

Since all of the ℓ_i 's are in $\mathbb{C}(x)$ and $\gcd(x^2 + y^2, y^m) = 1$ for any $m \in \mathbb{N}$, the residue of $\sum_{i=0}^d \ell_i a_i / t_1$ is not in $\mathbb{C}(x)$, which implies that $L(f)$ is not elementarily integrable over $F(t_1)$, a contradiction.

We now show that $p = f t_2$ has no telescoper with certificate in any elementary extension of $F(t_1, t_2)$. Since t_2 is also a primitive monomial over $F(t_1)$, we have $E_{D_y} = \mathbb{C}(x)$. Assume that $L := \sum_{i=0}^d \ell_i D_x^i$ with $\ell_i \in \mathbb{C}(x)$ not all zero is a telescoper for p . Then $L(p)$ is elementarily integrable over E . By a direct calculation, we get $L(p) = L(f)t_2 + r$ with $r \in F(t_1)$. The elementary integrability of $L(p)$ implies that $L(f) = c D_y(t_2) + D_y(b)$ for some $c \in \mathbb{C}(x)$ and $b \in F(t_1)$ by the formula (5.13) in the proof of Theorem 5.8.1 in [9, page 157]. We claim that $c = 0$. Since $D_x^i(f) = u_i / t_1^{i+1}$ with $u_i \in F[t_1]$ and $\deg_{t_1}(u_i) < i + 1$ and $D_y(t_2) = D_y(t_1)/(1 + t_1)$, the orders of $D_x^i(f)$ and $D_y(t_2)$ at $1 + t_1$ are equal to 0 and 1, respectively. If c is nonzero, the order of $c D_y(t_2)$ at $1 + t_1$ is equal to 1, which does not match with that of $L(f) - D_y(b)$ by Lemma 4.4.2 (i) in [9], a contradiction. Then $L(f) = D_y(b)$, i.e., L is a telescoper for f , which contradicts with the first assertion.

The next example shows that additive decompositions in Theorems 4.8 and 5.15 are useful for detecting the existence of telescopers for elementary functions that are not D -finite.

EXAMPLE 7.2. Let $F = \mathbb{C}(x, y)$ and $E = F(t)$ be a differential field extension of F with $t = \log(x^2 + y^2)$. Consider the function $f = t + 1 - \frac{2y}{(x^2 + y^2)t^2}$. Since the derivatives $D_x^i(1/t^2) = a_i/t^{i+2}$ with $a_i \in F \setminus \{0\}$ are linearly independent over F , we see that $1/t^2$ is not D -finite over F , and neither is f . Note that f can be decomposed as

$$f = D_y(1/t) + t + 1.$$

Since $t + 1$ is D -finite, it has a telescoper, and so does f .

8 CONCLUSION

In this paper, we developed additive decompositions in straight and flat towers, which enable us to determine in-field integrability and elementary integrability in a straightforward manner. It is natural to ask whether one can develop an additive decomposition in a general primitive tower. Moreover, we plan to investigate about the existence and the construction of telescopers for elementary functions using additive decompositions.

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