

Additive Decompositions in Primitive Extensions *

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Abstract

This paper extends the classical Ostrogradsky–Hermite reduction for rational functions to more general functions in primitive extensions of certain types. For an element f in such an extension K , the extended reduction decomposes f as the sum of a derivative in K and another element r such that f has an antiderivative in K if and only if $r = 0$; and f has an elementary antiderivative over K if and only if r is a linear combination of logarithmic derivatives over the constants when K is a logarithmic extension. Moreover, r is minimal in some sense. Additive decompositions may lead to reduction-based creative-telescoping methods for nested logarithmic functions, which are not necessarily D -finite.

1 Introduction

Symbolic integration, together with its discrete counterpart symbolic summation, nowadays has played a crucial role in building the infrastructure for applying computer algebra tools to solve problems in combinatorics and mathematical physics [15, 16, 27]. The early history of symbolic integration starts from the first tries of developing programs in LISP to evaluate integrals in freshman calculus symbolically in the 1960s. Two representative packages at the time were Slagle’s SAINT [28] and Moses’s SIN [19] which were both based on integral transformation rules and pattern recognition. The algebraic approach for symbolic integration is initialized by Ritt [25] in terms of differential algebra [14], which eventually leads to the Risch algorithm for the integration of elementary

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functions [23, 24]. The efficiency of the Risch algorithm is further improved by Rothstein [26], Davenport [11], Trager [29], Bronstein [6, 7] etc. Some standard references on this topic are Bronstein’s book [8] and Raab’s survey [22] that gives an overview of the Risch algorithm and its recent developments.

The central problem in symbolic integration is whether the integral of a given function can be written in “closed form”. Its algebraic formulation is given in terms of differential fields and their extensions [14, 8]. A differential field F is a field together with a derivation $'$ that is an additive map on F satisfying the product rule $(fg)' = f'g + fg'$ for all $f, g \in F$. A given element f in F is said to be *integrable* in F if $f = g'$ for some $g \in F$. The problem of deciding whether a given element is integrable or not in F is called the *integrability problem* in F . For example, if F is the field of rational functions, then for $f = 1/x^2$ we can find $g = -1/x$, while for $f = 1/x$ no suitable g exists in F . When f is not integrable in F , there are several other questions we may ask. One possibility is to ask whether there is a pair (g, r) in $F \times F$ such that $f = g' + r$, where r is minimal in some sense and $r = 0$ if f is integrable. This problem is called the *decomposition problem* in F . Extensive work has been done to solve the integrability and decomposition problems in differential fields of various kinds.

Abel and Liouville pioneered the early work on the integrability problem in the 19th century [25]. In 1833, Liouville provided a first decision procedure for solving the integrability problem on algebraic functions [18]. For other classes of functions, complete algorithms for solving the integrability problem are much more recent: 1) the Risch algorithm [23, 24] in the case of elementary functions was presented in 1969; 2) the Almkvist–Zeilberger algorithm [2] (also known as the differential Gosper algorithm) in the case of hyperexponential functions was given in 1990; 3) Abramov and van Hoeij’s algorithm [1] generalized the previous algorithm to the general D -finite functions of arbitrary order in 1997.

The decomposition problem was first considered by Ostrogradsky [20] in 1845 and later by Hermite [13] for rational functions. The idea of Ostrogradsky and Hermite is crucial for algorithmic treatments of the problem, since it avoids the root-finding of polynomials and only uses the extended Euclidean algorithm and squarefree factorization to obtain the additive decomposition of a rational function. This reduction is a basic tool for the integration of rational functions and also plays an important role in the base case of our work. We will refer this reduction as to the *rational reduction* in this paper. The rational reduction has been extended to more general classes of functions including algebraic functions [29, 9], hyperexponential functions [12, 4], multivariate rational functions [5, 17], and more recently including D -finite functions [10, 30]. Blending reductions with creative telescoping [2, 31] leads to the fourth and most recent generation of creative telescoping algorithms, which are called reduction-based algorithms [3, 4, 5, 9, 10].

The telescoping problem can also be formulated for elementary functions. Two related problems are how to decide the existence of telescopers for elementary functions and how to compute one if telescopers exist. Reduction algorithms have been shown to be crucial for solving these two problems. This naturally motivates us to design reduction algorithms for elementary functions.

In this paper, we extend the rational reduction to elements in straight and flat towers of primitive extensions (see Definition 3.5). Our extended reductions solve the decomposition problems in such towers without solving any Risch equations (Theorems 4.8 and 5.15), and determine elementary integrability in such towers when primitive extensions are logarithmic (Theorem 6.1).

The remainder of this paper is organized as follows. We present basic notions and terminologies on differential fields, and collect some useful facts about integrability in primitive extensions in Section 2. We define the notions of straight and flat towers, and describe some straightforward reduction processes in Section 3. Additive decompositions in straight and flat towers are given in Sections 4 and 5, respectively. The two decompositions are used to determine elementary integrability in Section 6. Examples are given in Section 7 to illustrate that the decompositions may be useful to study the telescoping problem for elementary functions that are not D -finite.

2 Preliminaries

Let $(F, ')$ be a differential field of characteristic zero, and let C_F denote the subfield of constants in F . Let E be a differential field extension of F . An element z of E is said to be *primitive* over F if z' belongs to F . If z is primitive and transcendental over F with $C_{F(z)} = C_F$, then it is called a *primitive monomial* over F , which is a special instance of Liouvillian monomials [8, Definition 5.1.2].

Let z be a primitive monomial over F in the rest of this section. An element $f \in F(z)$ is said to be *proper with respect to z or z -proper* for brevity if the degree of its numerator in z is lower than that of its denominator. In particular, zero is z -proper. It is well-known that f can be uniquely written as the sum of a z -proper element and a polynomial in z . They are called the fractional and polynomial parts of f , and denoted by $\text{fp}_z(f)$ and $\text{pp}_z(f)$, respectively.

Let p be a polynomial in $F[z]$. The degree and leading coefficient of p are denoted by $\text{deg}_z(p)$ and $\text{lc}_z(p)$, respectively. By [8, Theorem 5.1.1], p is squarefree if and only if $\text{gcd}(p, p') = 1$. A z -proper element is z -simple if its denominator is squarefree. Note that z -simple elements are not necessarily z -proper in [8], but they are assumed to be z -proper in this paper without loss of generality.

For $S \subset E$, we use S' to denote the set $\{f' \mid f \in S\}$. If S is a C_E -linear subspace, so is S' . For $m \in \mathbb{N}$, let

$$F[z]^{(m)} = \{p \in F[z] \mid \text{deg}_z(p) < m\}.$$

In particular, $F[z]^{(0)} = \{0\}$.

For $f \in F(z)$, Algorithm `HermiteReduce` in [8, page 139] computes a z -simple element $g \in F(z)$ and a polynomial $p \in F[z]$ such that $f \equiv g + p \pmod{F(z)'}$. This algorithm is an extension of the rational reduction by Ostrogradsky and Hermite. For rational functions, we have $p = 0$ since all polynomials have polynomial antiderivatives. Algorithm `HermiteReduce` is fundamental for our approach to additive decompositions in primitive extensions.

Lemma 2.1. *Let g be a z -simple element in $F(z)$. Then $g = 0$ if $g \in F(z)' + F[z]$.*

Proof. Suppose that $g \neq 0$. Since g is z -proper, there exists a nontrivial irreducible polynomial $p \in F[z]$ dividing the denominator of g . Since $g \in F(z)' + F[z]$, there exist $a \in F(z)$ and $b \in F[z]$ such that $g = a' + b$. The order of g at p is equal to -1 . But the order of a' at p is either nonnegative or less than -1 by Theorem 4.4.2 (i) in [8], and the order of b at p is nonnegative, a contradiction. ■

Every element $f \in F(z)$ is congruent to a unique z -simple element g modulo $F(z)' + F[z]$ by Algorithm `HermiterReduce` and Lemma 2.1. We call g the *Hermitian part* of f with respect to z , denoted by $\text{hp}_z(f)$. The map hp_z is C_F -linear on $F(z)$. Its kernel is equal to $F(z)' + F[z]$. Thus, two elements have the same Hermitian parts if they are congruent modulo $F(z)' + F[z]$. This observation is frequently used in the sequel.

Now, we collect some basic facts about primitive monomials. They are either straightforward or scattered in [8]. We list them below for the reader's convenience.

Lemma 2.2. *If $p \in F[z]$ and $p \in F(z)'$, then there exists $c \in C_F$ such that $\text{lc}_z(p) \equiv cz' \pmod{F'}$.*

Proof. Assume $p = r'$ for some $r \in F(z)$. Then $r \in F[z]$ by Theorem 4.4.2 (i) in [8]. Set $d = \deg_z(p)$ and $\ell = \text{lc}_z(p)$. Then $\deg_z(r) \leq d + 1$ by Lemma 5.1.2 in [8]. Assume that $r \equiv az^{d+1} + bz^d \pmod{F[z]^{(d)}}$ for some $a, b \in F$. Then

$$r' \equiv a'z^{d+1} + ((d+1)az' + b')z^d \pmod{F[z]^{(d)}}.$$

Since $p = r'$, we have that $a' = 0$ and $\ell = (d+1)az' + b'$. Hence, $\ell \equiv cz' \pmod{F'}$ with $c = (d+1)a \in C_F$. ■

The next lemma will be used to decrease the degree of a polynomial modulo $F(z)'$. Its proof is a straightforward application of integration by parts.

Lemma 2.3. *We have $f'z^d \equiv 0 \pmod{F(z)' + F[z]^{(d)}}$ for all $f \in F$ and $d \in \mathbb{N}$.*

Recall that an element f in F is said to be a *logarithmic derivative in F* if $f = a'/a$ for some nonzero element $a \in F$.

Lemma 2.4. *If f is a C_F -linear combination of logarithmic derivatives in $F(z)$, then $f = \text{hp}_z(f) + r$, where r is a C_F -linear combination of logarithmic derivatives in F .*

Proof. It suffices to assume that f is a logarithmic derivative in $F(z)$, because the map hp_z is C_F -linear.

If $f = 0$, then we choose $r = 0$, which equals $1'/1$. Otherwise, there exist two monic polynomials $u, v \in F[z]$ and $w \in F$ such that $f = u'/u - v'/v + w'/w$ by the logarithmic derivative identity in [8, page 104]. Note that $u'/u - v'/v$ is z -simple by Lemma 5.1.2 in [8] and w'/w is in F . Thus, $\text{hp}_z(f) = u'/u - v'/v$ and $r = w'/w$. ■

3 Primitive extensions

Let $(K_0, ')$ be a differential field of characteristic zero. Set $C = C_{K_0}$. Consider a tower of differential fields

$$K_0 \subset K_1 \subset \cdots \subset K_n, \quad (3.1)$$

where $K_i = K_{i-1}(t_i)$ for all i with $1 \leq i \leq n$. The tower given in (3.1) is said to be *primitive over K_0* if t_i is a primitive monomial over K_{i-1} for all i with $1 \leq i \leq n$. The notation introduced in (3.1) will be used in the rest of the paper.

Remark 3.1. *The derivatives t'_1, \dots, t'_n are linearly independent over C , since $C_{K_n} = C$ in (3.1).*

The following lemma tells us a way to modify the leading coefficient of a polynomial in $K_{n-1}[t_n]$ via integration by parts and Algorithm `HermiteReduce`.

Lemma 3.2. *Let the tower (3.1) be primitive with $n \geq 1$. Then, for all $\ell \in K_{n-1}$ and $d \in \mathbb{N}$, there exist a t_{n-1} -simple element $g \in K_{n-1}$ and a polynomial $h \in K_{n-2}[t_{n-1}]$ such that*

$$\ell t_n^d \equiv (g + h)t_n^d \pmod{K'_n + K_{n-1}[t_n]^{(d)}}.$$

Proof. By Algorithm `HermiteReduce`, there are $f, g \in K_{n-1}$ with g being t_{n-1} -simple, and $h \in K_{n-2}[t_{n-1}]$ such that $\ell = f' + g + h$. Then $\ell t_n^d = f' t_n^d + (g + h)t_n^d$. Applying Lemma 2.3 to the term $f' t_n^d$, we see that the lemma holds. ■

Let \prec be the purely lexicographic ordering on the set of monomials in t_1, t_2, \dots, t_n with $t_1 \prec t_2 \prec \dots \prec t_n$. For all i with $0 \leq i \leq n-1$ and $p \in K_i[t_{i+1}, \dots, t_n]$ with $p \neq 0$, the head monomial of p , denoted by $\text{hm}(p)$, is defined to be the highest monomial in t_{i+1}, \dots, t_n appearing in p with respect to \prec . The head coefficient of p , denoted by $\text{hc}(p)$, is defined to be the coefficient of $\text{hm}(p)$, which belongs to K_i . The head coefficient of zero is set to be zero. The monomial ordering \prec induces a partial ordering on $K_i[t_{i+1}, \dots, t_n]$, which is also denoted by \prec .

Example 3.3. *Let $\xi = t_1 t_2 t_3$. Viewing ξ as an element of $K_0[t_1, t_2, t_3]$, we have $\text{hm}(\xi) = \xi$ and $\text{hc}(\xi) = 1$, while, viewing ξ as an element of $K_1[t_2, t_3]$, we have $\text{hm}(\xi) = t_2 t_3$ and $\text{hc}(\xi) = t_1$.*

The next lemma will be used in Section 5. We present it below because it holds for primitive towers.

Lemma 3.4. *Let $n \geq 1$. For a polynomial $p \in K_{n-1}[t_n]$, there are polynomials $p_i \in K_i[t_{i+1}, \dots, t_n]$ such that $p \equiv \sum_{i=0}^{n-1} p_i \pmod{K'_n}$, and that $\text{hc}(p_i)$ is t_i -simple for all i with $1 \leq i \leq n-1$. Moreover, $\deg_{t_n}(p_i) \leq \deg_{t_n}(p)$ for all i with $0 \leq i \leq n-1$.*

Proof. We proceed by induction on n . If $n = 1$, then set $p_0 = p$, because there is no requirement on $\text{hc}(p_0)$. Assume that $n > 1$ and that the lemma holds for $n - 1$.

Let $p \in K_{n-1}[t_n]$ and $d = \deg_{t_n}(p)$. By Lemma 3.2,

$$p \equiv (g + h)t_n^d \pmod{K'_n + K_{n-1}[t_n]^{(d)}},$$

where $g \in K_{n-1}$ is t_{n-1} -simple and $h \in K_{n-2}[t_{n-1}]$. Then there exist $h_j \in K_j[t_{j+1}, \dots, t_{n-1}]$ such that $h = \sum_{j=0}^{n-2} h_j + u'$ for some u in K_{n-1} and $\text{hc}(h_j)$ is t_j -simple for all j when $1 \leq j \leq n-2$ by the induction hypothesis. Furthermore, we set $h_{n-1} = g$. By Lemma 2.3,

$$p \equiv \sum_{j=0}^{n-1} h_j t_n^d \pmod{K'_n + K_{n-1}[t_n]^{(d)}}. \quad (3.2)$$

We need to argue inductively on d . If $d = 0$, then it is sufficient to set $p_j = h_j$ for all j with $0 \leq j \leq n-1$, as $K_{n-1}[t_n]^{(0)} = \{0\}$. Assume that $d > 0$ and that the lemma holds for all polynomials in $K_{n-1}[t_n]^{(d)}$. By (3.2) and the induction hypothesis on d , we have

$$p \equiv \sum_{j=0}^{n-1} h_j t_n^d + \sum_{j=0}^{n-1} \tilde{p}_j \pmod{K'_n},$$

where \tilde{p}_j is in $K_j[t_{j+1}, \dots, t_n]$, $\text{hc}(\tilde{p}_j)$ is t_j -simple when $j \geq 1$, and $\deg_{t_n}(\tilde{p}_j) < d$. Set $p_j = h_j t_n^d + \tilde{p}_j$. Then $p \equiv \sum_{j=0}^{n-1} p_j \pmod{K'_n}$. Since $\text{hc}(p_j)$ is $\text{hc}(h_j)$ if $h_j \neq 0$ and $\text{hc}(p_j)$ is $\text{hc}(\tilde{p}_j)$ if $h_j = 0$, the requirements on each $\text{hc}(p_j)$ with $j \geq 1$ is fulfilled. The induction on d is completed, and so is the induction on n . \blacksquare

Definition 3.5. *The tower given in (3.1) is said to be straight if $\text{hp}_{t_{i-1}}(t'_i) \neq 0$ for all i with $2 \leq i \leq n$. The tower is said to be flat if $t'_i \in K_0$ for all i with $1 \leq i \leq n$.*

Example 3.6. *Let $K_0 = \mathbb{C}(x)$ with the usual derivation in x . Let*

$$\log(x) = \int x^{-1} dx \quad \text{and} \quad \text{Li}(x) = \int \log(x)^{-1} dx.$$

Then the tower

$$K_0 \subset K_0(\log(x)) \subset K_0(\log(x), \text{Li}(x))$$

is straight, while the tower

$$K_0 \subset K_0(\log(x)) \subset K_0(\log(x), \log(x+1))$$

is flat. They contain no new constants by Theorem 5.1.1 in [8].

In this paper, we consider additive decompositions for elements in either straight or flat towers with $K_0 = C(t_0)$.

Lemma 3.7. *Let the tower (3.1) be primitive. Assume that K_0 is equal to $C(t_0)$ with the usual derivation in t_0 . Then $\text{hp}_{t_0}(t'_1)$ is nonzero. Moreover, $\text{hp}_{t_0}(t'_i)$ is nonzero for all i with $2 \leq i \leq n$ if (3.1) is flat.*

Proof. By the rational reduction, $t'_1 = u' + v$ for some $u, v \in K_0$ with v being t_0 -simple. Then v is nonzero, since $C = C_{K_n}$. The second assertion can be proved similarly. ■

Example 3.8. *Let $K_{-1} = C$, $K_0 = K_{-1}(t_0)$ with the usual derivation in t_0 , and each t_i be logarithmic in (3.1) for all i with $1 \leq i \leq n$. By Lemma 2.4, $t'_i = \text{hp}_{t_{i-1}}(t'_i) + r_i$, where $r_i \in K_{n-2}$ for all i with $1 \leq i \leq n$. The tower is straight if and only if $t'_i \in K_{i-1} \setminus K_{i-2}$ for all i with $2 \leq i \leq n$. In addition, $t'_1 \in K_0 \setminus K_{-1}$ as $K_0 = K_{-1}(t_0)$.*

4 Straight towers

In this section, we assume that the tower (3.1) is straight and that $K_0 = C(t_0)$ with the usual derivation with respect to t_0 . The subfield C of constants is denoted by K_{-1} in recursive definitions and induction proofs to be carried out.

Our idea is reducing a polynomial in $K_{n-1}[t_n]$ to another of lower degree via integration by parts, whenever it is possible. The notion of t_n -rigid elements describes $r \in K_{n-1}$ such that rt_n^d cannot be congruent to a polynomial of degree lower than d modulo K'_n .

Definition 4.1. *An element $r \in K_{-1}$ is said to be t_0 -rigid if $r = 0$. Let $r \in K_{n-1}$ with*

$$f = \text{fp}_{t_{n-1}}(r) \quad \text{and} \quad p = \text{pp}_{t_{n-1}}(r).$$

We say that r is t_n -rigid if f is t_{n-1} -simple, $f \neq c \text{hp}_{t_{n-1}}(t'_n)$ for any nonzero $c \in C$, and $\text{lc}_{t_{n-1}}(p)$ is t_{n-1} -rigid.

Note that zero is t_n -rigid, because $\text{hp}_{t_{n-1}}(t'_n)$ is nonzero.

Example 4.2. *Let $t_0 = x$, $t_1 = \log(x)$ and $t_2 = \text{Li}(x)$. Let*

$$\ell_1 = \frac{1}{x + k_1} \quad \text{and} \quad \ell_2 = \frac{1}{t_1 + k_2} + \ell_1 t_1^2 + x t_1 + x^2.$$

Then ℓ_1 is t_1 -rigid if $k_1 \neq 0$ and ℓ_2 is t_2 -rigid if $k_1 k_2 \neq 0$.

The next lemma, together with Lemma 2.2, reveals that a nonzero polynomial p in $K_{n-1}[t_n]$ with a t_n -rigid leading coefficient has no antiderivative in K_n .

Lemma 4.3. *Let $r \in K_{n-1}$ be t_n -rigid. If*

$$r \equiv ct'_n \pmod{K'_{n-1}} \tag{4.1}$$

for some $c \in C$, then both r and c are zero.

Proof. We proceed by induction on n . If $n = 0$, then $r = 0$ by Definition 4.1. Thus, $ct'_0 \equiv 0 \pmod{K'_{-1}}$. Consequently, $c = 0$ because $K'_{-1} = \{0\}$ and $t'_0 = 1$.

Assume that $n > 0$ and that the lemma holds for $n - 1$. Set $f = \text{fp}_{t_{n-1}}(r)$. Then $f = \text{hp}_{t_{n-1}}(r)$, since f is t_{n-1} -simple by Definition 4.1. Applying the map $\text{hp}_{t_{n-1}}$ to (4.1), we have $f = c \text{hp}_{t_{n-1}}(t'_n)$ by Lemma 2.1. Hence, $c = 0$ and $f = 0$ by Definition 4.1.

Set $p = \text{pp}_{t_{n-1}}(r)$. Then (4.1) becomes $p \equiv 0 \pmod{K'_{n-1}}$, which, together with Lemma 2.2, implies that $\text{lc}_{t_{n-1}}(p) \equiv \tilde{c}t'_{n-1} \pmod{K'_{n-2}}$ for some $\tilde{c} \in C$. It follows from the induction hypothesis that $\text{lc}_{t_{n-1}}(p)$ is zero, and so is p . Thus, r is zero. \blacksquare

In $K_{n-1}[t_n]$, we define a class of polynomials that have no antiderivatives in K_n .

Definition 4.4. For $n \geq 0$, a polynomial in $K_{n-1}[t_n]$ is said to be t_n -straight if its leading coefficient is t_n -rigid.

Proposition 4.5. Let $p \in K_{n-1}[t_n]$ be a t_n -straight polynomial. Then $p = 0$ if $p \in K'_n$.

Proof. If $n = 0$, then $p = 0$ by Definition 4.1. Otherwise, $\text{lc}_{t_n}(p) \equiv ct'_n \pmod{K'_{n-1}}$ for some $c \in C$ by Lemma 2.2. Then $\text{lc}_{t_n}(p) = 0$ by Lemma 4.3. Consequently, $p = 0$. \blacksquare

Next, we reduce a polynomial to a t_n -straight one.

Lemma 4.6. For $p \in K_{n-1}[t_n]$, there exists a t_n -straight polynomial $q \in K_{n-1}[t_n]$ with $\deg_{t_n}(q) \leq \deg_{t_n}(p)$ such that $p \equiv q \pmod{K'_n}$.

Proof. If $p = 0$, then we choose $q = 0$. Assume that p is nonzero. We proceed by induction on n .

If $n = 0$, then $p \equiv 0 \pmod{K'_0}$, as every element of $K_{-1}[t_0]$ has an antiderivative in the same ring.

Assume that $n > 0$ and that the lemma holds for $n - 1$. Let $p \in K_{n-1}[t_n]$ with degree d and leading coefficient ℓ . By Algorithm `HermiteReduce`, there are $f, g, u, v \in K_{n-1}$ with g, v being t_{n-1} -simple, and $h, w \in K_{n-2}[t_{n-1}]$ such that

$$\ell = f' + g + h \quad \text{and} \quad t'_n = u' + v + w.$$

We are going to concoct a new expression for ℓ such that

$$\ell = (ct_n + a)' + r, \tag{4.2}$$

where $c \in C$, $a \in K_{n-1}$ and $r \in K_{n-1}$ is t_n -rigid. The expression helps us decrease degrees. To this end, we consider two cases.

Case 1. Assume $g \neq cv$ for any $c \in C \setminus \{0\}$. By the induction hypothesis, there exists a t_{n-1} -straight polynomial $\tilde{h} \in K_{n-2}[t_{n-1}]$ such that $h = b' + \tilde{h}$ for some $b \in K_{n-1}$. Then $\ell = a' + r$, where $a = f + b$ and $r = g + \tilde{h}$.

Case 2. Assume $g = cv$ for some $c \in C \setminus \{0\}$. By the induction hypothesis, there exists a t_{n-1} -straight polynomial $\tilde{h} \in K_{n-2}[t_{n-1}]$ such that $h - cw = b' + \tilde{h}$ for some $b \in K_{n-1}$. Then $\ell = (ct_n + a)' + r$, where $a = f - cu + b$ and $r = \tilde{h}$.

In both cases, r is t_n -rigid by Definition 4.1.

If $d = 0$, then $p = \ell$. By (4.2), we have $p \equiv r \pmod{K'_n}$. Let $q = r$, which is t_n -straight by Definition 4.4.

Assume that $d > 0$ and each polynomial in $K_{n-1}[t_n]^{(d)}$ is congruent to a t_n -straight polynomial modulo K'_n . By (4.2), Lemma 2.3 and the equality $ct'_n t_n^d = \left(\frac{c}{d+1} t_n^{d+1}\right)'$, we have

$$p \equiv rt_n^d + \tilde{q} \pmod{K'_n}$$

for some $\tilde{q} \in K_{n-1}[t_n]^{(d)}$. If $r \neq 0$, then set $q = rt_n^d + \tilde{q}$. Otherwise, applying the induction hypothesis on d to \tilde{q} yields a t_n -straight polynomial q with $p \equiv q \pmod{K'_n}$. ■

Example 4.7. Consider the integral

$$\int \log(x) \operatorname{Li}(x)^2 dx.$$

With the notation introduced in Example 4.2, we reduce the integrand $t_1 t_2^2$. We have that $\operatorname{lc}_{t_2}(t_1 t_2^2) = t_1$. Since t_1 is not t_2 -rigid, $t_1 t_2^2$ can be reduced. In fact, $t_1 t_2^2 = x' t_1 t_2^2$. By Lemma 2.3 and a straightforward calculation, we get

$$t_1 t_2^2 = (x t_1 t_2^2 - x t_2^2 - x^2 t_2)' + \frac{2x}{t_1} t_2 + \frac{x^2}{t_1}.$$

Since $2x/t_1$ is t_2 -rigid, we have that $(2x/t_1)t_2 + (x^2/t_1)$ is t_2 -straight. Hence, $t_1 t_2^2$ has no antiderivative in $C(x, t_1, t_2)$ by Proposition 4.5.

Below is an additive decomposition in a straight tower.

Theorem 4.8. For $f \in K_n$, the following assertions hold.

(i) There exist a t_n -simple element $g \in K_n$ and a t_n -straight polynomial $p \in K_{n-1}[t_n]$ such that

$$f \equiv g + p \pmod{K'_n}. \quad (4.3)$$

(ii) $f \in K'_n$ if and only if both g and p in (4.3) are zero.

(iii) If $f \equiv \tilde{g} + \tilde{p} \pmod{K'_n}$, where $\tilde{g} \in K_n$ is a t_n -simple element and $\tilde{p} \in K_{n-1}[t_n]$, then $g = \tilde{g}$ and $\deg_{t_n}(p) \leq \deg_{t_n}(\tilde{p})$.

Proof. (i) By Algorithm `HermiteReduce`, there exist a t_n -simple element $g \in K_n$ and a polynomial $h \in K_{n-1}[t_n]$ such that

$$f \equiv g + h \pmod{K'_n}.$$

By Lemma 4.6, h can be replaced by a t_n -straight polynomial p .

(ii) Since $f \in K'_n$, the congruence (4.3) becomes $g + p \equiv 0 \pmod{K'_n}$. Applying the map hp_{t_n} to the new congruence, we have $g = 0$, because $g = \text{hp}_{t_n}(g+p)$. Thus, $p = 0$ by Proposition 4.5.

(iii) Since $g - \tilde{g} \equiv \tilde{p} - p \pmod{K'_n}$, we have $g = \tilde{g}$ by Lemma 2.1. If $\deg_{t_n}(\tilde{p}) < \deg_{t_n}(p)$, then $p - \tilde{p}$ is t_n -straight, because $\text{lc}_{t_n}(p - \tilde{p})$ equals $\text{lc}_{t_n}(p)$. So $p - \tilde{p} = 0$ by Proposition 4.5, a contradiction. ■

Example 4.9. Consider the integral

$$\int \frac{1}{\text{Li}(x)^2} + \log(x) \text{Li}(x)^2 dx.$$

The integrand is $f := 1/t_2^2 + t_1 t_2^2$, in which the notation is introduced in Example 4.2. By Algorithm `HermiteReduce`, we have

$$f = (-t_1/t_2)' + x/t_2 + t_1 t_2^2.$$

By Theorem 4.8 and Example 4.7, f has no antiderivative in $C(x, t_1, t_2)$.

5 Flat towers

In this section, we let the tower (3.1) be flat. The ground field K_0 will be specialized to $C(t_0)$ later in this section. We are not able to fully carry out the same idea in Section 4, because $\text{hp}_{t_{i-1}}(t'_i) = 0$ for all $i = 2, \dots, n$. This spoils Lemma 4.3 and Proposition 4.5. So we need to study integrability in a flat tower differently.

This section is divided into two parts. First, we extend Lemma 2.3 to the differential ring $K_0[t_1, \dots, t_n]$. Second, we present a flat counterpart of the results in Section 4.

5.1 Scales

Let us denote $K_0[t_1, \dots, t_n]$ by R_n . For a monomial ξ in t_1, \dots, t_n , the C -linear subspace $\{p \in R_n \mid p \prec \xi\}$ is denoted by $R_n^{(\xi)}$. The notion of scales is motivated by the following example.

Example 5.1. Let $n = 2$, and $\xi_0 = 1, \xi_1 = t_1$ and $\xi_2 = t_2$. And let $\ell = t'_1 + t'_2$. Using integration by parts, we find three congruences

$$\ell \xi_0 \equiv 0 \pmod{K'_2}, \quad \ell \xi_1 \equiv -t'_1 t_2 \pmod{K'_2}, \quad \ell \xi_2 \equiv -t'_2 t_1 \pmod{K'_2}.$$

The first and third congruences lead to monomials lower than ξ_0 and ξ_2 , respectively. But the second one leads to t_2 , which is higher than ξ_1 . The notion of scales aims to prevent the second congruence from the reduction to be carried out.

Definition 5.2. Let $p \in R_n \setminus \{0\}$ and $\text{hm}(p) = t_1^{e_1} \cdots t_n^{e_n}$. The scale of p with respect to n is defined to be s if $e_1 = 0, \dots, e_{s-1} = 0$ and $e_s > 0$. Let $p \in K_0$. The scale of p with respect to n is defined to be n . The scale of p with respect to n is denoted by $\text{scale}_n(p)$.

Example 5.3. Let $\xi_0 = 1$, $\xi_1 = t_1 t_2$ and $\xi_2 = t_3^2$. Regarding ξ_0, ξ_1 and ξ_2 as elements in $K_0[t_1, t_2, t_3]$, we have that $\text{scale}_3(\xi_0) = 3$, $\text{scale}_3(\xi_1) = 1$ and $\text{scale}_3(\xi_2) = 3$; while, regarding them as elements in $K_0[t_1, t_2, t_3, t_4]$, we have that $\text{scale}_4(\xi_0) = 4$, $\text{scale}_4(\xi_1) = 1$ and $\text{scale}_4(\xi_2) = 3$.

Notably, if $p \in K_0$, then the scale of p with respect to n is equal to n , which varies as n does. Otherwise, the scale is fixed by $\text{hm}(p)$ no matter in which ring p lives.

The next lemma extends Lemma 2.3 and indicates what kind of integration by parts will be used for reduction.

Lemma 5.4. Let ξ be a monomial in t_1, \dots, t_n and $f \in K_0$. Then the followings hold.

(i) $f'\xi \equiv 0 \pmod{K'_n + R_n^{(\xi)}}$.

(ii) Let $s = \text{scale}_n(\xi)$. Then, for all $c_1, \dots, c_s \in C$,

$$(c_1 t'_1 + \cdots + c_s t'_s)\xi \equiv 0 \pmod{K'_n + R_n^{(\xi)}}.$$

Proof. (i) It follows from integration by parts and the fact that ξ' belongs to $R_n^{(\xi)}$.

(ii) Set $L_0 = 0$ and $L_i = \sum_{j=1}^i c_j t_j$ for $i = 1, \dots, n$.

If $\xi = 1$, then $s = n$ and $L'_n \xi \in K'_n$. The assertion clearly holds. Assume that $\xi = t_s^{e_s} \cdots t_n^{e_n}$ with $e_s > 0$. Then $L'_s \xi = L'_{s-1} \xi + c_s t'_s \xi$. Note that $L'_{s-1} \xi$ belongs to $K'_n + R_n^{(\xi)}$ by a direct use of integration by parts. Set $\eta = \xi/t_s^{e_s}$. Then the term $c_s t'_s \xi$ is equal to $\frac{c_s}{e_s+1} (t_s^{e_s+1})' \eta$. Integration by parts leads to

$$c_s t'_s \xi \equiv \frac{-c_s}{e_s+1} t_s^{e_s+1} \eta' \pmod{K'_n}. \quad (5.1)$$

Then $\eta = 1$ if $e_j = 0$ for all j with $j > s$. So $c_s t'_s \xi$ belongs to K'_n by (5.1). Otherwise, $e_j > 0$ for some j with $s < j \leq n$. Then each monomial in $t_s^{e_s+1} \eta'$ is of total degree $\sum_{j=s}^n e_j$ and is of degree $e_s + 1$ in t_s . So $t_s^{e_s+1} \eta' \prec \xi$. Consequently, $c_s t'_s \xi \in K'_n + R_n^{(\xi)}$ by (5.1). ■

In the rest of this section, we let $K_0 = C(t_0)$ with the usual derivation in t_0 . By Lemma 3.7, we may further assume that t'_i is nonzero and t_0 -simple for all i with $1 \leq i \leq n$.

Definition 5.5. For every k with $1 \leq k \leq n$, an element of K_0 is said to be k -rigid if either it is equal to zero or it is t_0 -simple and is not a C -linearly combination of t'_1, \dots, t'_k .

Proposition 5.6. *For $p \in R_n$, there exists $q \in R_n$ such that*

$$p \equiv q \pmod{K'_n}$$

and that $\text{hc}(q)$ is s -rigid, where $s = \text{scale}_n(q)$. Moreover, $q \preceq p$.

Proof. Set $q = 0$ if $p = 0$. Assume $p \neq 0$ and $\xi = \text{hm}(p)$. By the rational reduction, $\text{hc}(p) = f' + g$ for some $f, g \in K_0$ with g being t_0 -simple. Then $p = f'\xi + g\xi \pmod{R_n^{(\xi)}}$. By Lemma 5.4 (i), $p \equiv g\xi + r \pmod{K'_n}$ for some $r \in R_n^{(\xi)}$. Set $s = \text{scale}_n(\xi)$. If g is nonzero and s -rigid, then set $q = g\xi + r$. Otherwise, $p \equiv \tilde{r} \pmod{K'_n}$ for some $\tilde{r} \in R_n^{(\xi)}$ by Lemma 5.4 (ii). The proposition follows from a direct Noetherian induction on $\text{hm}(\tilde{r})$ with respect to \prec . \blacksquare

Example 5.7. *Let $K_0 = \mathbb{C}(x)$, $t_1 = \log(x)$, $t_2 = \log(x + 1)$. and*

$$p = t_1^2 t_2 + (2/x)t_1 t_2 + ((2/(x + 1))t_1.$$

Then $\text{hc}(p) = 1$, which is not 1-rigid. Since $t_1^2 t_2 = x' t_1 t_2^2$, integration by parts leads to $p = (x t_1^2 t_2)' + q$, where $q = (\frac{2}{x} - 2)t_1 t_2 - \frac{x}{x+1} t_1^2 + \frac{2}{x+1} t_1$. We can then reduce q further, because $\text{hc}(q) = (2t_1 - 2x)'$, which is not 1-rigid either. Repeating this reduction a finite number of times, we see that p is equal to the derivative of $(x + 1)t_1^2 t_2 - 2x t_1 t_2 - x t_1^2 + (2x + 2)t_2 + 4x t_1 - 6x$.

5.2 Reduction

A flat analogue of straight polynomials is given below.

Definition 5.8. *A polynomial in $C[t_0]$ is said to be t_0 -flat if it is zero. For $n \geq 1$, $p \in K_{n-1}[t_n]$ is called a t_n -flat polynomial if there exist $p_i \in K_i[t_{i+1}, \dots, t_n]$ for all i with $0 \leq i \leq n - 1$ such that $p = \sum_{i=0}^{n-1} p_i$, $\text{hc}(p_i)$ is t_i -simple for all $i \geq 1$, and $\text{hc}(p_0)$ is s -rigid, where $s = \text{scale}_n(p_0)$. The sequence $\{p_i\}_{i=0,1,\dots,n-1}$ is called a sequence associated to p .*

Example 5.9. *Let $n = 3$ and $t_0 = x$, $t_1 = \log(x)$, $t_2 = \log(x + 1)$ and $t_3 = \log(x + 2)$. Consider $p \in K_2[t_3]$*

$$p = \underbrace{\frac{1}{t_2} t_3^2}_{p_2} + \underbrace{\frac{1}{t_1} t_2 t_3}_{p_1} + \underbrace{\frac{1}{x+k} t_3^3 + x t_2 t_3}_{p_0},$$

where $k \in \mathbb{Z}$. Obviously, $\text{hc}(p_2)$ is t_2 -simple and $\text{hc}(p_1)$ is t_1 -simple. Moreover, $\text{scale}_3(p_0) = 3$ and $\text{hc}(p_0)$ is 3-rigid if $k \notin \{0, 1, 2\}$. So p is t_3 -flat if $k \notin \{0, 1, 2\}$.

We are going to extend the results in Section 4 to the flat case, based on the following technical lemma.

Lemma 5.10. *Let $n \geq 1$ and $p \in K_{n-1}[t_n]$ be t_n -flat. Set ℓ to be $\text{lc}_{t_n}(p)$. Then $\text{fp}_{t_{n-1}}(\ell)$ is t_{n-1} -simple and $\text{pp}_{t_{n-1}}(\ell)$ is t_{n-1} -flat. Moreover, $\text{pp}_{t_{n-1}}(\ell) - c t'_n$ is t_{n-1} -flat for all $c \in C$ if $n > 1$.*

Proof. The lemma is trivial if $p = 0$.

Assume that p is nonzero and $d = \deg_{t_n}(p)$. Let $\{p_i\}_{i=0,1,\dots,n-1}$ be a sequence associated to p , and let ℓ_i be the coefficient of t_n^d in p_i . Evidently, $\ell = \sum_{i=0}^{n-1} \ell_i$, $\text{fp}_{t_{n-1}}(\ell) = \ell_{n-1}$ and $\text{pp}_{t_{n-1}}(\ell) = \sum_{i=0}^{n-2} \ell_i$. Moreover, $\ell_i = 0$ if $\deg_{t_n}(p_i) < d$, and $\text{hc}(\ell_i) = \text{hc}(p_i)$ otherwise. This is because \prec is purely lexicographic with $t_{i+1} \prec \dots \prec t_n$ for all i with $0 \leq i \leq n-1$. Thus, $\text{fp}_{t_{n-1}}(\ell)$ is t_{n-1} -simple and $\text{pp}_{t_{n-1}}(\ell)$ is t_{n-1} -flat by Definition 5.8.

It remains to show the second assertion. Assume $n > 1$. Then

$$\text{pp}_{t_{n-1}}(\ell) - ct'_n = \ell_{n-2} + \dots + \ell_1 + \tilde{\ell}_0 \quad \text{with } \tilde{\ell}_0 = \ell_0 - ct'_n, \quad (5.2)$$

and $\text{hc}(\ell_j)$ is t_j -simple for all j with $1 \leq j \leq n-2$.

Set $s = \text{scale}_n(p_0)$ and $\tilde{s} = \text{scale}_{n-1}(\tilde{\ell}_0)$. It suffices to prove that $\text{hc}(\tilde{\ell}_0)$ is \tilde{s} -rigid by (5.2) and Definition 5.8.

Case 1. $\ell_0 \notin K_0$. Then $s < n$,

$$\text{hm}(p_0) = t_s^{e_s} \dots t_{n-1}^{e_{n-1}} t_n^d \quad \text{and} \quad \text{hm}(\ell_0) = t_s^{e_s} \dots t_{n-1}^{e_{n-1}},$$

where $e_s > 0$. Moreover,

$$s = \text{scale}_{n-1}(\ell_0), \quad \text{hm}(\ell_0) = \text{hm}(\tilde{\ell}_0), \quad \text{and} \quad \text{hc}(p_0) = \text{hc}(\ell_0) = \text{hc}(\tilde{\ell}_0).$$

In particular, $\tilde{s} = s$. Hence, $\text{hc}(\tilde{\ell}_0)$ is \tilde{s} -rigid, because $\text{hc}(p_0)$ is s -rigid.

Case 2. $\ell_0 \in K_0$ with $\ell_0 \neq 0$. Then $\text{hm}(p_0) = t_n^d$ and $s = n$. Moreover, $\tilde{s} = n-1$, since $\tilde{\ell}_0 \in K_0$. Note that p is t_n -flat. So $\text{hc}(p_0)$ is not a C -linear combination of $\{t'_1, \dots, t'_{n-1}, t'_n\}$, and neither is ℓ_0 because $\ell_0 = \text{hc}(p_0)$. Consequently, $\tilde{\ell}_0$ is not a C -linear combination of $\{t'_1, \dots, t'_{n-1}\}$, and neither is $\text{hc}(\tilde{\ell}_0)$, because $\text{hc}(\tilde{\ell}_0) = \tilde{\ell}_0$. Thus, $\text{hc}(\tilde{\ell}_0)$ is $(n-1)$ -rigid.

Case 3. $\ell_0 = 0$. Then $\tilde{s} = n-1$ and $\text{hc}(\tilde{\ell}_0) = \tilde{\ell}_0 = -ct'_n$, which is $(n-1)$ -rigid by Remark 3.1.

The second assertion is proved. ■

The next lemma is a flat-analogue of Lemma 4.3

Lemma 5.11. *Let $n \geq 1$ and $p \in K_{n-1}[t_n]$ be t_n -flat. If*

$$\text{lc}_{t_n}(p) \equiv ct'_n \pmod{K'_{n-1}} \quad (5.3)$$

for some $c \in C$, then both p and c are zero.

Proof. If $n = 1$, then the tower $K_0 \subset K_1$ is also straight, and p is t_1 -straight by Definition 4.4 and Lemma 3.7. Both p and c are zero by Lemma 4.3.

Assume that $n > 1$ and the lemma holds for $n-1$. Set $\ell = \text{lc}_{t_n}(p)$. Applying the map $\text{hp}_{t_{n-1}}$ to (5.3), we have $\text{hp}_{t_{n-1}}(\ell) = 0$. Then $\text{fp}_{t_{n-1}}(\ell) = 0$ by the first assertion of Lemma 5.10. Consequently, we have $\ell = \text{pp}_{t_{n-1}}(\ell)$. Let $q = \ell - ct'_n$. Then q is t_{n-1} -flat by the second assertion of Lemma 5.10. On the other hand,

$q \in K'_{n-1}$ by (5.3). Then $\text{lc}_{t_{n-1}}(q) \equiv \tilde{c}t'_{n-1} \pmod{K'_{n-2}}$ for some $\tilde{c} \in C$ by Lemma 2.2. So $q = 0$ by the induction hypothesis. Accordingly,

$$\ell = ct'_n \in K_0. \quad (5.4)$$

Let $\{p_i\}_{i=0,1,\dots,n-1}$ be a sequence associated to p . By (5.4), we have $\text{hc}(p_0) = ct'_n$, because ct'_n is not t_i -simple for all i with $1 \leq i \leq n$. Hence, $\text{hm}(p_0)$ is a non-negative power of t_n and $\text{scale}_n(p_0) = n$. Then ct'_n is n -rigid by Definition 5.8. We have $c = 0$. By (5.4), we conclude that ℓ is zero, and so is p . ■

The following proposition corresponds to Proposition 4.5.

Proposition 5.12. *Let $n \geq 1$ and p be a t_n -flat polynomial in $K_{n-1}[t_n]$. If $p \in K'_n$, then $p = 0$.*

Proof. Since $p \in K'_n$, we have $\text{lc}_{t_n}(p) \equiv ct'_n \pmod{K'_{n-1}}$ for some $c \in C$ by Lemma 2.2. Then $p = 0$ by Lemma 5.11. ■

The next lemma corresponds to Lemma 4.6.

Lemma 5.13. *For $p \in K_{n-1}[t_n]$, there exists a t_n -flat polynomial $q \in K_{n-1}[t_n]$ such that $p \equiv q \pmod{K'_n}$. Moreover, $\text{deg}_{t_n}(q)$ is no more than $\text{deg}_{t_n}(p)$.*

Proof. By Lemma 3.4, there exist $p_i \in K_i[t_{i+1}, \dots, t_n]$ for all i with $1 \leq i \leq n-1$ and $p_0 \in R_n$ such that

$$p \equiv \sum_{i=1}^{n-1} p_i + p_0 \pmod{K'_n}.$$

Moreover, $\text{hc}(p_i) \in K_i$ is t_i -simple for all $i \geq 1$. By Proposition 5.6, there exists $r \in R_n$ with $s = \text{scale}_n(r)$ such that $p_0 \equiv r \pmod{K'_n}$ and that $\text{hc}(r)$ is s -rigid. Set q to be $\sum_{i=1}^{n-1} p_i + r$. Then q is t_n -flat and $p \equiv q \pmod{K'_n}$. ■

Example 5.14. *Let p be given as in Example 5.9, in which $k = 2$. By integration by parts, we have*

$$p \equiv p_2 + p_1 + \underbrace{-3t'_3 t_2 t_3^2 + x t_2 t_3}_{q_0} \pmod{K'_3}.$$

Then $\text{scale}_3(q_0) = 2$ and $\text{hc}(q_0) = -3t'_3 = -3/(x+2)$, which is 2-rigid. Hence, $p_2 + p_1 + q_0$ is t_3 -flat.

We are ready to present the main result of this section.

Theorem 5.15. *For $f \in K_n$, the following assertions hold.*

(i) *There exist a t_n -simple element $g \in K_n$ and a t_n -flat polynomial $p \in K_{n-1}[t_n]$ such that*

$$f \equiv g + p \pmod{K'_n}. \quad (5.5)$$

(ii) $f \equiv 0 \pmod{K'_n}$ if and only if both g and p are zero.

(iii) If $f \equiv \tilde{g} + \tilde{p} \pmod{K'_n}$, where $\tilde{g} \in K_n$ is t_n -simple and $\tilde{p} \in K_{n-1}[t_n]$, then $g = \tilde{g}$ and $\deg_{t_n}(p) \leq \deg_{t_n}(\tilde{p})$.

Proof. (i) Applying Algorithm `HermiteReduce` to f with respect to t_n , we get a t_n -simple element g of K_n and an element h of $K_{n-1}[t_n]$ such that $f \equiv g + h \pmod{K'_n}$. We can replace h with a t_n -flat polynomial p by Lemma 5.13.

(ii) Assume $f \in K'_n$. Then (5.5) becomes $g + p \equiv 0 \pmod{K'_n}$. Applying the map hp_{t_n} to the above congruence yields $g = 0$ by Lemma 2.1. Thus, $p \equiv 0 \pmod{K'_n}$. Consequently, $p = 0$ by Proposition 5.12.

(iii) Since $(g - \tilde{g}) + (p - \tilde{p}) \equiv 0 \pmod{K'_n}$ and $g - \tilde{g}$ is t_n -simple, we have $g = \tilde{g}$ by Lemma 2.1. So $p - \tilde{p} \equiv 0 \pmod{K'_n}$. By Lemma 2.2, we have $\text{lc}_{t_n}(p - \tilde{p}) \equiv ct'_n \pmod{K'_{n-1}}$ for some $c \in C$. If $\deg_{t_n}(\tilde{p})$ is smaller than $\deg_{t_n}(p)$, then $\text{lc}_{t_n}(p) = \text{lc}_{t_n}(p - \tilde{p}) \equiv ct'_n \pmod{K'_{n-1}}$. By Lemma 5.11, we conclude $p = 0$, a contradiction. ■

6 Elementary integrability

In this section, we study elementary integrability of elements in a straight or flat tower by Theorems 4.8 and 5.15.

Theorem 6.1. *Let the tower be given in (3.1), in which C is algebraically closed, $K_0 = C(t_0)$ and t_i be a C -linear combination of logarithmic monomials over K_{i-1} for all i with $1 \leq i \leq n$. Assume that, for $f \in K_n$,*

$$f \equiv g + p \pmod{K'_n}, \quad (6.1)$$

where g and p are described in (4.3) if (3.1) is straight, and in (5.5) if (3.1) is flat, respectively. Then f is elementarily integrable over K_n if and only if $g + p$ is a C -linear combination of logarithmic derivatives in K_n .

Proof. We denote by \mathbb{L}_i the C -linear subspace spanned by the logarithmic derivatives in K_i for all i with $0 \leq i \leq n$.

Clearly, f is elementarily integrable over K_n if $g + p \in \mathbb{L}_n$.

Conversely, there exists $r \in \mathbb{L}_n$ such that $f \equiv r \pmod{K'_n}$ by Liouville's theorem ([8, Theorem 5.5.1]). By (6.1),

$$g + p \equiv r \pmod{K'_n}. \quad (6.2)$$

Note that $\text{hp}_{t_n}(g + p) = g$, as g is t_n -simple. So $g = \text{hp}_{t_n}(r)$ by (6.2) and Lemma 2.1. Hence, $g \in \mathbb{L}_n$ by Lemma 2.4. Set \tilde{r} to be $r - \text{hp}_{t_n}(r)$. Then $\tilde{r} \in \mathbb{L}_{n-1}$ by Lemma 2.4. Moreover, (6.2) becomes

$$p \equiv \tilde{r} \pmod{K'_n}. \quad (6.3)$$

We show that (6.2) implies $g + p \in \mathbb{L}_n$ by induction. If $n = 0$, then p is zero. The assertion holds. Assume that the assertion holds for $n - 1$. Let $d = \deg_{t_n}(p)$ and $\ell = \text{lc}_{t_n}(p)$.

Case 1. $d > 0$. Then $\ell = \text{lc}_{t_n}(p - \tilde{r})$, which, together with (6.3) and Lemma 2.2, implies that $\ell \equiv ct'_n \pmod{K'_{n-1}}$ for some $c \in C$. Then ℓ is equal to 0 by Lemma 4.3 in the straight case and by Lemma 5.11 in the flat case, a contradiction.

Case 2. $d = 0$. Then $\ell = p$. So $\ell \equiv \tilde{r} + ct'_n \pmod{K'_{n-1}}$ for some $c \in C$ by (6.3) and Lemma 2.2. Consequently, ℓ is elementarily integrable over K_{n-1} , because $\tilde{r}, t'_n \in \mathbb{L}_{n-1}$. Moreover, the above congruence can be rewritten as

$$\text{fp}_{t_{n-1}}(\ell) + \text{pp}_{t_{n-1}}(\ell) \equiv \tilde{r} + ct'_n \pmod{K'_{n-1}} \quad (6.4)$$

Note that $\text{fp}_{t_{n-1}}(\ell)$ is t_{n-1} -simple and $\text{pp}_{t_{n-1}}(\ell)$ is t_{n-1} -straight (resp. flat) by Definition 4.4 (resp. Lemma 5.10). By (6.4) and the induction hypothesis, we see that $\ell \in \mathbb{L}_{n-1}$. Therefore, p belongs to \mathbb{L}_{n-1} . Accordingly, $g + p \in \mathbb{L}_n$. ■

Determining whether an element r in K_n is a C -linear combination of logarithmic derivatives amounts to computing partial fraction decompositions and Rothstein–Trager resultants by Theorem 4.4.3 in [8]. So elementary integrability in straight and flat towers can be checked by merely algebraic computation whenever a decomposition in the form (6.1) is available.

Example 6.2. Let K_0, t_1 and t_2 be given as in Example 5.7. We compute an additive decomposition for

$$f = \frac{1}{xt_1} + \frac{1}{xt_2 + t_2} + t_1^2 t_2 + \frac{2}{x} t_1 t_2 + \frac{2}{x+1} t_1 + \frac{1}{x+2}.$$

By Theorem 5.15 and Example 5.7, we have

$$f = a' + \underbrace{\frac{1}{xt_2 + t_2}}_g + \underbrace{\frac{1}{xt_1} + \frac{1}{x+2}}_p,$$

where $a = (x+1)t_1^2 t_2 - 2xt_1 t_2 - xt_1^2 + (2x+2)t_2 + 4xt_1 - 6x$. As the Rothstein–Trager resultant of each fraction in $g + p$ has only constant roots, $g + p$ is a C -linear combination of logarithmic derivatives in K_2 . So f is elementarily integrable over K_2 by Theorem 6.1. Indeed,

$$\int f dx = a + \log(t_2) + \log(t_1) + \log(x+2).$$

7 Telescopers for elementary functions

The problem of creative telescoping is classically formulated for D -finite functions in terms of linear differential operators [2, 31]. Raab in his thesis [21] has studied the telescoping problem viewed as a special case of the parametric integration problem in differential fields. However, there are no theoretical results concerning the existence of telescopers for elementary functions. To be more

precise, let F be a differential field with two derivations D_x and D_y that commute with each other and let F_∂ be the set $\{f \in F \mid \partial(f) = 0\}$ for $\partial \in \{D_x, D_y\}$. For a given element $f \in F$, the telescoping problem asks whether there exists a nonzero linear differential operator $L = \sum_{i=0}^d \ell_i D_x^i$ with $\ell_i \in F_{D_y}$ such that $L(f) = D_y(g)$ for some g in a specific differential extension E of F . We call L a *telescoper* for f and g the corresponding *certificate* for L in E . Usually, we take E to be the field F itself or an elementary extension of F . In contrast to D -finite functions, telescopers may not exist for elementary functions as shown in the following example.

Example 7.1. Let $F = \mathbb{C}(x, y)$ and $E = F(t_1, t_2)$ be a differential field extension of F with

$$t_1 = \log(x^2 + y^2) \quad \text{and} \quad t_2 = \log(1 + t_1).$$

We first show that $f = 1/t_1 \in F(t_1)$ has no telescoper with certificate in any elementary extension of $F(t_1)$. Since t_1 is a primitive monomial over F , we have $F_{D_y} = \mathbb{C}(x)$. We claim that for any $i \in \mathbb{N}$, $D_x^i(f)$ can be decomposed as

$$D_x^i(f) = D_y(g_i) + \frac{a_i}{t_1},$$

where $g_i \in F(t_1)$, and $a_i \in F$ satisfies the recurrence relation

$$a_{i+1} = D_x(a_i) - D_y\left(\frac{xa_i}{y}\right) \quad \text{with } a_0 = 1.$$

For $n = 0$, the claim holds by taking $g_0 = 0$. Assume that the claim holds for all $i < k$. Applying the induction hypothesis and Algorithm `HermiteReduce` to $D_x^k(f)$ yields

$$\begin{aligned} D_x^k(f) &= D_x(D_x^{k-1}(f)) = D_x\left(D_y(g_{k-1}) + \frac{a_{k-1}}{t_1}\right) \\ &= D_y\left(D_x(g_{k-1}) + \frac{a_{k-1}x}{yt_1}\right) + \frac{D_x(a_{k-1}) - D_y\left(\frac{xa_{k-1}}{y}\right)}{t_1}. \end{aligned}$$

This completes the induction. A straightforward calculation shows that $a_i = A_i/y^{2i}$ for some $A_i \in \mathbb{C}[x, y] \setminus \{0\}$ with $\deg_y(A_i) < 2i$. Using the notion of residues in [8, page 118], we have

$$\text{residue}_{t_1}\left(\frac{a_i}{t_1}\right) = \frac{a_i}{D_y(t_1)} = \frac{(x^2 + y^2)A_i}{2y^{2i+1}},$$

which is not in $\mathbb{C}(x)$. Then $D_x^i(f)$ is not elementarily integrable over $F(t_1)$ for any $i \in \mathbb{N}$ by the residue criterion in [8, Theorem 5.6.1]. Assume that f has a telescoper $L := \sum_{i=0}^d \ell_i D_x^i$ with $\ell_i \in \mathbb{C}(x)$ not all zero. Then $L(f)$ is elementarily integrable over $F(t_1)$. However,

$$L(f) = D_y\left(\sum_{i=0}^d \ell_i g_i\right) + \frac{\sum_{i=0}^d \ell_i a_i}{t_1}.$$

Since all of the ℓ_i 's are in $\mathbb{C}(x)$ and $\gcd(x^2 + y^2, y^m) = 1$ for any $m \in \mathbb{N}$, the residue of $\sum_{i=0}^d \ell_i a_i / t_1$ is not in $\mathbb{C}(x)$, which implies that $L(f)$ is not elementarily integrable over $F(t_1)$, a contradiction.

We now show that $p = ft_2 + 1 \in F(t_1)[t_2]$ has no telescoper with certificate in any elementary extension of $F(t_1, t_2)$. Since t_2 is also a primitive monomial over $F(t_1)$, we have $E_{D_y} = \mathbb{C}(x)$. Assume that $L := \sum_{i=0}^d \ell_i D_x^i$ with $\ell_i \in \mathbb{C}(x)$ not all zero is a telescoper for p . Then $L(p)$ is elementarily integrable over E . By a direct calculation, we get $L(p) = L(f)t_2 + r$ with $r \in F(t_1)$. The elementary integrability of $L(p)$ implies that $L(f) = cD_y(t_2) + D_y(b)$ for some $c \in \mathbb{C}(x)$ and $b \in F(t_1)$ by the formula (5.13) in the proof of Theorem 5.8.1 in [8, page 157]. We claim that $c = 0$. Since $D_x^i(f) = u_i/t_1^{i+1}$ with $u_i \in F[t_1]$ and $\deg_{t_1}(u_i) < i + 1$ and $D_y(t_2) = D_y(t_1)/(1 + t_1)$, the orders of $D_x^i(f)$ and $D_y(t_2)$ at $1 + t_1$ are equal to 0 and 1, respectively. If c is not zero, the order of $cD_y(t_2)$ at $1 + t_1$ is equal to 1, which does not match with that of $L(f) - D_y(b)$ by Theorem 4.4.2 (i) in [8], a contradiction. Then $L(f) = D_y(b)$, i.e., L is a telescoper for f . This contradicts with the first assertion we have shown.

The next example shows that additive decompositions in Theorems 4.8 and 5.15 are useful for detecting the existence of telescopers for elementary functions that are not D -finite.

Example 7.2. Let $F = \mathbb{C}(x, y)$ and $E = F(t)$ be a differential field extension of F with $t = \log(x^2 + y^2)$. Consider the function $f = t + 1 - \frac{2y}{(x^2 + y^2)t^2}$. Since the derivatives $D_x^i(1/t^2) = a_i/t^{i+2}$ with $a_i \in F \setminus \{0\}$ are linearly independent over F , we see that $1/t^2$ is not D -finite over F , and neither is f . Note that f can be decomposed as

$$f = D_y(1/t) + t + 1.$$

Since $t + 1$ is D -finite, it has a telescoper, and so does f .

8 Conclusion

In this paper, we developed additive decompositions in straight and flat towers, which enable us to determine in-field and elementary integrability in a straightforward manner. It is natural to ask whether one can develop an additive decomposition in a general primitive tower. Moreover, we plan to investigate about the existence and the construction of telescopers for elementary functions using additive decompositions.

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