreducible polynomials $P_0, P_1, P_2, \ldots$ in $\mathbb{F}(t)[x,y]$ s.t. $P_i = \sigma^{m_i}\tau^{-1}(P_{i-1})$ and $P_i \mid \mathrm{den}(a)\mathrm{num}(a)$. Therefore, there are two $\mathbb{F}(t)$-linearly dependent members in the sequence. Using these two members, we get $P_0 = c\sigma^m\tau^n(P_0)$ for some c in $\mathbb{F}(t)$ and m, n in $\mathbb{Z}$ with $n \neq 0$. Write

$$P_0 = p_d(x)y^d + p_{d-1}(x)y^{d-1} + \cdots + p_0(x),$$

where $d > 0$, $p_i \in \mathbb{F}(t)[x]$ and $p_d \neq 0$. Then

$$p_d(x) = cp_d(x+m)q^{-dn} \quad \text{and} \quad p_0(x) = cp_0(x+m).$$

Since P_0 is irreducible and of positive degree in x, p_0 is also nonzero. We see that $1 = cq^{-dn}$ and $1 = c$ when comparing the leading coefficients in the above two equalities. Consequently, q is a root of unity, a contradiction. This proves that all irreducible factors of $\mathrm{den}(a)\mathrm{num}(a)$ are either in $\mathbb{F}(t)[x]$ or $\mathbb{F}(t)[y]$. Therefore, a is a product of an element in $\mathbb{F}(t,x)$ and an element in $\mathbb{F}(t,y)$.

So we can write $a = a_1\, a_2$ for some a_1 in $\mathbb{F}(t,x)$ and a_2 in $\mathbb{F}(t,y)$. By $\ell\sigma(b) = \ell\tau(a)$, the equation $\sigma(z) = \ell\tau(a_2)z$ has a rational solution b. Since $\ell\tau(a_2)$ is a constant w.r.t. σ, we conclude $\ell\tau(a_2) = 1$, for otherwise, $\sigma(z) = \ell\tau(a_2)z$ would have no rational solution. So $b \in \mathbb{F}(t,y)$ and $a \in \mathbb{F}(t,x)$. $\square$

Similar to Lemmas 3.1 and 3.2, the above lemma implies that an H-solution h can be written as a product of a constant w.r.t. σ and τ, a rational function, a hypergeometric term, and a q-hypergeometric term.

We shall extend these lemmas to multivariate cases in Section 5. Before closing this section, we present three examples to illustrate calculations involving compatibility conditions. These calculations are useful in Section 5.

EXAMPLE 3.2. *Assume*

$$u = \ell\delta(f) + \ell\delta(a)\,x + b \quad \text{and} \quad v = \ell\sigma(f)\,a\,c,$$

where $f, c \in \mathbb{F}(t,x,y)^\times$, $a \in \mathbb{F}(t,y)^\times$, and $b \in \mathbb{F}(t,x,y)$. By the logarithmic derivative identity: for all r, s in $\mathbb{F}(t,x,y)^\times$, $\ell\delta(r\,s) = \ell\delta(r) + \ell\delta(s)$, we get

$$\ell\delta(v) = \ell\delta \circ \ell\sigma(f) + \ell\delta(a) + \ell\delta(c).$$

Since $\ell\delta(a)$ is constant w.r.t. σ, and $\sigma \circ \ell\delta = \ell\delta \circ \sigma$, we have

$$\begin{aligned}\sigma(u) - u &= \sigma \circ \ell\delta(f) - \ell\delta(f) + \ell\delta(a) + \sigma(b) - b \\ &= \ell\delta \circ \ell\sigma(f) + \ell\delta(a) + \sigma(b) - b.\end{aligned}$$

If (9) *holds, then* $\ell\delta(c)=\sigma(b)-b$. *Hence,* $\delta(c)=0$ *iff* $\sigma(b)=b$, *i.e.,* $c \in \mathbb{F}(x,y)$ *iff* $b \in \mathbb{F}(t,y)$.

Example 3.3. *Assume* $u = \ell\delta(f) + a$ *and* $w = \ell\tau(f)\, b$, *where* $a \in \mathbb{F}(t,x,y)$ *and* $f, b \in \mathbb{F}(t,x,y)^\times$. *If* (10) *holds, then a similar calculation as above yields* $\ell\delta(b) = \tau(a) - a$. *Hence,* $\delta(b) = 0$ *iff* $\tau(a) = a$, *i.e.,* $b \in \mathbb{F}(x,y)$ *iff* $a \in \mathbb{F}(t,x)$.

Example 3.4. *Assume* $v = \ell\sigma(f)\, a$ *and* $w = \ell\tau(f)\, b$, *where* $f, a, b \in \mathbb{F}(t,x,y)^\times$. *Applying* $\ell\sigma, \ell\tau$ *to* w, v, *respectively, we see that*

$$\ell\sigma(w) = \ell\sigma \circ \ell\tau(f)\, \ell\sigma(b), \quad \ell\tau(v) = \ell\tau \circ \ell\sigma(f)\, \ell\tau(a).$$

If (11) *holds, then* $\ell\sigma(b) = \ell\tau(a)$, *because* $\ell\sigma \circ \ell\tau = \ell\tau \circ \ell\sigma$. *Hence,* $\sigma(b) = b$ *iff* $\tau(a) = a$, *i.e.,* $b \in \mathbb{F}(t,y)$ *iff* $a \in \mathbb{F}(t,x)$.

4. PREPARATION LEMMAS

To extend Lemmas 3.1, 3.2, and 3.3 to multivariate cases, we will proceed by induction on the number of variables. There arise different expressions for a rational function in our induction. Lemmas given in this section will be used to eliminate redundant indeterminates in these expressions.

We define a few additive subgroups of $\mathbb{F}(\mathbf{t},\mathbf{x},\mathbf{y})$ to avoid complicated expressions.

$$L_i = \left\{ \ell\delta_i(f) \mid f \in \mathbb{F}(\mathbf{t},\mathbf{x},\mathbf{y})^\times \right\}, \ i = 1, \ldots, l,$$

$$M_i = \left\{ \sum_{j=1}^{m} \ell\delta_i(g_j)\, x_j \mid g_j \in \mathbb{F}(\mathbf{t},\mathbf{y})^\times \right\}, \ i = 1, \ldots, l.$$

For $i = 1, \ldots, l$ and $j = 1, \ldots, m$, $M_{i,j}$ denotes the group

$$\left\{ \sum_{k=1}^{j-1} \ell\delta_i(g_k)\, x_k + \sum_{k=j+1}^{m} \ell\delta_i(g_k)\, x_k \mid g_k \in \mathbb{F}(\mathbf{t}, x_j, \mathbf{y})^\times \right\}.$$

Moreover, we set

$$N_i = L_i + M_i + \mathbb{F}(\mathbf{t},\mathbf{y}) \quad \text{and} \quad N_{i,j} = L_i + M_{i,j} + \mathbb{F}(\mathbf{t}, x_j, \mathbf{y}).$$

Let $Z = \{t_1, \ldots, t_l, x_1, \ldots, x_m, y_1, \ldots, y_n\}$. We will use an evaluation trick in the sequel. Let $Z' = \{z_1, \ldots, z_s\}$ be a subset of Z. For $f \in \mathbb{F}(\mathbf{t},\mathbf{x},\mathbf{y})^\times$, there exist $\xi_1, \ldots, \xi_s$ in $\mathbb{F}$ s.t. f evaluated at $z_1 = \xi_1, \ldots, z_s = \xi_s$ is a well-defined and nonzero rational function f'. We say that f' is a *proper evaluation* of f w.r.t. Z'. A proper evaluation can be carried out for finitely many rational functions as well. In addition, we say that a rational function f is free of Z' if it is free of every indeterminate in Z'.

Remark 4.1. *If* $Z' \subset Z$, $f \in L_i$ *and* $t_i \notin Z'$, *then all proper evaluations of* f *w.r.t.* Z' *are also in* L_i.

In the next example, we illustrate two typical proper evaluations to be used later.

Example 4.2. *Let* $f = \ell\delta_i(r)$ *for some* $f, r \in \mathbb{F}(\mathbf{t},\mathbf{x},\mathbf{y})^\times$. *Assume that both* $f(\mathbf{t},\boldsymbol{\xi},\mathbf{y})$ *and* $r(\mathbf{t},\boldsymbol{\xi},\mathbf{y})$ *are well-defined and nonzero, where* $\boldsymbol{\xi} \in \mathbb{F}^m$. *Then* $f(\mathbf{t},\boldsymbol{\xi},\mathbf{y})$ *is still in* L_i.

Let $g \in \mathbb{F}(\mathbf{t},\mathbf{y})^\times$. *Then* $\delta_i(z) = gz$ *has a rational solution in* $\mathbb{F}(\mathbf{t},\mathbf{y})^\times$ *if it has a rational solution in* $\mathbb{F}(\mathbf{t},\mathbf{x},\mathbf{y})^\times$. *This can also be shown by a proper evaluation.*

The following lemma helps us merge rational expressions involving logarithmic derivatives.

Lemma 4.1. *Let* $i \in \{1, \ldots, l\}$.

(i) Let $Z_1, Z_2 \subset Z$ *with* $Z_1 \cap Z_2 = \emptyset$. *If* $\mathbb{A}$ *is any subfield of* $\mathbb{F}(\mathbf{t},\mathbf{x},\mathbf{y})$ *whose elements are free of* t_i *and free of* $Z_1 \cup Z_2$, *then*

$$L_i + \mathbb{A}(t_i) = (L_i + \mathbb{A}(t_i, Z_1)) \cap (L_i + \mathbb{A}(t_i, Z_2)).$$

(ii) If $d, e \in \{1, \ldots, m\}$ *with* $d \neq e$, *then* $N_i = N_{i,d} \cap N_{i,e}$.

Proof. To prove the first assertion, note that $L_i + \mathbb{A}(t_i)$ is a subset of $(L_i + \mathbb{A}(t_i, Z_1)) \cap (L_i + \mathbb{A}(t_i, Z_2))$. Assume that a is in $(L_i + \mathbb{A}(t_i, Z_1)) \cap (L_i + \mathbb{A}(t_i, Z_2))$. Then there exist $a_1 \in \mathbb{A}(t_i, Z_1)$ and $a_2 \in \mathbb{A}(t_i, Z_2)$ s.t.

$$a \equiv a_1 \mod L_i \quad \text{and} \quad a \equiv a_2 \mod L_i.$$

Hence, $a_1 - a_2 \in L_i$. Let $Z_2' = Z_2 \setminus \{t_i\}$, and a_2' be a proper evaluation of a_2 w.r.t. Z_2'. Then $a_1 - a_2'$ is a proper evaluation of $a_1 - a_2$ w.r.t. Z_2', because a_1 is free of Z_2'. Thus, $a_1 - a_2'$ belongs to L_i by Remark 4.1. Since a_2' is in $\mathbb{A}(t_i)$, a_1 is in $L_i + \mathbb{A}(t_i)$, and so is a.

To prove the second assertion, assume $i = 1$, $d = 1$ and $e = m$. Note that $N_1 \subset N_{1,1} \cap N_{1,m}$, because M_1 is contained in $(M_{1,1} + \mathbb{F}(\mathbf{t}, x_1, \mathbf{y})) \cap (M_{1,m} + \mathbb{F}(\mathbf{t}, x_m, \mathbf{y}))$. It remains to show $N_{1,1} \cap N_{1,m} \subset N_1$. Let $a \in N_{1,1} \cap N_{1,m}$. Then

$$a = \ell\delta_1(f) + \left(\textstyle\sum_{j=2}^{m-1} \ell\delta_1(g_j)\, x_j \right) + \ell\delta_1(g_m)\, x_m + r \quad (14)$$

$$= \ell\delta_1(\tilde{f}) + \ell\delta_1(\tilde{g}_1)\, x_1 + \left(\textstyle\sum_{j=2}^{m-1} \ell\delta_1(\tilde{g}_j)\, x_j \right) + \tilde{r}, \quad (15)$$

where $f, \tilde{f} \in \mathbb{F}(\mathbf{t},\mathbf{x},\mathbf{y})$, $g_j, r \in \mathbb{F}(\mathbf{t}, x_1, \mathbf{y})$, $\tilde{g}_j, \tilde{r} \in \mathbb{F}(\mathbf{t}, x_m, \mathbf{y})$ and $f\tilde{f}g_j\tilde{g}_j \neq 0$. For all j with $1 \leq j \leq m$, let P_j be the polynomial part of a w.r.t. x_j. Then $\deg_{x_j} P_j \leq 1$ for all j with $1 \leq j \leq m-1$ by Remark 2.2 and (15), and $\deg_{x_m} P_m \leq 1$ by the same Remark and (14).

Claim. Let b_j denote the coefficient of x_j in P_j. Then there exists $s_j \in \mathbb{F}(\mathbf{t},\mathbf{y})$ s.t. $b_j = \ell\delta_1(s_j)$ for all j with $1 \leq j \leq m$.

Proof of Claim. By (14) and Remark 2.2, b_1 is the coefficient of x_1 in the polynomial part of r w.r.t. x_1. So b_1 is in $\mathbb{F}(\mathbf{t},\mathbf{y})$. By (15) and the same remark, $b_1 = \ell\delta_1(\tilde{g}_1)$. Let s_1 be a proper evaluation of $\tilde{g}_1$ w.r.t. x_m. Then $b_1 = \ell\delta_1(s_1)$ as b_1 is free of x_m. By the same argument, $b_m = \ell\delta_1(s_m)$ for some s_m in $\mathbb{F}(\mathbf{t},\mathbf{x})$. By (14) and (15), $b_j = \ell\delta_1(g_j) = \ell\delta_1(\tilde{g}_j)$ for all j with $2 \leq j \leq m-1$. Let s_j be a proper evaluation of $\tilde{g}_j$ w.r.t. x_m. Then $\ell\delta_j(g_j) = \ell\delta_j(s_j)$, because g_j is free of x_m. Hence, $b_j = \ell\delta_1(s_j)$. The claim holds.

Set $b = \sum_{j=1}^{m} b_j x_j$. Then $a - b$ is in $L_1 + \mathbb{F}(\mathbf{t}, x_1, \mathbf{y})$ and $L_1 + \mathbb{F}(\mathbf{t}, x_m, \mathbf{y})$ by (14), (15) and the claim. Thus, $a - b$ is in $L_1 + \mathbb{F}(\mathbf{t},\mathbf{y})$ by the first assertion (setting $Z_1 = \{x_1\}$, $Z_2 = \{x_m\}$, and $\mathbb{A} = \mathbb{F}(t_2, \ldots, t_l, \mathbf{y})$). By the claim, b is in M_1. Thus, a is in $L_1 + M_1 + \mathbb{F}(\mathbf{t},\mathbf{y})$. $\square$

We define a few multiplicative subgroups in $\mathbb{F}(\mathbf{t},\mathbf{x},\mathbf{y})^\times$. Let $G_j = \{\ell\sigma_j(f) \mid f \in \mathbb{F}(\mathbf{t},\mathbf{x},\mathbf{y})^\times\}$ for $j = 1, \ldots, m$. Similarly, let $H_k = \{\ell\tau_k(f) \mid f \in \mathbb{F}(\mathbf{t},\mathbf{x},\mathbf{y})^\times\}$ for $k = 1, \ldots, n$.

Remark 4.3. *If* $Z' \subset Z$, $f \in H_k$ *and* $y_k \notin Z'$, *then all proper evaluations of* f *w.r.t.* Z' *are in* H_k. *The same holds for* G_j.

The next lemma helps us merge rational expressions involving shift or q-shift quotients.

Lemma 4.2. *Let* $j \in \{1, \ldots, m\}$, $k \in \{1, \ldots, n\}$. *Assume that* Z_1 *and* Z_2 *are disjoint subsets of* Z.

(i) If $\mathbb{A}$ is any subfield of $\mathbb{F}(\mathbf{t},\mathbf{x},\mathbf{y})$ whose elements are free of x_j and free of $Z_1 \cup Z_2$, then

$$G_j\mathbb{A}(x_j)^\times = \left(G_j\mathbb{A}(x_j,Z_1)^\times\right) \cap \left(G_j\mathbb{A}(x_j,Z_2)^\times\right).$$

(ii) If $\mathbb{A}$ is any subfield of $\mathbb{F}(\mathbf{t},\mathbf{x},\mathbf{y})$ whose elements are free of y_k and free of $Z_1 \cup Z_2$, then

$$H_k\mathbb{A}(y_k)^\times = \left(H_k\mathbb{A}(y_k,Z_1)^\times\right) \cap \left(H_k\mathbb{A}(y_k,Z_2)^\times\right).$$

(iii) If $\mathbb{A} = \mathbb{F}(\mathbf{t},\mathbf{y})$ and $\mathbb{B} = \mathbb{F}(\mathbf{x},\mathbf{y})$, then

$$G_j\mathbb{A}^\times\mathbb{B}^\times = \left(G_j\mathbb{A}^\times\mathbb{B}(Z_1)^\times\right) \cap \left(G_j\mathbb{A}^\times\mathbb{B}(Z_2)^\times\right).$$

PROOF. The proofs of the first two assertions are similar to that of Lemma 4.1 (i). So we only outline the proof of the second assertion. Clearly,

$$H_k\mathbb{A}(y_k)^\times \subset \left(H_k\mathbb{A}(y_k,Z_1)^\times\right) \cap \left(H_k\mathbb{A}(y_k,Z_2)^\times\right).$$

For an element $a \in \left(H_k\mathbb{A}(y_k,Z_1)^\times\right) \cap \left(H_k\mathbb{A}(y_k,Z_2)^\times\right)$, there exist $a_1 \in \mathbb{A}(y_k,Z_1)^\times$ and $a_2 \in \mathbb{A}(y_k,Z_2)^\times$ s.t.

$$a \equiv a_1 \mod H_k \quad \text{and} \quad a \equiv a_2 \mod H_k.$$

Using a proper evaluation, one sees that a is in $H_k\mathbb{A}(y_k)^\times$.

We present a detailed proof of the third assertion due to the presence of both $\mathbb{A}$ and $\mathbb{B}$, though the idea goes along the same line as before. It suffices to show that the intersection of $G_j\mathbb{A}^\times\mathbb{B}(Z_1)^\times$ and $G_j\mathbb{A}^\times\mathbb{B}(Z_2)^\times$ is a subset of $G_j\mathbb{A}^\times\mathbb{B}^\times$. Assume that a is in the intersection. Then

$$a \equiv a_1 b_1 \mod G_j \quad \text{and} \quad a \equiv a_2 b_2 \mod G_j \tag{16}$$

for some a_1, a_2 in $\mathbb{A}^\times$, b_1 in $\mathbb{B}(Z_1)^\times$, and b_2 in $\mathbb{B}(Z_2)^\times$. Let $Z_2' = Z_2 \setminus \mathbb{B}$, and c be a proper evaluation of $a_1/(a_2b_2)$ w.r.t. Z_2'. Then $c\,b_1$ is a proper evaluation of $a_1b_1/(a_2b_2)$ w.r.t. Z_2', as b_1 is free of Z_2'. So $c\,b_1$ is in G_j by Remark 4.3. Since c is in $\mathbb{A}^\times\mathbb{B}^\times$, b_1 is in $G_j\mathbb{A}^\times\mathbb{B}^\times$, and so is a. $\square$

The next lemma says that some compatible rational functions belong to a common coset.

LEMMA 4.3. *Let $v_1, \ldots, v_m, w_1, \ldots, w_n \in \mathbb{F}(\mathbf{t},\mathbf{x},\mathbf{y})^\times$. Assume that the compatibility conditions in (4) and (5) hold.*

(i) If v_j is in $G_j\mathbb{F}(\mathbf{t},\mathbf{y})^\times\mathbb{F}(\mathbf{x},\mathbf{y})^\times$ for all j with $1 \le j \le m$, then there exists $f \in \mathbb{F}(\mathbf{t},\mathbf{x},\mathbf{y})$ s.t. each v_j is in the coset $\ell\sigma_j(f)\mathbb{F}(\mathbf{t},\mathbf{y})^\times\mathbb{F}(\mathbf{x},\mathbf{y})^\times$.

(ii) Let $\mathbb{E}$ be a subfield of $\mathbb{F}(\mathbf{t},\mathbf{x})$. If $w_k \in H_k\mathbb{E}(\mathbf{y})^\times$ for all k with $1 \le k \le n$, then there exists $f \in \mathbb{F}(\mathbf{t},\mathbf{x},\mathbf{y})$ s.t. each w_k is in the coset $\ell\tau_k(f)\mathbb{E}(\mathbf{y})^\times$.

PROOF. We are going to show the second assertion. The first one can be proved in the same fashion.

The second assertion clearly holds when $n = 1$. Assume that $n > 1$ and the lemma holds for $n-1$. Then there exist $g \in \mathbb{F}(\mathbf{t},\mathbf{x},\mathbf{y})$ and $b_1, \ldots, b_{n-1} \in \mathbb{E}(\mathbf{y})$ s.t. $w_k = \ell\tau_k(g)b_k$ for all k with $1 \le k \le n-1$. Assume

$$w_n = \ell\tau_n(g)\,a \quad \text{for some } a \in \mathbb{F}(\mathbf{t},\mathbf{x},\mathbf{y}). \tag{17}$$

Then the compatibility conditions in (5) imply that the first-order system $\{\tau_k(z) = \ell\tau_n(b_k)\,z \mid k = 1, \ldots, n-1\}$ has a solution a in $\mathbb{F}(\mathbf{t},\mathbf{x},\mathbf{y})^\times$. It follows from the hypothesis $b_k \in \mathbb{E}(\mathbf{y})$ for all k with $1 \le k \le n-1$ that the above system has a solution a' in $\mathbb{E}(\mathbf{y})^\times$. Thus, $a = c\,a'$ for some constant c w.r.t. $\tau_1, \ldots, \tau_{n-1}$. Consequently, c belongs to $\mathbb{F}(\mathbf{t},\mathbf{x},y_n)$. On one hand, (17) leads to

$$w_n = \ell\tau_n(g)\,ca'. \tag{18}$$

On the other hand, $w_n \in H_n\mathbb{E}(\mathbf{y})^\times$ implies $c = \ell\tau_n(s)\,r$ for some r in $\mathbb{E}(\mathbf{y})$ and s in $\mathbb{F}(\mathbf{t},\mathbf{x},\mathbf{y})$. Let $Z' = \{y_1, \ldots, y_{n-1}\}$, and let s' and r' be two proper evaluations of s and r w.r.t. Z' at a point in $\mathbb{F}^{n-1}$, respectively. Then $c = \ell\tau_n(s')\,r'$ since c is free of Z'. By (18), $w_n = \ell\tau_n(s'g)\,r'a'$. Set $f = s'g$ and $b_n = r'a'$. Then $w_k = \ell\tau_k(f)\,b_k$ for all k with $1 \le k \le n$, as s' is a constant w.r.t. $\tau_1, \ldots, \tau_{n-1}$. $\square$

5. A STRUCTURE THEOREM

In this section, we extend Lemmas 3.1, 3.2 and 3.3, and then combine these results to a structure theorem on Δ-compatible rational functions.

The first proposition extends Lemma 3.1.

PROPOSITION 5.1. *Let $u_1, \ldots, u_l, v_1, \ldots, v_m$ be rational functions in $\mathbb{F}(\mathbf{t},\mathbf{x},\mathbf{y})$ with $v_1 \cdots v_m \neq 0$. If the compatibility conditions in (3), (4) and (6) hold, then there exist f in $\mathbb{F}(\mathbf{t},\mathbf{x},\mathbf{y})$, $a_1, \ldots, a_m, b_1, \ldots, b_l$ in $\mathbb{F}(\mathbf{t},\mathbf{y})$, and $c_1, \ldots, c_m$ in $\mathbb{F}(\mathbf{x},\mathbf{y})$ s.t., for all i with $1 \le i \le l$,*

$$u_i = \ell\delta_i(f) + \ell\delta_i(a_1)\,x_1 + \cdots + \ell\delta_i(a_m)\,x_m + b_i,$$

and, for all j with $1 \le j \le m$,

$$v_j = \ell\sigma_j(f)\,a_j\,c_j.$$

Moreover, the sequence $b_1, \ldots, b_l, c_1, \ldots, c_m$ is compatible w.r.t. $\{\delta_1, \ldots, \delta_l, \sigma_1, \ldots, \sigma_m\}$.

PROOF. First, we consider the case in which $l = 1$ and m arbitrary. The proposition holds when $m = 1$ by Lemma 3.1. Assume that $m > 1$ and the proposition holds for the values lower than m. Applying the induction hypothesis to $t_1, x_1, \ldots, x_{m-1}$ and to $t_1, x_2, \ldots, x_m$, respectively, we see that both $u_1 \in N_{1,m}$ and $u_1 \in N_{1,1}$. Since $m > 1$, $u_1 \in N_1$ by Lemma 4.1 (ii). Hence,

$$u_1 = \ell\delta_1(f) + \ell\delta_1(a_1)\,x_1 + \cdots + \ell\delta_1(a_m)x_m + b_1$$

for some $f \in \mathbb{F}(t_1,\mathbf{x},\mathbf{y})$ and $a_1, \ldots, a_m, b_1 \in \mathbb{F}(t_1,\mathbf{y})$. Assume that $v_j = \ell\sigma_j(f)\,a_jc_j$. Then $c_1, \ldots, c_m$ are in $\mathbb{F}(\mathbf{x},\mathbf{y})$ by the compatibility conditions in (6) (see Example 3.2). The proposition holds for $l = 1$ and m arbitrary.

Second, we show that the proposition holds for all l and m by induction on l. It holds if $l = 1$ by the preceding paragraph. Assume that $l > 1$ and that the proposition holds for the values lower than l. Applying the induction hypothesis to $t_1, \ldots, t_{l-1}, \mathbf{x}$ and to $t_2, \ldots, t_l, \mathbf{x}$, respectively, we have

$$v_j \in \left(G_j\mathbb{A}^\times\mathbb{B}(Z_1)^\times\right) \cap \left(G_j\mathbb{A}^\times\mathbb{B}(Z_2)^\times\right),$$

where $\mathbb{A}=\mathbb{F}(\mathbf{t},\mathbf{y})$, $\mathbb{B}=\mathbb{F}(\mathbf{x},\mathbf{y})$, $Z_1=\{t_l\}$, and $Z_2=\{t_1\}$. We see that $v_j \in G_j\mathbb{A}^\times\mathbb{B}^\times$ by Lemma 4.2 (iii). So $v_j \in \ell\sigma_j(f)\mathbb{A}^\times\mathbb{B}^\times$ for some f in $\mathbb{F}(\mathbf{t},\mathbf{x},\mathbf{y})$ by Lemma 4.3 (i). Thus,

$$v_j = \ell\sigma_j(f)\,a_j\,c_j,$$

where $a_j \in \mathbb{A}$, $c_j \in \mathbb{B}$ and $j = 1, \ldots, m$. Assume that, for all i with $1 \le i \le l$, $u_i = \ell\delta_i(f) + \sum_{j=1}^m \ell\delta_i(a_j)\,x_j + b_i$. All the b_i's belong to $\mathbb{F}(\mathbf{t},\mathbf{y})$ by the compatibility conditions in (6) (see Example 3.2). The sequence $b_1, \ldots, b_l, c_1, \ldots, c_m$ is compatible because of (3), (4) and (6). $\square$

The second proposition extends Lemma 3.2.

PROPOSITION 5.2. *Let $u_1, \ldots, u_l, w_1, \ldots, w_n$ be rational functions in $\mathbb{F}(\mathbf{t},\mathbf{x},\mathbf{y})$ with $w_1 \cdots w_n \neq 0$. Assume that the compatibility conditions (3), (5) and (7) hold. Then* *there exist f in $\mathbb{F}(\mathbf{t},\mathbf{x},\mathbf{y})$, $a_1,\ldots,a_l$ in $\mathbb{F}(\mathbf{t},\mathbf{x})$, and $b_1,\ldots,b_n$ in $\mathbb{F}(\mathbf{x},\mathbf{y})$ s.t.*

$$u_i = \ell\delta_i(f) + a_i \quad \text{and} \quad w_k = \ell\tau_k(f)\, b_k$$

for all i with $1 \le i \le l$ and k with $1 \le k \le n$. Moreover, the sequence $a_1,\ldots,a_l$, b_1, ..., b_n is compatible w.r.t. the set $\{\delta_1,\ldots,\delta_l,\tau_1,\ldots,\tau_n\}$.

The proof of this proposition goes along the same line as in that of Proposition 5.1.

The last proposition extends Lemma 3.3.

Proposition 5.3. *Let $v_1,\ldots,v_m,w_1,\ldots,w_n$ be rational functions in $\mathbb{F}(\mathbf{t},\mathbf{x},\mathbf{y})^\times$. Assume that the compatibility conditions in (4), (5) and (8) hold. Then there exist a rational function f in $\mathbb{F}(\mathbf{t},\mathbf{x},\mathbf{y})$, $a_1,\ldots,a_m$ in $\mathbb{F}(\mathbf{t},\mathbf{x})$, and $b_1,\ldots,b_n$ in $\mathbb{F}(\mathbf{t},\mathbf{y})$ s.t., for all j with $1 \le j \le m$ and k with $1 \le k \le n$,*

$$v_j = \ell\sigma_j(f)\, a_j \quad \text{and} \quad w_k = \ell\tau_k(f)\, b_k.$$

Furthermore, the sequence $a_1,\ldots,a_m$, b_1, ..., b_n is compatible w.r.t. $\{\sigma_1,\ldots,\sigma_m,\tau_1,\ldots,\tau_n\}$.

Proof. First, we consider the case, in which $m = 1$ and n arbitrary. We proceed by induction on n. The proposition holds when $n = 1$ by Lemma 3.3. Assume that $n>1$, and the proposition holds for the values lower than n. Applying the induction hypothesis to $x_1, y_1, \ldots, y_{n-1}$ and to $x_1, y_2, \ldots, y_n$, respectively, we get $v_1 \in G_1\mathbb{F}(\mathbf{t}, x_1, y_n)^\times \cap G_1\mathbb{F}(\mathbf{t}, x_1, y_1)^\times$. Setting $\mathbb{A}{=}\mathbb{F}(\mathbf{t})$, $Z_1{=}\{y_n\}$ and $Z_2{=}\{y_1\}$ in Lemma 4.2 (i), we see that $v_1 \in G_1\mathbb{F}(\mathbf{t}, x_1)^\times$, which, together with the definition of $G_1\mathbb{F}(\mathbf{t}, x_1)^\times$, there exist f in $\mathbb{F}(\mathbf{t}, x_1, \mathbf{y})$ and a in $\mathbb{F}(\mathbf{t}, x_1)$ s.t. $v_1 = \ell\sigma_1(f)\, a$. Assume that $w_k = \ell\tau_k(f)\, b_k$ for some b_k in $\mathbb{F}(\mathbf{t}, x_1, \mathbf{y})$ and for all k with $1 \le k \le n$. By (8), $\sigma_1(b_k){=}b_k$, i.e., $b_k \in \mathbb{F}(\mathbf{t},\mathbf{y})$ (see Example 3.4). The proposition holds for $m = 1$ and n arbitrary.

Second, assume that $m > 1$ and the proposition holds for values lower than m and arbitrary n. Applying this induction hypothesis to $x_1, \ldots, x_{m-1}, \mathbf{y}$ and to $x_2, \ldots, x_m, \mathbf{y}$, respectively, we have

$$w_k \in \left(H_k\mathbb{A}(y_k, Z_1)^\times\right) \cap \left(H_k\mathbb{A}(y_k, Z_2)^\times\right),$$

where $\mathbb{A}{=}\mathbb{F}(\mathbf{t}, y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n)$, Z_1 and Z_2 are equal to $\{x_m\}$ and $\{x_1\}$, respectively. Thus, $w_k \in H_k\mathbb{A}(y_k)^\times$ by Lemma 4.2 (ii), and $w_k \in \ell\tau_k(f)\mathbb{A}(y_k)^\times$ for some f in $\mathbb{F}(\mathbf{t},\mathbf{x},\mathbf{y})$ by Lemma 4.3 (ii). Let $w_k = \ell\tau_k(f)\, b_k$, where b_k is in $\mathbb{A}(y_k)^\times$, and $k = 1, \ldots, n$. Let $a_j = v_j/\ell\sigma_j(f)$ for all j with $1 \le j \le m$. Then $\tau_k(a_j) = a_j$ for all k with $1 \le k \le n$ and j with $1 \le j \le m$ by the compatibility conditions in (8) (see Example 3.4). Hence, all the a_j's are in $\mathbb{F}(\mathbf{t},\mathbf{x})$. The sequence $a_1, \ldots, a_m$, $b_1, \ldots, b_n$ is compatible because of (4), (5) and (8). $\square$

Now, we present a theorem describing the structure of compatible rational functions.

Theorem 5.4. *Let*

$$u_1, \ldots, u_l,\ v_1, \ldots, v_m,\ w_1, \ldots, w_n \tag{19}$$

be a sequence of rational functions in $\mathbb{F}(\mathbf{t},\mathbf{x},\mathbf{y})$. If the sequence is Δ-compatible, then there exist f in $\mathbb{F}(\mathbf{t},\mathbf{x},\mathbf{y})$, α_1, ..., α_m, β_1, ..., β_l in $\mathbb{F}(\mathbf{t})$, λ_1, ..., λ_m in $\mathbb{F}(\mathbf{x})$, and μ_1, ..., μ_n in $\mathbb{F}(\mathbf{y})$ s.t., for all i with $1 \le i \le l$,

$$u_i = \ell\delta_i(f) + \ell\delta_i(\alpha_1)\, x_1 + \cdots + \ell\delta_i(\alpha_m)\, x_m + \beta_i, \tag{20}$$

for all j with $1 \le j \le m$, and, for all k with $1 \le k \le n$,

$$v_j = \ell\sigma_j(f)\, \alpha_j\lambda_j \quad \text{and} \quad w_k = \ell\tau_k(f)\, \mu_k. \tag{21}$$

Moreover, the sequence β_1, ..., β_l, λ_1, ..., λ_m, μ_1, ..., μ_n is Δ-compatible.

Proof. By Propositions 5.2 and 5.3,

$$w_k = \ell\tau_k(g')\, a'_k = \ell\tau_k(\tilde{g})\, \tilde{a}_k$$

for some $g', \tilde{g} \in \mathbb{F}(\mathbf{t},\mathbf{x},\mathbf{y})$, $a'_k \in \mathbb{F}(\mathbf{x},\mathbf{y})$, and $\tilde{a}_k \in \mathbb{F}(\mathbf{t},\mathbf{y})$ with $1 \le k \le n$. Set $Z_1 = \{t_1, \ldots, t_l\}$, $Z_2 = \{x_1, \ldots, x_m\}$, and $\mathbb{A} = \mathbb{F}(y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n)$ in Lemma 4.2 (ii). Then the lemma implies that there exist μ_k in $\mathbb{F}(\mathbf{y})$ and g_k in $\mathbb{F}(\mathbf{t},\mathbf{x},\mathbf{y})$ s.t. $w_k = \ell\tau_k(g_k)\, \mu_k$. Setting $\mathbb{E} = \mathbb{F}$ in the second assertion of Lemma 4.3, we may further assume that all the g_k's are equal to a rational function, say g. Let

$$u_i = \ell\delta_i(g) + r_i \ (1 \le i \le l) \text{ and } v_j = \ell\sigma_j(g)\, s_j \ (1 \le j \le m).$$

Then the compatibility conditions in (7) imply that the r_i's are in $\mathbb{F}(\mathbf{t},\mathbf{x})$ (see Example 3.3). Similarly, those conditions in (8) imply that the s_j's are in $\mathbb{F}(\mathbf{t},\mathbf{x})$ (see Example 3.4). Furthermore, $r_1, \ldots, r_l, s_1, \ldots, s_m$ are compatible w.r.t. the set $\{\delta_1, \ldots, \delta_l, \sigma_1, \ldots, \sigma_m\}$. By Proposition 5.1, we get

$$r_i = \ell\delta_i(b) + \ell\delta_i(\alpha_1)\, x_1 + \cdots + \ell\delta_i(\alpha_m)\, x_m + \beta_i,$$

and $s_j = \ell\sigma_j(b)\, \alpha_j\lambda_j$ for some b in $\mathbb{F}(\mathbf{t},\mathbf{x})$, α_j, β_i in $\mathbb{F}(\mathbf{t})$, λ_j in $\mathbb{F}(\mathbf{x})$, $1 \le i \le l$, and $1 \le j \le m$. Note that b belongs to $\mathbb{F}(\mathbf{t},\mathbf{x})$. Setting $f = gb$, we get the desired form for u_i's, v_j's and w_k's. The compatibility of the sequence β_1, ..., β_l, λ_1, ..., λ_m, μ_1, ..., μ_n follows from that of u_1, ..., u_l, v_1, ..., v_m, w_1, ..., w_n. $\square$

With the notation introduced in Theorem 5.4, we say that the sequence:

$$f,\ \alpha_1,\ \ldots,\ \alpha_m,\ \beta_1,\ \ldots,\ \beta_l, \lambda_1,\ \ldots,\ \lambda_m, \mu_1,\ \ldots,\ \mu_n \tag{22}$$

is a *representation* of Δ-compatible rational functions given in (19) if the equalities in (20) and (21) hold.

A rational function $\mathbb{F}(\mathbf{t},\mathbf{x},\mathbf{y})$ is said to be *nonsplit* w.r.t. $\mathbf{t}$ if its denominator and numerator have no irreducible factors in $\mathbb{F}[\mathbf{t}]$. Similarly, we define the notion of nonsplitness w.r.t. $\mathbf{x}$ or $\mathbf{y}$. Let $\prec$ be a fixed monomial ordering on $\mathbb{F}[\mathbf{t},\mathbf{x},\mathbf{y}]$. A nonzero rational function in $\mathbb{F}(\mathbf{t},\mathbf{x},\mathbf{y})$ is said to be *monic* w.r.t. $\prec$ if its denominator and numerator are both monic w.r.t. $\prec$. A representation (22) of Δ-compatible rational functions in (19) is said to be *standard* w.r.t. $\prec$ if

(i) f is nonsplit w.r.t. $\mathbf{t}$, $\mathbf{x}$, and $\mathbf{y}$, that is, the nontrivial irreducible factors of $\mathrm{den}(f)\mathrm{num}(f)$ are neither in $\mathbb{F}[\mathbf{t}]$, nor in $\mathbb{F}[\mathbf{x}]$, nor in $\mathbb{F}[\mathbf{y}]$;

(ii) both f and α_j are monic w.r.t. $\prec$, $j = 1, 2, \ldots, m$.

Assume that the sequence (22) is a representation of (19). Factor $f{=}f_1f_2f_3f_4$, where f_1 is monic and nonsplit w.r.t. $\mathbf{t}$, $\mathbf{x}$ and $\mathbf{y}$, f_2 is in $\mathbb{F}(\mathbf{t})$, f_3 in $\mathbb{F}(\mathbf{x})$, and f_4 in $\mathbb{F}(\mathbf{y})$. Set $\alpha_j{=}c_j\alpha'_j$, where $c_j \in \mathbb{F}$, and α'_j is monic. Then

$$f_1,\ \alpha'_1,\ \ldots,\ \alpha'_m,\ \beta_1 + \ell\delta_1(f_2),\ \ldots,\ \beta_l + \ell\delta_l(f_2),$$

$$\ell\sigma_1(f_3)c_1\lambda_1,\ \ldots,\ \ell\sigma_m(f_3)c_m\lambda_m,\ \ell\tau_1(f_4)\mu_1,\ \ldots,\ \ell\tau_n(f_4)\mu_n$$

is also a representation of (19). This proves the existence of standard representations. Its uniqueness follows from the uniqueness of factorization of rational functions.

Corollary 5.5. *A Δ-compatible sequence has a unique standard representation w.r.t. a given monomial ordering.*

6. ALGORITHMS AND APPLICATIONS

In this section, we discuss how to compute a representation of compatible rational functions, and present two applications in analyzing H-solutions. Let us fix a monomial ordering on $\mathbb{F}[\mathbf{t}, \mathbf{x}, \mathbf{y}]$ for standard representations.

Let the sequence given in (19) be Δ-compatible. We compute a representation of the sequence in the form of (22).

First, we compute $\mu_1(\mathbf{y}), \ldots, \mu_n(\mathbf{y})$ in the sequence (22). By gcd-computation, we write $w_k = a_k b_k$, where a_k is nonsplit w.r.t. $\mathbf{y}$, b_k is in $\mathbb{F}(\mathbf{y})$, and $k = 1, \ldots, n$. By Theorem 5.4, $w_k = \ell\tau_k(f)\,\mu_k$, where f is nonsplit w.r.t. $\mathbf{y}$ and μ_k is in $\mathbb{F}(\mathbf{y})$. Thus, $b_k = c_k\mu_k$ for some $c_k \in \mathbb{F}^\times$.

To determine c_k, write $a_k = \ell\tau_k(g_k)\,r_k$, where g_k and r_k are in $\mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$ with r_k being τ_k-reduced. By the two expressions of w_k, $c_k r_k = \ell\tau_k(f/g_k)$. Since a_k is nonsplit w.r.t. $\mathbf{y}$ and r_k is τ_k-reduced, g_k can be chosen to be nonsplit w.r.t. $\mathbf{y}$, and so is f/g_k. Thus, f/g_k is free of y_k, because $c_k r_k$ is τ_k-reduced. Accordingly, $c_k r_k{=}1$ and $\mu_k{=}r_k b_k$. As a byproduct, we obtain g_k with $\ell\tau_k(f) = \ell\tau_k(g_k)$.

Second, we compute $\alpha_1, \ldots, \alpha_m$ and $\lambda_1, \ldots, \lambda_m$. Assume that j is an integer with $1 \le j \le m$. By gcd-computation, we write $v_j = s_j a_j b_j$, where s_j is nonsplit w.r.t. $\mathbf{t}$ and $\mathbf{x}$, a_j is in $\mathbb{F}(\mathbf{t})$, and b_j in $\mathbb{F}(\mathbf{x})$. Moreover, set a_j to be monic. By Theorem 5.4, $v_j = \ell\sigma_j(f)\,\alpha_j\lambda_j$, where f is nonsplit w.r.t. $\mathbf{t}$ and $\mathbf{x}$, α_j is a monic element in $\mathbb{F}(\mathbf{t})$, and λ_j is in $\mathbb{F}(\mathbf{x})$. Hence, $a_j = \alpha_j$ and $b_j = c_j\lambda_j$ for some $c_j \in \mathbb{F}^\times$. As in the preceding paragraph, we write $s_j = \ell\sigma_j(g_j')\,r_j$ with r_j being σ_j-reduced. Then $c_j r_j = \ell\sigma_j(f/g_j')$. Since $c_j r_j$ is σ_j-reduced, $c_j r_j = 1$. Hence, $\lambda_j = r_j b_j$. As a byproduct, we find g_j' with $\ell\sigma_j(f) = \ell\sigma_j(g_j')$.

Third, we compute f. Note that f is a nonzero rational solution of the system $\{\sigma_j(z) = \ell\sigma_j(g_j')\,z, \tau_k(z) = \ell\tau_k(g_k)\,z\}$, where $1 \le j \le m$, $1 \le k \le n$, and g_j', g_k are obtained in the first two steps. So f can be computed by several methods, e.g., the method in the proof of [14, Proposition 3].

At last, we set $\beta_i = u_i - \ell\delta_i(f) - \sum_{j=1}^m \ell\delta_i(\alpha_j)\,x_j$, for all i with $1 \le i \le l$. Using $v_j = \ell\sigma_j(f)\,\alpha_j\lambda_j$ and $w_k = \ell\tau_k(f)\,\mu_k$ and the compatibility conditions in (6) and (7), we see that all the β_i's are in $\mathbb{F}(\mathbf{t})$, as required.

Example 6.1. *Consider the case $l = m = n = 1$. Let u, v and w be compatible rational functions, where*

$$u = \frac{(4t + 2x + y^2)(t + 1) + (t + x + 1)(t + x)(2t + y^2)}{(t + 1)(t + x)(2t + y^2)},$$
$$v = \frac{2(2x + 3)(x + 1)(t + 1)(t + x + 1)(5x + y)}{(5x + y + 5)(t + x)},$$
$$w = \frac{(5x + y)(2t + q^2y^2)(1 + qy)}{(5x + qy)(2t + y^2)}.$$

A representation of u, v, w is of the form

$$\left(\frac{(2t + y^2)(t + x)}{5x + y},\ t + 1,\ 1,\ 2(2x + 3)(x + 1),\ qy + 1\right).$$

From now on, we assume that our ground field $\mathbb{F}$ is algebraically closed. In general, Δ-extensions of $\mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$ are rings. We recall that an H-solution over $\mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$ is a nonzero solution of system (1) and, given a finite number of H-solutions, there is a Δ-extension of $\mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$ containing these H-solutions and their inverses. The ring of constants of this Δ-extension is equal to $\mathbb{F}$ by Theorem 2 in [5]. We will only encounter finitely many pairwise dissimilar H-solutions. Hence, it makes sense to multiply and invert them in some Δ-extension, which will not be specified explicitly if no ambiguity arises. All H-solutions we consider will be over $\mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$. Denote by $\mathbf{0}_s$ and $\mathbf{1}_s$ the sequences consisting of s 0's and of s 1's, respectively.

An H-solution is said to be a *symbolic power* if its certificates are of the form

$$\sum_{j=1}^m x_j\ell\delta_1(\alpha_j),\ \ldots,\ \sum_{j=1}^m x_j\ell\delta_l(\alpha_j),\ \alpha_1,\ \ldots,\ \alpha_m,\ \mathbf{1}_n, \tag{23}$$

where $\alpha_1, \ldots, \alpha_m$ are monic elements in $\mathbb{F}(\mathbf{t})^\times$. It is easy to verify that such a sequence is Δ-compatible. Such a symbolic power is denoted $\alpha_1^{x_1}\cdots\alpha_m^{x_m}$. The monicity of the α_i's excludes the case, in which some α_i is a constant different from one. By an E-solution, we mean an H-solution whose certificates are of the form $\beta_1, \ldots, \beta_l, \mathbf{1}_{m+n}$, where $\beta_1, \ldots, \beta_l$ are in $\mathbb{F}(\mathbf{t})$. An E-solution is a hyperexponential function w.r.t. the derivations, and a constant w.r.t. other operators. By a G-solution, we mean an H-solution whose certificates are of the form $\mathbf{0}_l, \lambda_1, \ldots, \lambda_m, \mathbf{1}_n$, where $\lambda_1, \ldots, \lambda_m$ are in $\mathbb{F}(\mathbf{x})^\times$. A G-solution is a hypergeometric term w.r.t. the shift operators, and a constant w.r.t. other operators. Similarly, by a Q-solution, we mean an H-solution whose certificates are of the form $\mathbf{0}_l$, $\mathbf{1}_m$, $\mu_1, \ldots, \mu_n$, where $\mu_1, \ldots, \mu_n$ are in $\mathbb{F}(\mathbf{y})^\times$. A Q-solution is a q-hypergeometric term w.r.t. the q-shift operators, and a constant w.r.t. other operators.

The next proposition describes a multiplicative decomposition of H-solutions.

Proposition 6.1. *An H-solution is a product of an element in $\mathbb{F}^\times$, a rational function in $\mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$, a symbolic power, an E-solution, a G-solution, and a Q-solution.*

Proof. Let h be an H-solution. Then its certificates are compatible. By Theorem 5.4, the certificates have a standard representation f, $\alpha_1, \ldots, \alpha_m$, $\beta_1, \ldots, \beta_l$, $\lambda_1, \ldots, \lambda_m$, $\mu_1, \ldots, \mu_n$. Moreover, the following three sequences:

$$\beta_1, \ldots, \beta_l, \mathbf{1}_{m+n};\quad \mathbf{0}_l, \lambda_1, \ldots, \lambda_m, \mathbf{1}_n;\quad \mathbf{0}_l, \mathbf{1}_m, \mu_1, \ldots, \mu_n$$

are Δ-compatible, respectively. Hence, there exist an E-solution $\mathcal{E}$, a G-solution $\mathcal{G}$, and a Q-solution $\mathcal{Q}$ s.t. their certificates are given in the above three sequences, respectively. It follows from Theorem 5.4 that h and the product $f\alpha_1^{x_1}\cdots\alpha_m^{x_m}\mathcal{E}\mathcal{G}\mathcal{Q}$ have the same certificates. So they differ by a multiplicative constant, which is in $\mathbb{F}$. $\square$

The H-solution in Example 6.1 can be decomposed as

$$\frac{(2t + y^2)(t + x)}{5x + y}\,(t + 1)^x\ \exp(t)\ (2x + 1)!\,\Gamma_q(1 + qy),$$

where $\Gamma_q(1+qy)$ is a Q-solution with certificates $0, 1, 1+qy$.

The next proposition characterizes rational H-solutions via their standard representations.

Proposition 6.2. *Let $\mathcal{P}$ be a symbolic power, $\mathcal{E}$ an E-solution, $\mathcal{G}$ a G-solution and $\mathcal{Q}$ a Q-solution. Then $\mathcal{P}\mathcal{E}\mathcal{G}\mathcal{Q}$ is in $\mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$ iff $\mathcal{P} \in \mathbb{F}$, $\mathcal{E} \in \mathbb{F}(\mathbf{t})$, $\mathcal{G} \in \mathbb{F}(\mathbf{x})$ and $\mathcal{Q} \in \mathbb{F}(\mathbf{y})$.*

Proof. ($\Leftarrow$) Clear.

($\Rightarrow$) Assume that f is rational and equal to $\mathcal{P}\mathcal{E}\mathcal{G}\mathcal{Q}$, where $\mathcal{P}, \mathcal{E}, \mathcal{G}, \mathcal{Q}$ are a symbolic power, an E-, a G-, and a Q-solution, respectively. Suppose that the certificates of $\mathcal{P}$ are given in (23). Applying $\ell\delta_i$ to f, $i = 1, \ldots, l$, we see that

$$\ell\delta_i(f) = \sum_{j=1}^m \ell\delta_i(\alpha_j)x_j + \ell\delta_i(\mathcal{E}).$$

Comparing the polynomial parts of the left and right hand-sides of the above equality w.r.t. x_j, we see that $\ell\delta_i(\alpha_j) = 0$ by Remark 2.2 and $\ell\delta_i(\mathcal{E}) \in \mathbb{F}(\mathbf{t})$ for all i and j. Hence, all the α_j's are in $\mathbb{F}$, and, consequently, all the α_j's are equal to one as they are monic. Hence, $\mathcal{P}$ is in $\mathbb{F}$. Moreover,

$$\ell\delta_i(f) = \ell\delta_i(\mathcal{E}) \quad \text{for all } i \text{ with } 1 \le i \le l.$$

Let g be a proper evaluation of f w.r.t. $\mathbf{x}$ and $\mathbf{y}$. Then

$$\ell\delta_i(g) = \ell\delta_i(\mathcal{E}) \quad \text{for all } i \text{ with } 1 \le i \le l,$$

since $\ell\delta_i(\mathcal{E})$ is in $\mathbb{F}(\mathbf{t})$. Hence, $\ell\delta_i(\mathcal{E}/g) = 0$, $\ell\sigma_j(\mathcal{E}/g) = 1$, and $\ell\tau_k(\mathcal{E}/g) = 1$, where $1 \le i \le l$, $1 \le j \le m$, and $1 \le k \le n$. We conclude that $\mathcal{E} = cg$ for some $c \in \mathbb{F}$. So $\mathcal{E}$ is in $\mathbb{F}(\mathbf{t})$.

Applying $\ell\sigma_j$ and $\ell\tau_k$ to f leads to $\ell\sigma_j(f) = \ell\sigma_j(\mathcal{G})$, and $\ell\tau_k(f) = \ell\tau_k(\mathcal{Q})$, respectively. One can show that $\mathcal{G}$ is in $\mathbb{F}(\mathbf{x})$ and $\mathcal{Q}$ is in $\mathbb{F}(\mathbf{y})$ by similar arguments. □

Now, we consider how to determine whether a finite number of H-solutions are algebraically dependent over $\mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$. Let $h_1, \cdots, h_s$ be H-solutions. By Proposition 6.1,

$$h_i \equiv \mathcal{P}_i\mathcal{E}_i\mathcal{G}_i\mathcal{Q}_i \mod \mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})^\times, \quad i = 1, \ldots, s, \qquad (24)$$

where $\mathcal{P}_i, \mathcal{E}_i, \mathcal{G}_i, \mathcal{Q}_i$ are a symbolic power, an E-solution, a G-solution, and a Q-solution, respectively.

COROLLARY 6.3. *Let $h_1, \ldots, h_s$ be H-solutions s.t. all the congruences in* (24) *hold. Then they are algebraically dependent over $\mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$ iff there exist integers $\omega_1, \ldots \omega_s$, not all zero, s.t. $\mathcal{P}_1^{\omega_1} \cdots \mathcal{P}_s^{\omega_s}$ is in $\mathbb{F}$, $\mathcal{E}_1^{\omega_1} \cdots \mathcal{E}_s^{\omega_s}$ in $\mathbb{F}(\mathbf{t})$, $\mathcal{G}_1^{\omega_1} \cdots \mathcal{G}_s^{\omega_s}$ in $\mathbb{F}(\mathbf{x})$ and $\mathcal{Q}_1^{\omega_1} \cdots \mathcal{Q}_s^{\omega_s}$ in $\mathbb{F}(\mathbf{y})$.*

PROOF. It follows from [15, Corollary 4.2] that $h_1, \cdots, h_s$ are algebraically dependent over $\mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$ iff there exist integers $\omega_1, \ldots \omega_s$, not all zero, s.t. $h_1^{\omega_1} \cdots h_s^{\omega_s}$ is in $\mathbb{F}(\mathbf{t}, \mathbf{x}, \mathbf{y})$. The corollary follows from (24) and Proposition 6.2. □

By the above corollary, one may determine the algebraic dependence of $h_1, \ldots, h_s$ using the decompositions in Proposition 6.1. By gcd-computation, one can find all nonzero integer vectors $(\omega_1, \ldots \omega_s)$ s.t. $\mathcal{P}_1^{\omega_1} \cdots \mathcal{P}_s^{\omega_s}$ is in $\mathbb{F}$. According to [19], one can find all nonzero integer vectors $(\omega_1, \ldots \omega_s)$ s.t. $\mathcal{E}_1^{\omega_1} \cdots \mathcal{E}_s^{\omega_s} \in \mathbb{F}(\mathbf{t})$ by seeking rational number solutions of a linear homogeneous system over $\mathbb{F}$. Computing all nonzero integer vectors $(\omega_1, \ldots \omega_s)$ s.t. $\mathcal{G}_1^{\omega_1} \cdots \mathcal{G}_s^{\omega_s} \in \mathbb{F}(\mathbf{x})$ reduces to the following subproblem: given $c_1, \ldots, c_s \in \mathbb{F}^\times$, compute integers $\omega_1, \ldots \omega_s$, not all zero, with $c_1^{\omega_1} \cdots c_s^{\omega_s} = 1$ (see [19]). Algorithms for tackling this subproblem and related discussions are contained in [13, §7.3] and the references given there. We are trying to develop an algorithm that finds integers $\omega_1, \ldots \omega_s$, not all zero, s.t. $\mathcal{Q}_1^{\omega_1} \cdots \mathcal{Q}_s^{\omega_s}$ belongs to $\mathbb{F}(\mathbf{y})$.

The reader is referred to [6] for an extended version of this paper, which contains a short proof of Fact 3.1 and a proof of Proposition 5.2. A Maple implementation is being written for decomposing H-solutions. We shall apply the structure theorem to study the existence of telescopers in the mixed cases in which any two of differential, shift and q-shift operators appear.

Acknowledgments. The authors thank Frédéric Chyzak, Bruno Salvy, Michael Singer and anonymous referees for helpful discussions and suggestions.

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