Existence Problem of Telescopers for Rational Functions in Three Variables: the Mixed Cases *

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Abstract

We present criteria on the existence of telescopers for trivariate rational functions in four mixed cases, in which discrete and continuous variables appear simultaneously. We reduce the existence problem in the trivariate case to the exactness testing problem, the separation problem and the existence problem in the bivariate case. The existence criteria we present help us determine the termination of Zeilberger's algorithm for the input functions studied in this paper.

1 Introduction

Creative telescoping plays a crucial role in the algorithmic proof theory of combinatorial identities developed by Wilf and Zeilberger in the early 1990s [28, 29, 27]. For a given function $f(x, y_1, \ldots, y_n)$, the process of creative telescoping constructs a nonzero linear differential or recurrence operator L in x such that

$$L(f) = \partial_{y_i}(g_1) + \cdots + \partial_{y_n}(g_n),$$

where ∂_{y_i} denotes the derivation or difference operator in y_i and the g_i's belong to the same class of functions as f. The operator L is then called a *telescoper*

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for f, and the g_i's are called the *certificates* of L. Two fundamental problems have been studied extensively related to creative telescoping. The first problem is the *existence problem of telescopers*, i.e., deciding the existence of telescopers for a given class of functions. The second one is the *construction problem of telescopers*, i.e., designing efficient algorithms for computing telescopers if they exist. In this paper, we will mainly focus on the existence problem of telescopers.

The existence of telescopers is connected to the termination of Zeilberger's algorithm [3] and the hypertranscendence and algebraic dependency of functions defined by indefinite sums or integrals [22, 26]. In 1990, Zeilberger first presented a sufficient condition on the existence of telescopers by showing that telescopers always exist for so-called *holonomic functions* in [28] using Bernstein's theory of algebraic D-modules. Soon after this work, Wilf and Zeilberger in [27] proved that telescopers exist for proper hypergeometric terms. However, holonomicity and properness are only sufficient conditions. Abramov and Le [4] gave a necessary and sufficient condition on the existence of telescopers for rational functions in two discrete variables. This work was soon extended to the hypergeometric case by Abramov [3], the q-hypergeometric case in [16], and the mixed rational and hypergeometric case in [14, 9]. All of the above work only focussed on the problem for bivariate functions of a special class. The first criterion on the existence of telescopers beyond the bivariate case was given in [12], in which a necessary and sufficient condition is presented on the existence problem of telescopers for rational functions in three discrete variables. The goal of this paper is continuing this project by considering four mixed cases, in which both the discrete and continuous variables appear.

The remainder of this paper is organized as follows. We define the existence problem of telescopers precisely in Section 2 and recall different types of reductions that are used in testing the exactness of bivariate rational functions in Section 3. Existence criteria are given for four types of telescopers for rational functions in three variables in Section 4.

2 Preliminaries

Let $\mathbb{K}$ be a field of characteristic zero and let $\mathbb{E} = \mathbb{K}(\mathbf{v})$ be the field of rational functions in $\mathbf{v} = \{x, y_1, \ldots, y_n\}$ over $\mathbb{K}$. We define the derivation δ_v on $\mathbb{E}$ in the variable $v \in \mathbf{v}$ as the usual partial derivation ∂/∂_v. The shift operator σ_v on $\mathbb{E}$ in the variable $v \in \mathbf{v}$ is defined as the $\mathbb{K}$-isomorphism such that $\sigma_v(v) = v + 1$ and $\sigma_v(w) = w$ for all $w \in \mathbf{v} \setminus \{v\}$. The ring of linear functional operators in $\mathbf{v}$ over $\mathbb{E}$ is denoted by $\mathbb{E}\langle\partial_{\mathbf{v}}\rangle$ with $\partial_{\mathbf{v}} = \{\partial_x, \partial_{y_1}, \ldots, \partial_{y_n}\}$, in which ∂_v is either the derivation D_v such that $D_v f = f D_v + \delta_v(f)$ or the shift operator S_v such that $S_v f = \sigma_v(f) S_v$ for any $f \in \mathbb{E}$ and $v \in \mathbf{v}$, and the ∂_v's commute pairwise. For $v \in \mathbf{v}$, we let Δ_v denote the difference operator $S_v - \mathbf{1}$, where $\mathbf{1}$ stands for the identity map on $\mathbb{E}$. Let $\overline{\mathbb{E}}$ be the algebraic closure of $\mathbb{E}$. The operators δ_v and σ_v on $\mathbb{E}$ can be extended to $\overline{\mathbb{E}}$, which will be denoted by the same symbols. The functions we consider will be in certain $\mathbb{E}\langle\partial_{\mathbf{v}}\rangle$-module, such as the fields $\mathbb{E}$ and $\overline{\mathbb{E}}$. The ring $\mathbb{K}(x)\langle\partial_x\rangle$ is a subring of $\mathbb{E}\langle\partial_{\mathbf{v}}\rangle$ that is also a left

Euclidean domain. Efficient algorithms for basic operations in $\mathbb{K}(x)\langle\partial_x\rangle$, such as computing the least common left multiple (LCLM) of operators, have been developed in [7, 5].

Lemma 2.1. *For an operator $L = \sum_{i=0}^{\rho} e_i D_x^i \in \overline{\mathbb{K}(x)}\langle D_x\rangle$ with $e_\rho = 1$, we let $\mathbb{F}$ be a finite normal extension of $\mathbb{K}(x)$ containing the coefficients e_i's and G be the Galois group of $\mathbb{F}$ over $\mathbb{K}(x)$. Let T be the LCLM of the operators $\sigma(L) = \sum_{i=0}^{\rho} \sigma(e_i) D_x^i$ for all $\sigma \in G$. Then T belongs to $\mathbb{K}(x)\langle D_x\rangle$.*

Proof. It suffices to show that $\tau(T) = T$ for all $\tau \in G$. Since D_x commutes with any isomorphism in G by [6, Theorem 3.2.4 (i)], we have $\tau(L_1L_2) = \tau(L_1)\tau(L_2)$ for all $L_1, L_2 \in \mathbb{F}\langle D_x\rangle$. For each $\sigma \in G$, we have $T = P_\sigma \sigma(L)$ for some $P_\sigma \in \mathbb{F}\langle D_x\rangle$, which implies that $\tau(\sigma(L))$ divides $\tau(T)$. When σ runs through all elements of G, so does $\tau\sigma$. Hence $\tau(T)$ is also a common left multiple of the operators $\sigma(L)$ for all $\sigma \in G$. Since $\tau(T)$ and T are both monic and of the same degree in D_x, we get $\tau(T) = T$. ∎

Remark 2.2. *The above assertion is not true in the shift case. For example, take $L = S_x + \sqrt{x}$. The LCLM of L and its conjugation $S_x - \sqrt{x}$ is $S_x^2 - \sqrt{x(x+1)}$, which is not in $\mathbb{K}(x)\langle S_x\rangle$.*

Definition 2.3 (Creative Telescoping)**.** *Let $\mathfrak{M}$ be an $\mathbb{E}\langle\partial_{\mathbf{v}}\rangle$-module and $f \in \mathfrak{M}$. A nonzero linear operator $L \in \mathbb{K}(x)\langle\partial_x\rangle$ is called a* telescoper *of type $(\partial_x, \partial_{y_1}, \ldots, \partial_{y_n})$ for f if there exist $g_1, \ldots, g_n \in \mathfrak{M}$ such that*

$$L(f) = \partial_{y_1}(g_1) + \cdots + \partial_{y_n}(g_n), \tag{2.1}$$

where $\partial_v \in \{D_v, \Delta_v\}$. The rational functions $g_1, \ldots, g_n$ are called certificates *of L in $\mathfrak{M}$.*

Note that all of the telescopers for a given function together with the zero operator form a left ideal of $\mathbb{K}(x)\langle\partial_x\rangle$ (see [17, Definition 1]). The following lemma summarizes closure properties related to the existence of telescopers.

Lemma 2.4. *Let $f, g \in \overline{\mathbb{E}}$, $a, b \in \mathbb{K}(x)$ and $\alpha, \beta \in \overline{\mathbb{K}(x)}$. Then we have*

(i) if both f and g have telescopers in $\mathbb{K}(x)\langle D_x\rangle$, so does $\alpha f + \beta g$;

(ii) if both f and g have telescopers in $\mathbb{K}(x)\langle S_x\rangle$, so does $af + bg$.

Proof. We first show that αf has a telescoper in $\mathbb{K}(x)\langle D_x\rangle$ if f does. Assume that $L = \sum_{i=0}^{\rho} e_i D_x^i \in \mathbb{K}(x)\langle D_x\rangle$ is a telescoper for f. Then $L(f) = \partial_{y_1}(g_1) + \cdots + \partial_{y_n}(g_n)$ with $g_i \in \overline{\mathbb{E}}$. Set $\tilde{L} = L \cdot \frac{1}{\alpha}$, which belongs to $\overline{\mathbb{K}(x)}\langle D_x\rangle$. Then we have $\tilde{L}(\alpha f) = \partial_{y_1}(g_1) + \cdots + \partial_{y_n}(g_n)$, which means $\tilde{L}$ is a telescoper for αf. By Lemma 2.1, there exists $T \in \mathbb{K}(x)\langle D_x\rangle$ such that T is a left multiple of $\tilde{L}$. So T is also a telescoper for αf. When telescopers are in $\mathbb{K}(x)\langle S_x\rangle$, the above argument works for af for any $a \in \mathbb{K}(x)$. It remains to show that $f + g$ has a telescoper in $\mathbb{K}(x)\langle\partial_x\rangle$ with $\partial_x \in \{D_x, S_x\}$ if both f and g do. Assume that $P, Q \in \mathbb{K}(x)\langle\partial_x\rangle$ are telescopers for f, g, respectively. Then the LCLM of P and Q is a telescoper for $f + g$ by the commutativity between operators in $\mathbb{K}(x)\langle\partial_x\rangle$ and the operators ∂_{y_i}'s. ∎

Let $V = (V_1, \ldots, V_m)$ be any set partition of the variables $\mathbf{x} = \{x_1, \ldots, x_n\}$. A rational function $f \in \mathbb{K}(\mathbf{x})$ is said to be *split* with respect to the partition V if $f = f_1 \cdots f_m$ with $f_i \in \mathbb{K}(V_i)$. Split polynomials and rational functions will be used to state our existence criteria for telescopers in Section 4.

The central problem studied in this paper is the following existence problem on telescopers for *rational functions* in three variables.

Problem 2.5. *Given $f \in \mathbb{K}(x, y, z)$, decide whether f has a telescoper of type $(\partial_x, \partial_y, \partial_z)$.*

In the pure continuous case, telescopers of type (D_x, D_y, D_z) always exist for rational functions in $\mathbb{K}(x, y, z)$ [13]. The above existence problem in the pure discrete case in which telescopers are of type $(S_x, \Delta_y, \Delta_z)$ has been solved in [12]. We will consider the remaining four mixed cases in Section 4.

Let $G = \langle \sigma_x, \sigma_y, \sigma_z \rangle$ be the free abelian group generated by $\sigma_x, \sigma_y, \sigma_z$. Let $f \in \mathbb{E}$ and H be a subgroup of G. We call

$$[f]_H := \{\sigma(f) \mid \sigma \in H\}$$

the H-orbit at f. Two elements $f, g \in \mathbb{E}$ are said to be H-equivalent if $[f]_H = [g]_H$, denoted by $f \sim_H g$. The relation $\sim_H$ is an equivalence relation.

3 Reductions and Exactness Criteria

The first necessary step for solving the existence problem of telescopers is the following exactness problem. Throughout this section, we let $\mathbb{F}$ be a field of characteristic zero.

Problem 3.1. *Given a rational function $f \in \mathbb{F}(\mathbf{y})$ with $\mathbf{y} = (y_1, \ldots, y_n)$, decide whether there exist $g_1, \ldots, g_n \in \mathbb{F}(\mathbf{y})$ such that $f = \partial_{y_1}(g_1) + \cdots + \partial_{y_n}(g_n)$. If such g_i's exist, we say that f is $(\partial_{y_1}, \ldots, \partial_{y_n})$-*exact *in $\mathbb{F}(\mathbf{y})$, or* exact *for short when no ambiguity arises.*

The following lemma shows that the exactness is unchanged even if we are looking for the g_i's in a larger field.

Lemma 3.2. *Let $f \in \mathbb{F}(\mathbf{y})$. Then f is exact in $\overline{\mathbb{F}(\mathbf{y})}$ if and only if it is exact in $\mathbb{F}(\mathbf{y})$.*

Proof. The sufficiency is obvious. For the necessity, we assume that there exist $u_1, \ldots, u_n \in \overline{\mathbb{F}(\mathbf{y})}$ such that

$$f = \partial_{y_1}(u_1) + \cdots + \partial_{y_n}(u_n).$$

Let $\mathbb{L}$ be a finite normal extension of $\mathbb{F}(\mathbf{y})$ containing the u_i's and $\partial_{y_i}(u_i)$'s and let $\mathrm{Tr}_{\mathbb{L}/\mathbb{F}(\mathbf{y})}$ be the trace from $\mathbb{L}$ to $\mathbb{F}(\mathbf{y})$, which commutes with ∂_{y_i} by [15, Lemma 3.1] for $\partial_{y_i} = \Delta_{y_i}$ and by [6, Theorem 3.2.4 (i)] for $\partial_{y_i} = D_{y_i}$. Then

$$\mathrm{Tr}_{\mathbb{L}/\mathbb{F}(\mathbf{y})}(f) = \mathrm{Tr}_{\mathbb{L}/\mathbb{F}(\mathbf{y})}\left(\sum_{i=1}^{n} \partial_{y_i}(u_i)\right) = \sum_{i=1}^{n} \partial_{y_i}(\mathrm{Tr}_{\mathbb{L}/\mathbb{F}(\mathbf{y})}(u_i)).$$

Since $f \in \mathbb{F}(\mathbf{y})$, we have $\mathrm{Tr}_{\mathbb{L}/\mathbb{F}(\mathbf{y})}(f)=mf$ with $m=[\mathbb{L} : \mathbb{F}(\mathbf{y})]$. Thus $f = \sum_{i=1}^{n} \partial_{y_i}(g_i)$ with $g_i = \frac{1}{m}\mathrm{Tr}_{\mathbb{L}/\mathbb{F}(\mathbf{y})}(u_i) \in \mathbb{F}(\mathbf{y})$. ∎

The exactness problem for bivariate rational functions can be determined by reductions (see [13, 15, 24, 8]). For a later convenience, we summarize these results below.

The Ostrogradsky–Hermite reduction in z [25, 23] decomposes a rational function $f \in \mathbb{F}(y,z)$ into the form

$$f = D_z(g) + \frac{a}{b}, \tag{3.1}$$

where $g \in \mathbb{F}(y,z)$ and $a, b \in \mathbb{F}(y)[z]$ with $\gcd(a,b) = 1$, $\deg_z(a) < \deg_z(b)$ and b being squarefree in z over $\mathbb{F}(y)$. Moreover, $f = D_z(u)$ for some $u \in \mathbb{F}(y,z)$ if and only if $a = 0$. As a discrete analogue of the Ostrogradsky–Hermite reduction, Abramov's reduction in z [1, 2] decomposes $f \in \mathbb{F}(y,z)$ into the form

$$f = \Delta_z(g) + \frac{a}{b}, \tag{3.2}$$

where $g \in \mathbb{F}(y,z)$ and $a, b \in \mathbb{F}(y)[z]$ with $\gcd(a,b) = 1$, $\deg_z(a) < \deg_z(b)$ and b being shift-free in z over $\mathbb{F}(y)$. Moreover, $f = \Delta_z(u)$ for some $u \in \mathbb{F}(y,z)$ if and only if $a = 0$. We recall the criterion on the (D_y, D_z)-exactness of bivariate rational functions from [13, Lemma 4].

Lemma 3.3. *Let $f \in \mathbb{F}(y,z)$ be of the form (3.1) and write*

$$\frac{a}{b} = \sum_{i=1}^{n} \frac{\alpha_i}{z - \beta_i},$$

where $\alpha_i, \beta_i \in \overline{\mathbb{F}(y)}$ with $\beta_i \neq \beta_j$ for $1 \leq i \neq j \leq n$. Then f is (D_y, D_z)-exact in $\mathbb{F}(y,z)$ if and only if for each i with $1 \leq i \leq n$, we have $\alpha_i = D_y(\gamma_i)$ for some $\gamma_i \in \overline{\mathbb{F}(y)}$.

For any isomorphism σ on $\mathbb{F}(y,z)$ and $a, b \in \mathbb{F}(y,z)$, we have the reduction formula

$$\frac{a}{\sigma^n(b)} = \sigma(g) - g + \frac{\sigma^{-n}(a)}{b}, \tag{3.3}$$

where $g = \sum_{i=0}^{n-1} \frac{\sigma^{i-n}(a)}{\sigma^i(b)}$ if $n \geq 0$ and $g = -\sum_{i=0}^{-n-1} \frac{\sigma^i(a)}{\sigma^{n+i}(b)}$ if $n < 0$. For a rational function $f \in \mathbb{F}(y,z)$ of the form (3.1), we can use the above reduction formula with $\sigma = \sigma_y$ to further decompose f as

$$f = \Delta_y(u) + D_z(v) + \sum_{i=1}^{I} \frac{a_i}{d_i}, \tag{3.4}$$

where $u, v \in \mathbb{F}(y,z), a_i \in \mathbb{F}(y)[z], d_i \in \mathbb{F}[y,z]$ with $\deg_z(a_i) < \deg_z(d_i)$ and the d_i's are irreducible polynomials in distinct $\langle \sigma_y \rangle$-orbits. We recall the criterion on the (Δ_y, D_z)-exactness in $\mathbb{F}(y,z)$ from [8, Theorem 2].

Lemma 3.4. *Let $f \in \mathbb{F}(y,z)$ be of the form (3.4). Then f is (Δ_y, D_z)-exact in $\mathbb{F}(y,z)$ if and only if for each $i \in \{1,\ldots,I\}$, $d_i \in \mathbb{F}[z]$ and $a_i = \Delta_y(b_i)$ for some $b_i \in \mathbb{F}(y)[z]$. In particular, if f is (Δ_y, D_z)-exact, so is each a_i/d_i.*

For a rational function $f \in \mathbb{F}(y,z)$ of the form (3.2), we can use the above reduction formula with $\sigma = \sigma_y$ to further decompose f as

$$f = \Delta_y(u) + \Delta_z(v) + \sum_{i=1}^{I}\sum_{j=1}^{J_i} \frac{a_{i,j}}{d_i^j}, \tag{3.5}$$

where $u, v \in \mathbb{F}(y,z)$, $a_{i,j} \in \mathbb{F}(y)[z]$, and $d_i \in \mathbb{F}[y,z]$ with $\deg_z(a_{i,j}) < \deg_z(d_i)$ and d_i's being irreducible such that d_i and d_j are in distinct $\langle \sigma_y, \sigma_z\rangle$-orbits for all $1 \le i \ne j \le I$. We recall the criterion on the (Δ_y, Δ_z)-exactness of bivariate rational functions by combining Lemma 3.2 and Theorem 3.3 in [24].

Lemma 3.5. *Let $f \in \mathbb{F}(y,z)$ be of the form (3.5). Then f is (Δ_y, Δ_z)-exact in $\mathbb{F}(y,z)$ if and only if for all i with $1 \le i \le I$, we have $\sigma_y^{m_i}(d_i) = \sigma_z^{n_i}(d_i)$ for some $m_i, n_i \in \mathbb{Z}$ with $m_i > 0$ and $a_{i,j} = \sigma_y^{m_i}\sigma_z^{-n_i}(b_{i,j}) - b_{i,j}$ for some $b_{i,j} \in \mathbb{K}(x,y)[z]$ with $\deg_z(b_{i,j}) < \deg_z(d_i)$. In particular, if f is (Δ_y, Δ_z)-exact, so is each $a_{i,j}/d_i^j$.*

We now present a vector version of the Hermite-like reduction in [19], which will be used in Section 4.1. Let $\vec{a} = \frac{1}{d}(a_1,\ldots,a_n) \in \mathbb{K}(x,y)^n$ with $a_i, d \in \mathbb{K}[x,y]$ satisfying that $\gcd(d, a_1, \ldots, a_n) = 1$ and $\mathbf{B} = \frac{1}{e}(b_{i,j}) \in \mathbb{K}(x,y)^{n\times n}$ with $e, b_{i,j} \in \mathbb{K}[x,y]$ and $\gcd(e, b_{1,1}, \ldots, b_{1,n}, \ldots, b_{n,n}) = 1$. Let $p \in \mathbb{K}[x,y]$ be any irreducible factor of d that is coprime with e. Then $d = p^m q$ with $q \in \mathbb{K}[x,y]$ and $\gcd(p,q) = 1$. Since $\gcd(p, D_y(p)) = 1$, we have $\gcd(p, D_y(p)q) = 1$ and then the Bézout relation

$$a_i = s_i p + t_i D_y(p) q,$$

where $s_i, t_i \in \mathbb{K}(x)[y]$. Using integration by part, we get

$$\frac{a_i}{p^m q} = \frac{s_i p + t_i D_y(p) q}{p^m q} = D_y\left(\frac{u_i}{p^{m-1}}\right) + \frac{v_i}{p^{m-1}q},$$

where $u_i = t_i(1-m)^{-1}$ and $v_i = s_i - (1-m)^{-1}D_y(t_i)q$. Let $\vec{u} = (u_1,\ldots,u_n)$ and $\vec{v} = (v_1,\ldots,v_n)$. Then we have

$$\vec{a} = D_y\left(\frac{\vec{u}}{p^{m-1}}\right) + \frac{\vec{v}}{p^{m-1}q} = D_y\left(\frac{\vec{u}}{p^{m-1}}\right) + \frac{\vec{u}}{p^{m-1}}\cdot\mathbf{B} + \frac{\vec{w}}{p^{m-1}qe},$$

where $\vec{w} \in \mathbb{K}(x)[y]^n$. Repeating this process yields

$$\vec{a} = D_y\left(\frac{\vec{g}}{p^{m-1}}\right) + \frac{\vec{g}}{p^{m-1}}\cdot\mathbf{B} + \frac{\vec{h}}{pqe},$$

where $\vec{g}, \vec{h} \in \mathbb{K}(x)[y]^n$. By reducing the multiplicity of each irreducible factor of d that is coprime with e in the above way, we obtain the additive decomposition

$$\vec{a} = D_y(\vec{b}) + \vec{b}\cdot\mathbf{B} + \vec{r}, \tag{3.6}$$

where $\vec{b} \in \mathbb{K}(x,y)^n$ and $\vec{r} = \frac{1}{pq}(r_1, \ldots, r_n)$ with $r_i \in \mathbb{K}(x)[y]$ and $p, q \in \mathbb{K}[x,y]$ be such that p is a squarefree polynomial and $\gcd(p, e) = 1$ and each irreducible factor of q divides e. We call the above process a *vector Hermite reduction* of $\vec{a}$ with respect to $\mathbf{B}$.

4 Existence Criteria

We will reduce the existence problem of telescopers in the trivariate case to that in the bivariate case and two related problems. To this end, we first recall the existence criterion on telescopers for bivariate rational functions from [14, 9].

Theorem 4.1. *A rational function $f \in \mathbb{K}(x,y)$ has a telescoper of type (S_x, D_y) (or (D_x, Δ_y)) if and only if f can be decomposed into the form $f = D_y(g) + r$ (or $f = \Delta_y(g) + r$), where $g, r \in \mathbb{K}(x,y)$ and the denominator of r is split with respect to the partition $(\{x\}, \{y\})$, i.e., it is of the form $p_1 p_2$ with $p_1 \in \mathbb{K}[x]$ and $p_2 \in \mathbb{K}[y]$.*

Example 4.2. *Let $f = 1/(x+y)$. Then f has no telescoper of type (S_x, D_y) nor type (D_x, Δ_y) since $x + y$ is not split.*

Problem 4.3 (Shift Equivalence Testing Problem)**.** *Let $\mathbb{F}$ be any computable field of characteristic zero. Given $p \in \mathbb{F}[x_1, ..., x_n]$, decide whether there exist $m_1, \ldots, m_n \in \mathbb{Z}$ with $m_1 > 0$ such that $p(x_1+m_1, \ldots, x_n+m_n) = p(x_1, \ldots, x_n)$.*

This problem is solved by Grigoriev in [20, 21] and more recently by Dvir et al. in [18] with better complexity.

Problem 4.4 (Separation Problem)**.** *Given an algebraic function $\alpha \in \overline{\mathbb{K}(x,y)}$, decide whether there exists a nonzero operator $L \in \mathbb{K}(x)\langle D_x \rangle$ such that $L(\alpha) = 0$. If such an operator exists, we say that α is* separable *in x and y.*

As a special case of [10, Proposition 10], a rational function is separable if and only if it is of the form $a/(bc)$ with $a \in \mathbb{K}[x,y], b \in \mathbb{K}[x]$ and $c \in \mathbb{K}[y]$. This motivates the nomenclature of Problem 4.4. We will solve the separation problem in the forthcoming paper [11] related to parallel telescoping for algebraic functions.

4.1 Telescopers of type (S_x, D_y, D_z)

We now consider the first mixed case of the existence problem of telescopers for rational functions in three variables.

Problem 4.5. *Given $f \in \mathbb{K}(x,y,z)$, decide whether there exists a nonzero operator $L \in \mathbb{K}(x)\langle S_x \rangle$ such that $L(f) = D_y(g) + D_z(h)$ for some $g, h \in \mathbb{K}(x,y,z)$.*

Let $f \in \mathbb{F}(y,z)$ be of the form (3.1) with $\mathbb{F} = \mathbb{K}(x)$. If f is (D_y, D_z)-exact in $\mathbb{K}(x,y,z)$, then 1 is a telescoper for f. From now on, we assume that f is not

(D_y, D_z)-exact. By collecting the roots of b into different $\langle\sigma_x\rangle$-orbits, we can write f as $f = D_z(u) + r$ with $u \in \mathbb{K}(x, y, z)$ and

$$r = \sum_{i=1}^{I} \sum_{j=0}^{J_i} \frac{\alpha_{i,j}}{z - \sigma_x^j(\beta_i)}, \tag{4.1}$$

where $\alpha_{i,j}, \beta_i \in \overline{\mathbb{K}(x, y)}$ and the β_i's are in distinct $\langle\sigma_x\rangle$-orbits. Note that f has a telescoper of type (S_x, D_y, D_z) if and only if r has a telescoper of the same type.

Lemma 4.6. *Let $r = \sum_{j=0}^{J} \alpha_j/(z - \sigma_x^j(\beta))$ with $\alpha_j, \beta \in \overline{\mathbb{K}(x, y)}$ and $\sigma_x^m(\beta) \neq \beta$ for any $m \in \mathbb{Z} \setminus \{0\}$. Then r is (D_y, D_z)-exact if it has a telescoper of type (S_x, D_y, D_z).*

Proof. Assume that $L = \sum_{\ell=0}^{\rho} e_\ell S_x^\ell \in \mathbb{K}(x)\langle S_x\rangle$ with $e_0 \neq 0$ is a telescoper for r of type (S_x, D_y, D_z). Then

$$L(r) = \sum_{j=0}^{J+\rho} \frac{\tilde{\alpha}_j}{z - \sigma_x^j(\beta)} = D_y(u) + D_z(v),$$

where $u, v \in \overline{\mathbb{K}(x, y)}(z)$ and $\tilde{\alpha}_j = \sum_{k=0}^{j} e_k \sigma_x^k(\alpha_{j-k})$ with $e_k = 0$ for $k > \rho$ and $\alpha_j = 0$ for $j > J$. Since $\sigma_x^m(\beta) \neq \beta$ for any $m \in \mathbb{Z} \setminus \{0\}$, we have $\tilde{\alpha}_j = D_y(\tilde{\gamma}_j)$ for some $\tilde{\gamma}_j \in \overline{\mathbb{K}(x, y)}$ by Lemma 3.3. We now show by induction the claim that for each j with $0 \leq j \leq J$, $\alpha_j = D_y(\gamma_j)$ for some $\gamma_j \in \overline{\mathbb{K}(x, y)}$. Since $\tilde{\alpha}_0 = e_0\alpha_0$ and $e_0 \in \mathbb{K}(x) \setminus \{0\}$, we have $\alpha_0 = D_y(\gamma_0)$ with $\gamma_0 = \tilde{\gamma}_0/e_0$. So the claim is true for α_0. Suppose that we have shown that $\alpha_j = D_y(\gamma_j)$ for $j = 0, \ldots, k-1$ with $k \leq J$. Note that

$$\tilde{\alpha}_k = e_0\alpha_k + e_1\sigma_x(\alpha_{k-1}) + \cdots + e_k\sigma_x^k(\alpha_0) = D_y(\tilde{\gamma}_k).$$

Then $\alpha_k = D_y(\gamma_k)$ with $\gamma_k = \frac{1}{e_0}(\tilde{\gamma}_k - \sum_{j=1}^{k} e_j\sigma_x^j(\gamma_{k-j}))$. This proves the claim. So r is (D_y, D_z)-exact. ∎

Theorem 4.7. *Let $r \in \mathbb{K}(x, y, z)$ be of the form (4.1). Then r has a telescoper of type (S_x, D_y, D_z) if and only if for each i with $1 \leq i \leq I$, either $\alpha_{i,j}/(z - \sigma_x^j(\beta_i))$ is (D_y, D_z)-exact or $\beta_i \in \overline{\mathbb{K}(y)}$ and there exists a nonzero $L_{i,j} \in \mathbb{K}(x)\langle D_x\rangle$ such that $L_{i,j}(\alpha_{i,j}) = D_y(\gamma_{i,j})$ for some $\gamma_{i,j} \in \mathbb{K}(x, y)(\beta_i)$.*

Proof. The sufficiency follows from Lemma 2.4 since each fraction $\alpha_{i,j}/(z - \sigma_x^j(\beta_i))$ is either (D_y, D_z)-exact or has a telescoper of type (S_x, D_y, D_z). To show the necessity, we assume that $L = \sum_{\ell=0}^{\rho} e_\ell S_x^\ell \in \mathbb{K}(x)\langle S_x\rangle$ with $e_0 \neq 0$ is a telescoper for r of type (S_x, D_y, D_z). Then we have

$$L(r) = \sum_{i=1}^{I} \sum_{j=0}^{J_i+\rho} \frac{\tilde{\alpha}_{i,j}}{z - \sigma_x^j(\beta_i)} = D_y(u) + D_z(v),$$

where $u, v \in \mathbb{K}(x, y, z)$ and $\tilde{\alpha}_{i,j} = \sum_{k=0}^{j} e_k \sigma_x^k(\alpha_{i,j-k})$ with $e_k = 0$ for $k > \rho$ and $\alpha_{i,j} = 0$ for $j > J_i$. By Lemma 3.3, we have $r_i = \sum_{j=0}^{J_i+\rho} \frac{\tilde{\alpha}_{i,j}}{z-\sigma_x^j(\beta_i)}$ is (D_y, D_z)-exact for each i with $1 \leq i \leq I$ since the β_i's are in distinct $\langle \sigma_x \rangle$-orbits. If there exists a nonzero $m_i \in \mathbb{N}$ such that $\sigma_x^{m_i}(\beta_i) = \beta_i$, then $\beta_i \in \overline{\mathbb{K}(y)}$ by [14, Lemma 3.4 (i)]. So $J_i = 0$ and $L(\alpha_{i,0}/(z - \beta_i)) = L(\alpha_{i,0})/(z - \beta_i)$ is (D_y, D_z)-exact, which implies that $L(\alpha_{i,0}) = D_y(\gamma_{i,0})$ for some $\gamma_{i,0} \in \overline{\mathbb{K}(x, y)}$. Since $\alpha_{i,0} \in \mathbb{K}(x, y)(\beta_i)$, we can choose $\gamma_{i,0} \in \mathbb{K}(x, y)(\beta_i)$ by the trace argument. If there is no nonzero $m_i \in \mathbb{N}$ such that $\sigma_x^{m_i}(\beta_i) = \beta_i$, then the theorem follows from Lemma 4.6. ∎

Problem 4.5 now has been reduced to the exactness testing problem and the following existence problem.

Problem 4.8. *Given $\alpha \in \mathbb{K}(x, y)(\beta)$ with β algebraic over $\mathbb{K}(y)$, decide whether α has a telescoper of type (S_x, D_y), i.e., there exists a nonzero $L \in \mathbb{K}(x)\langle S_x \rangle$ such that $L(\alpha) = D_y(\gamma)$ for some $\gamma \in \mathbb{K}(x, y)(\beta)$.*

Let $\beta \in \overline{\mathbb{K}(y)}$ and $n = [\mathbb{K}(y, \beta) : \mathbb{K}(y)]$. Assume that $\{\beta_1, \ldots, \beta_n\}$ is a basis for $\mathbb{K}(y, \beta)$ as a linear space over $\mathbb{K}(y)$. Since $D_y(\beta_i) \in \mathbb{K}(y, \beta)$, we have $D_y(\beta_i) = \frac{1}{e} \sum_{j=1}^{n} b_{j,i} \beta_j$ with $e, b_{j,i} \in \mathbb{K}[y]$. Set $\mathbf{B} = \frac{1}{e}(b_{i,j}) \in \mathbb{K}(y)^{n \times n}$. Then $D_y(\vec{\beta}) = \vec{\beta} \cdot \mathbf{B}$ with $\vec{\beta} = (\beta_1, \ldots, \beta_n)$. Since $\alpha \in \mathbb{K}(x, y)(\beta)$, we can write $\alpha = \vec{a} \cdot \vec{\beta}^T$ for some $\vec{a} = \frac{1}{d}(a_1, \ldots, a_n) \in \mathbb{K}(x, y)^n$ with $d, a_i \in \mathbb{K}[x, y]$. Applying the vector Hermite reduction to $\vec{a}$ with respect to $\mathbf{B}$ yields the additive decomposition (3.6), which is equivalent to

$$\alpha = D_y(\vec{b} \cdot \vec{\beta}^T) + \tilde{\alpha} \text{ with } \tilde{\alpha} = \frac{1}{pq} \sum_{i=1}^{n} r_i \beta_i, \tag{4.2}$$

where $r_i, p \in \mathbb{K}[x, y]$ with p being squarefree and $\gcd(p, e) = 1$ and each irreducible factor of q divides $e \in \mathbb{K}[y]$.

Theorem 4.9. *Let $\alpha \in \mathbb{K}(x, y)(\beta)$ be of the form (4.2). Then α has a telescoper of type (S_x, D_y) if and only if the polynomial p in (4.2) is split in x and y.*

Proof. Assume that p is split in x and y, i.e., $p = p_1 p_2$ for some $p_1 \in \mathbb{K}[x]$ and $p_2 \in \mathbb{K}[y]$. Then $\tilde{\alpha}$ can be written as $\tilde{\alpha} = \sum_{j=1}^{m} f_j \cdot g_j$ with $f_j \in \mathbb{K}(x)$ and $g_j \in \mathbb{K}(y)(\beta)$ since $\beta_i \in \mathbb{K}(y)(\beta)$ and $q \in \mathbb{K}[y]$. Let $L_j = f_j(x) S_x - f_j(x + 1) \in \mathbb{K}(x)\langle S_x \rangle$. Then $L_j(f_j \cdot g_j) = 0$. So the LCLM of the L_j's annihilates $\tilde{\alpha}$, which then is a telescoper for α of type (S_x, D_y). To show the necessity, we assume that $L = \sum_{\ell=0}^{\rho} e_\ell S_x^\ell$ with $e_0 e_\rho \neq 0$ is a telescoper for α of type (S_x, D_y). Then $L(\tilde{\alpha}) = D_y(\tilde{\gamma})$ for some $\tilde{\gamma} \in \mathbb{K}(x, y)(\beta)$. Write $\tilde{\gamma} = \vec{s} \cdot \vec{\beta}^T$ with $\vec{s} \in \mathbb{K}(x, y)^n$ and $\vec{r} = (r_1, \ldots, r_n)$. Then we have

$$L\left(\frac{1}{pq}\vec{r}\right) = \sum_{\ell=0}^{\rho} \frac{e_\ell}{\sigma_x^\ell(p) q} \sigma_x^\ell(\vec{r}) = D_y(\vec{s}) + \vec{s} \cdot \mathbf{B}.$$

Suppose that p is not split in x and y. Then there exists a non-split irreducible factor p_0 of p such that $\sigma_x(p_0) \nmid p$. Then $\sigma_x^\rho(p_0)$ is also a non-split irreducible polynomial and only divides the denominator $\sigma_x^\rho(p)q$. Since p is squarefree, the valuation of the left-hand side of the above equality at $\sigma_x^\rho(p_0)$ is -1. However, the valuation of the right-hand side is either ≥ 0 or < -1 since $\mathbf{B} \in \mathbb{K}(y)^{n\times n}$. This leads to a contradiction. So p is split in x and y. ∎

Example 4.10. *Let $f = x/(z^2 - y)$. Then*

$$f = \frac{\alpha}{z-\beta} + \frac{-\alpha}{z+\beta},$$

where $\alpha = x/(2\sqrt{y})$ and $\beta = \sqrt{y}$. By Theorem 4.7, f has a telescoper of type (S_x, D_y, D_z) since $\beta \in \overline{\mathbb{K}(y)}$ and $L = xS_x - (x+1)$ is a telescoper for α of type (S_x, D_y). Indeed, L is also a telescoper for f of type (S_x, D_y, D_z).

4.2 Telescopers of type (S_x, Δ_y, D_z)

We address the second mixed case of the existence problem of telescopers for rational functions in three variables.

Problem 4.11. *Given $f \in \mathbb{K}(x, y, z)$, decide whether there exists a nonzero operator $L \in \mathbb{K}(x)\langle S_x\rangle$ such that $L(f) = \Delta_y(g) + D_z(h)$ for some $g, h \in \mathbb{K}(x, y, z)$.*

By the Ostrogradsky–Hermite reduction in z and the reduction formula (3.3) with $\sigma = \sigma_y$, we can decompose f as

$$f = \Delta_y(u) + D_z(v) + r, \text{ where } r = \sum_{i=1}^{I}\sum_{j=0}^{J_i} \frac{a_{i,j}}{\sigma_x^j(d_i)} \tag{4.3}$$

with $a_{i,j} \in \mathbb{K}(x, y)[z], d_i \in \mathbb{K}[x, y, z]$ such that $\deg_z(a_{i,j}) < \deg_z(d_i)$ and the d_i's are irreducible polynomials in distinct $\langle\sigma_x, \sigma_y\rangle$-orbits. Note that f has a telescoper of type (S_x, Δ_y, D_z) if and only if r does.

Lemma 4.12. *Let $r \in \mathbb{K}(x, y, z)$ be as in (4.3). Then r has a telescoper of type (S_x, Δ_y, D_z) if and only if for each i with $1 \leq i \leq I$, we have $r_i = \sum_{j=0}^{J_i} \frac{a_{i,j}}{\sigma_x^j(d_i)}$ has a telescoper of the same type.*

Proof. The sufficiency follows from Lemma 2.4. For the necessity we assume that $L = \sum_{k=0}^{\rho} \ell_k S_x^k$ with $\ell_0 \neq 0$ is a telescoper for r of type (S_x, Δ_y, D_z). Then

$$L(r) = \sum_{i=1}^{I} L(r_i) = \sum_{i=1}^{I}\left(\sum_{j=0}^{J_i+\rho} \frac{\sum_{k=0}^{j} \ell_k \sigma_x^k(a_{i,j-k})}{\sigma_x^j(d_i)}\right)$$

with $\ell_k = 0$ if $k > \rho$ and $a_{i,j} = 0$ if $j > J_i$ is (Δ_y, D_z)-exact. Since the d_i's are in distinct $\langle\sigma_x, \sigma_y\rangle$-orbits, the $\sigma_x^j(d_i)$'s are in distinct $\langle\sigma_y\rangle$-orbits. By Lemma 3.4, we have $L(r_i)$ is (Δ_y, D_z)-exact for each $i \in \{1, \ldots, I\}$. Thus each r_i has a telescoper of the same type as r. ∎

Now the existence problem is reduced to that for rational functions of the form

$$f = \sum_{i=0}^{I} \frac{a_i}{\sigma_x^i(d)}, \tag{4.4}$$

where $a_i \in \mathbb{K}(x,y)[z], d \in \mathbb{K}[x,y,z]$ with $\deg_z(a_i) < \deg_z(d)$ and d is irreducible in z over $\mathbb{K}(x,y)$. We will proceed by a case distinction according to whether or not d satisfies the condition: there exist integers m,n with $m > 0$ such that

$$\sigma_x^m(d) = \sigma_y^n(d). \tag{4.5}$$

This condition can be checked by solving the bivariate case of Problem 4.3.

Lemma 4.13. *Let $f \in \mathbb{K}(x,y,z)$ be of the form (4.4) and d does not satisfy the condition (4.5). Then f has a telescoper of type (S_x, Δ_y, D_z) if and only if f is (Δ_y, D_z)-exact.*

Proof. The sufficiency is clear by definition. Assume that $L = \sum_{k=0}^{\rho} \ell_k S_x^k$ with $\ell_0 \neq 0$ is a telescoper for f of type (S_x, Δ_y, D_z). Then we have

$$L(f) = \sum_{i=0}^{\rho+I} \left(\frac{\sum_{j=0}^{i} \ell_j \sigma_x^j(a_{i-j})}{\sigma_x^i(d)} \right)$$

is (Δ_y, D_z)-exact. Since d does not satisfy the condition (4.5), we have $\sigma_x^i(d)$ and $\sigma_x^{i'}(d)$ are in distinct $\langle \sigma_y \rangle$-orbits for all $i \neq i'$. By Lemma 3.4, for any i with $0 \leq i \leq \rho + I$, there exist $u_i, v_i \in \mathbb{K}(x,y,z)$ such that

$$\frac{\sum_{j=0}^{i} \ell_j \sigma_x^j(a_{i-j})}{\sigma_x^i(d)} = \Delta_y(u_i) + D_z(v_i). \tag{4.6}$$

To show that all fractions $a_i/\sigma_x^i(d)$ are (Δ_y, D_z)-exact, we proceed by induction. The assertion is true for $i = 0$ since $a_0/d = \Delta_y(u_0/\ell_0) + D_z(v_0/\ell_0)$. Suppose that we have shown that $a_i/\sigma_x^i(d)$ is (Δ_y, D_z)-exact for $i = 0, \ldots, s-1$ with $s \leq I$. By the equality (4.6) with $i = s$, we get

$$\frac{a_s}{\sigma_x^s(d)} = \Delta_y \left(\frac{u_s}{\ell_0} \right) + D_z \left(\frac{v_s}{\ell_0} \right) - \sum_{j=1}^{s} \frac{\ell_j}{\ell_0} \sigma_x^j \left(\frac{a_{s-j}}{\sigma_x^{s-j}(d)} \right).$$

By the commutativity between σ_x and σ_y, σ_z and Lemma 3.4, we have $a/\sigma_x^i(d)$ is (Δ_y, D_z)-exact for any $i \in \mathbb{N}$ if a/d is. By the induction hypothesis, we have $\frac{\ell_j}{\ell_0}\sigma_x^j(a_{s-j}/\sigma_x^{s-j}(d))$ is (Δ_y, D_z)-exact for all $1 \leq j \leq s$. So are $a_s/\sigma_x^s(d)$ and f. ∎

We now deal with the case in which d satisfies the condition (4.5). From now on, we will always assume that m is the smallest positive integer such that

$\sigma_x^m(d) = \sigma_y^n(d)$ for some $n \in \mathbb{Z}$. By the reduction formula (3.3) with $\sigma = \sigma_y$, the existence problem is further reduced to that for rational function of the form

$$f = \sum_{i=0}^{m-1} \frac{a_i}{\sigma_x^i(d)}, \tag{4.7}$$

where $a_i \in \mathbb{K}(x, y)[z], d \in \mathbb{K}[x, y, z]$ with $\deg_z(a_i) < \deg_z(d)$ and d is irreducible in z over $\mathbb{K}(x, y)$.

The following lemma is similar to Lemma 5.3 in [12].

Lemma 4.14. *Let $f \in \mathbb{K}(x, y, z)$ be of the form (4.7) and d satisfy the condition (4.5). Then f has a telescoper of type (S_x, Δ_y, D_z) if and only if for each i with $0 \le i \le I$, the fraction $a_i/\sigma_x^i(d)$ has a telescoper of the same type.*

Proof. The sufficiency follows from Lemma 2.4. For the necessity direction, one can adapt the second part of the proof of [12, Lemma 5.3] to the setting of telescopers of type (S_x, Δ_y, D_z) literally by interpreting $\equiv_{y,z} 0$ as being (Δ_y, D_z)-exact. ∎

The above lemma reduces the existence problem to that for simple fractions of the form

$$f = \frac{a}{bd}, \tag{4.8}$$

where $a, d \in \mathbb{K}[x, y, z], b \in \mathbb{K}[x, y]$ with $\gcd(a, bd) = 1$ and $\deg_z(a) < \deg_z(d)$, and d is irreducible and satisfies the condition (4.5). We will consider two cases according to whether d is in $\mathbb{K}[x, z]$ or not. If $d \in \mathbb{K}[x, z]$, then $\sigma_y^i(d) = d$ for all $i \in \mathbb{N}$. The condition $\sigma_x^m(d) = \sigma_y^n(d)$ implies that d is also free of x, i.e., $d \in \mathbb{K}[z]$. Thus $L \in \mathbb{K}(x)\langle S_x\rangle$ is a telescoper for f of type (S_x, Δ_y, D_z) if and only if $L(a/b) = \Delta_y(u)$ for some $u \in \mathbb{K}(x, y)[z]$ with $\deg_z(u) < \deg_z(d)$. Write $a = \sum_{i=0}^{\deg_z(d)-1} a_i z^i$ and $u = \sum_{i=0}^{\deg_z(d)-1} u_i z^i$. Then for each i with $0 \le i \le \deg_z(d) - 1$, we have $L(a_i/b) = \Delta_y(u_i)$, i.e., L is a telescoper for all a_i/b of type (S_x, Δ_y). The existence problem is then reduced to that in the bivariate case, for which Theorem 4.1 applies. So it remains to deal with the case when d is not in $\mathbb{K}[x, z]$.

Lemma 4.15. *Let $f = a/b$ with $a, b \in \mathbb{K}[x, y]$ and $\gcd(a, b) = 1$ and let $e_0, \ldots, e_r \in \mathbb{K}(x)$ be such that $e_0 e_r \neq 0$. Then*

(i) $b = b_1 b_2$ with $b_1 \in \mathbb{K}[x], b_2 \in \mathbb{K}[y]$ if $\sum_{i=0}^r e_i \sigma_x^i(f) = 0$;

(ii) $b = b_1 b_2$ with $b_1 \in \mathbb{K}[x]$ and $b_2 \in \mathbb{K}[v]$ with $v = nx+my$ if $\sum_{i=0}^r e_i \tau^i(a/b) = 0$ with $\tau := \sigma_x^m \sigma_y^{-n}$.

Proof. (i) Assume that $\sum_{i=0}^r e_i \sigma_x^i(f) = 0$. Let b_1 and b_2 be the content and primitive part of b as a polynomial in y over $\mathbb{K}[x]$. If $b_2 \in \mathbb{K}[x, y]\setminus\mathbb{K}[y]$, then there exists at least one irreducible factor p such that $\deg_x(p) > 0$ and $\sigma_x^i(p) \nmid b_2$ for all $i < 0$. Then $\sigma_x^i(p)$ is also irreducible for all $i \in \mathbb{Z}$ and $\gcd(\sigma_x^i(p), \sigma_x^j(p)) = 1$ if $i \neq j$. Let s be the largest integer such that $\sigma_x^s(p) \mid b_2$. Then the irreducible

polynomial $\sigma_x^{r+s}(p)$ only divides the denominator $\sigma_x^r(b)$ and not others, which implies that $\sum_{i=0}^r e_i\sigma_x^i(f) \neq 0$. A contradiction. So we must have $b_2 \in \mathbb{K}[y]$.
(*ii*) Note that $\mathbb{K}(x,y) = \mathbb{K}(\bar{x},\bar{y})$ with $\bar{x} = x/m$ and $\bar{y} = nx+my$. For any $f \in \mathbb{K}(x,y)$, we have $\tau(f) = \sigma_{\bar{x}}(\bar{f})$ if $f(x,y) = \bar{f}(\bar{x},\bar{y})$. Then $\sum_{i=0}^r e_i\tau^i(a/b) = 0$ if and only if $\sum_{i=0}^r \bar{e}_i\sigma_{\bar{x}}^i(\bar{a}/\bar{b}) = 0$. By the first assertion, we have $\bar{b} = \bar{b}_1\bar{b}_2$ for some $\bar{b}_1 \in \mathbb{K}[\bar{x}]$ and $\bar{b}_2 \in \mathbb{K}[\bar{y}]$. Thus $b = b_1b_2$ for some $b_1 \in \mathbb{K}[x]$ and $b_2 \in \mathbb{K}[nx+my]$. ∎

Lemma 4.16. *Let $a \in \mathbb{K}(x)[y,z]$ and $b \in \mathbb{K}[x,y,z]$ be such that $b \neq 0$ and $\sigma_x^m(b) = \sigma_y^n(b)$ for some $m,n \in \mathbb{Z}$ with $m > 0$. Then a/b has a telescoper of type (S_x,Δ_y,D_z).*

Proof. Set $f = a/b$. It suffices to show that for sufficiently large $I \in \mathbb{N}$, there exist $\ell_0,\ldots,\ell_I \in \mathbb{K}(x)$, not all zero, and $g \in \mathbb{K}(x,y,z)$ such that $L(f) = \Delta_y(g)$ with $L = \sum_{i=0}^I \ell_i S_x^{im}$. By the reduction formula (3.3) with $\sigma = \sigma_y$, we have

$$S_x^{im}(f) = \frac{\sigma_x^{im}(a)}{\sigma_x^{im}(b)} = \frac{\sigma_x^{im}(a)}{\sigma_y^{in}(b)} = \Delta_y(g_i) + \frac{\sigma_y^{-in}\sigma_x^{im}(a)}{b}$$

for some $g_i \in \mathbb{K}(x,y,z)$. Note that the degrees of the polynomials $\sigma_y^{-in}\sigma_x^{im}(a)$ in y and z are the same as that of a. So all the polynomials $\sigma_y^{-in}\sigma_x^{im}(a)$ lie in a finite dimensional linear space over $\mathbb{K}(x)$. Therefore, for sufficiently large I, there exist $\ell_0,\ldots,\ell_I \in \mathbb{K}(x)$, not all zero, such that $\sum_{i=0}^I \ell_i\sigma_y^{-in}\sigma_x^{im}(a) = 0$. This implies that L is a telescoper for f of type (S_x,Δ_y,D_z). ∎

Theorem 4.17. *Let $f \in \mathbb{K}(x,y,z)$ be of the form (4.8). Assume that d is not in $\mathbb{K}[x,z]$. Then f has a telescoper of type (S_x,Δ_y,D_z) if and only if $b = b_1b_2$ for some $b_1 \in \mathbb{K}[x]$ and $b_2 \in \mathbb{K}[x,y]$ satisfying $\sigma_x^m(b_2) = \sigma_y^n(b_2)$.*

Proof. The sufficiency follows from Lemma 4.16. For the necessity, we assume that $L \in \mathbb{K}(x)\langle S_x\rangle$ is a telescoper for f of type (S_x,Δ_y,D_z). Write $L = L_0 + L_1 + \cdots + L_{m-1}$ with $L_i = \sum_{j=0}^{r_i} \ell_{i,j}S_x^{jm+i}$. Since $\sigma_x^i(d)$ and $\sigma_x^j(d)$ are in distinct $\langle\sigma_y\rangle$-orbits for all $0 \leq i \neq j \leq m-1$, Lemma 3.4 implies that L_i is also a telescoper for f of the same type for each i with $0 \leq i \leq m-1$. A direct calculation yields

$$L_0(f) = \Delta_y(g_0) + \frac{A}{d},$$

where $A = \sum_{j=0}^{r_0} \ell_{0,j}\tau^j(a/b)$ with $\tau = \sigma_y^{-n}\sigma_x^m$. By Lemma 3.4, we have $A = 0$ since $d \notin \mathbb{K}[x,z]$. So the necessity follows from Lemma 4.15 (ii). ∎

Example 4.18. *Let $f = 1/(bd)$ with $b = x+y$ and $d = z^2-x-y$. Note that d satisfies the condition $\sigma_x(d) = \sigma_y(d)$ and is not in $\mathbb{K}[x,z]$. By Theorem 4.17, f has a telescoper of type (S_x,Δ_y,D_z) since b satisfies the same condition as d. Indeed, $L = S_x - 1$ is a telescoper for f since $L(f) = \Delta_y(f) + D_z(0)$.*

4.3 Telescopers of type (D_x, Δ_y, D_z)

We consider the third mixed case of the existence problem of telescopers for rational functions in three variables.

Problem 4.19. *Given $f \in \mathbb{K}(x, y, z)$, decide whether there exists a nonzero operator $L \in \mathbb{K}(x)\langle D_x\rangle$ such that $L(f) = \Delta_y(g)+D_z(h)$ for some $g, h \in \mathbb{K}(x, y, z)$.*

By the Ostrogradsky–Hermite reduction and the reduction formula (3.3), we can decompose $f \in \mathbb{K}(x, y, z)$ as

$$f = \Delta_y(u) + D_z(v) + r \text{ with } r = \sum_{i=1}^{I} \frac{\alpha_i}{z - \beta_i}, \tag{4.9}$$

where $u, v \in \mathbb{K}(x, y, z)$ and $\alpha_i, \beta_i \in \overline{\mathbb{K}(x, y)}$ with $\alpha_i \neq 0$ and the β_i's are in distinct $\langle\sigma_y\rangle$-orbits. Then f has a telescoper of type (D_x, Δ_y, D_z) if and only if r has a telescoper of the same type.

Lemma 4.20. *For any $L = \sum_{j=0}^{\rho} \ell_j D_x^j \in \mathbb{K}(x)\langle D_x\rangle$ and $\alpha, \beta \in \overline{\mathbb{K}(x, y)}$, there exists $g \in \overline{\mathbb{K}(x, y)}(z)$ such that*

$$L\left(\frac{\alpha}{z-\beta}\right) = \frac{L(\alpha)}{z-\beta} + D_z(g). \tag{4.10}$$

Proof. Let $\mathrm{res}_z(f, \beta)$ denote the residue of f at $z = \beta$ in z. The map $\mathrm{res}_z(\cdot, \beta)$ is $\mathbb{K}(x, y)$-linear and commutes with the operator D_x by [13, Proposition 3]. Then we have

$$\mathrm{res}_z\left(L\left(\frac{\alpha}{z-\beta}\right), \beta\right) = L\left(\mathrm{res}_z\left(\frac{\alpha}{z-\beta}, \beta\right)\right) = L(\alpha).$$

So all residues of $h := L(\alpha/(z - \beta)) - L(\alpha)/(z - \beta)$ at all of its poles are zero. By Proposition 2.2 in [14], we have h is D_z-exact, i.e., $h = D_z(g)$ for some $g \in \overline{\mathbb{K}(x, y)}(z)$. ∎

The next theorem reduces Problem 4.19 to the separation problem for algebraic functions (Problem 4.4) and the existence problem of telescopers in $\mathbb{K}(x, y)(\beta)$ with $\beta \in \overline{\mathbb{K}(x)}$.

Theorem 4.21. *Let $f \in \mathbb{K}(x, y, z)$ be of the form (4.9). Then f has a telescoper of type (D_x, Δ_y, D_z) if and only if for each i with $1 \leq i \leq I$, either α_i is separable in x and y or $\beta_i \in \overline{\mathbb{K}(x)}$ and $\alpha_i \in \mathbb{K}(x, y)(\beta_i)$ has a telescoper of type (D_x, Δ_y).*

Proof. If for each i with $1 \leq i \leq I$, either α_i is separable or $\beta_i \in \overline{\mathbb{K}(x)}$ and $\alpha_i \in \mathbb{K}(x, y)(\beta_i)$ has a telescoper of type (D_x, Δ_y), then there exists a nonzero $L_i \in \mathbb{K}(x)\langle D_x\rangle$ such that either $L_i(\alpha_i) = 0$ or $L_i(\alpha_i) = \Delta_y(\gamma_i)$ for some $\gamma_i \in \mathbb{K}(x, y)(\beta_i)$. By Lemmas 4.20 and 3.4, we have

$$\begin{aligned} L_i\left(\frac{\alpha_i}{z-\beta_i}\right) &= D_z(g_i) + \frac{L_i(\alpha_i)}{z-\beta_i} = D_z(g_i) + \frac{\Delta_y(\gamma_i)}{z-\beta_i} \\ &= D_z(g_i) + \Delta_y\left(\frac{\gamma_i}{z-\beta_i}\right), \end{aligned}$$

where $g_i \in \overline{\mathbb{K}(x,y)}(z)$. So for each i with $1 \leq i \leq I$, the fraction $\alpha_i/(z-\beta_i)$ has a telescoper of type (D_x, Δ_y, D_z). Then f has a telescoper of the same type by Lemmas 2.4 and 3.2. To show the necessity, we assume that $L \in \mathbb{K}(x)\langle D_x\rangle$ is a telescoper for f of type (D_x, Δ_y, D_z). By Lemma 4.20, there exists $w \in \overline{\mathbb{K}(x,y)}(z)$ such that

$$\begin{aligned} L(f) &= \Delta_y(L(u)) + D_z(L(v) + w) + \sum_{i=1}^{I} \frac{L(\alpha_i)}{z-\beta_i} \\ &= \Delta_y(g) + D_z(h) \end{aligned}$$

for some $g, h \in \mathbb{K}(x,y,z)$. For each i with $1 \leq i \leq I$, either α_i is separable if $L(\alpha_i) = 0$ or $L(\alpha_i)/(z-\beta_i)$ is (Δ_y, D_z)-exact if $L(\alpha_i) \neq 0$. In the later case we have $\beta_i \in \overline{\mathbb{K}(x)}$ and $L(\alpha_i) = \Delta_y(\gamma_i)$ for some $\gamma_i \in \mathbb{K}(x,y)(\beta_i)$ by Lemma 3.4. ∎

Remark 4.22. *The existence problem of telescopers of type (D_x, Δ_y) can be verified by [14, Theorem 4.9], whose statement is for functions in $\mathbb{K}(x,y)$, but its proof also works for functions in $\overline{\mathbb{K}(x)}(y)$. In particular, this covers the case in which the functions are in $\mathbb{K}(x,y)(\beta)$ with $\beta \in \overline{\mathbb{K}(x)}$.*

Example 4.23. *Let f be as in Example 4.18. Then*

$$f = \frac{\alpha}{z-\beta} + \frac{-\alpha}{z+\beta},$$

where $\alpha = \frac{1}{2(x+y)\sqrt{x+y}}$ and $\beta = \sqrt{x+y}$. Note that α is not separable in x and y since its successive derivatives $D_x^i(\alpha) = (-1)^i \prod_{j=0}^{i}(j+1/2)(x+y)^{-(i+3/2)}$ are linearly independent over $\mathbb{K}(x)$. Since β is not in $\overline{\mathbb{K}(x)}$, we have f has no telescoper of type (D_x, Δ_y, D_z) by Theorem 4.21.

4.4 Telescopers of type $(D_x, \Delta_y, \Delta_z)$

We now address the last mixed case of the existence problem of telescopers for rational functions in three variables.

Problem 4.24. *Given $f \in \mathbb{K}(x,y,z)$, decide whether there exists a nonzero operator $L \in \mathbb{K}(x)\langle D_x\rangle$ such that $L(f) = \Delta_y(g) + \Delta_z(h)$ for some $g, h \in \mathbb{K}(x,y,z)$.*

As mentioned in Section 3, any rational function $f \in \mathbb{K}(x,y,z)$ can be decomposed as

$$f = \Delta_y(u) + \Delta_z(v) + r \text{ with } r = \sum_{i=1}^{I}\sum_{j=1}^{J_i} \frac{a_{i,j}}{d_i^j}, \tag{4.11}$$

where $u, v \in \mathbb{K}(x,y,z)$, $a_{i,j} \in \mathbb{K}(x,y)[z]$, and $d_i \in \mathbb{K}[x,y,z]$ with $\deg_z(a_{i,j}) < \deg_z(d_i)$ and d_i's being irreducible polynomials in distinct $\langle\sigma_y, \sigma_z\rangle$-orbits. Then f has a telescoper of type $(D_x, \Delta_y, \Delta_z)$ if and only if r has a telescoper of the same type. Next we will check whether a polynomial $d \in \mathbb{K}[x,y,z]$ satisfies the condition:

$$\sigma_y^m(d) = \sigma_z^n(d) \quad \text{for some } m, n \in \mathbb{Z} \text{ with } m > 0. \tag{4.12}$$

Theorem 4.25. *Let $r \in \mathbb{K}(x,y,z)$ be as in (4.11). Then r has a telescoper of type $(D_x, \Delta_y, \Delta_z)$ if and only if the fraction $a_{i,j}/d_i^j$ has a telescoper of the same type for all i,j with $1 \leq i \leq I$ and $1 \leq j \leq J_i$.*

Proof. The sufficiency follows from Lemma 2.4. For the necessity, we assume that $L = \sum_{\ell=0}^{\rho} e_\ell D_x^\ell \in \mathbb{K}(x)\langle D_x\rangle$ with $e_\rho \neq 0$ is a telescoper for r of type $(D_x, \Delta_y, \Delta_z)$. If r is (Δ_y, Δ_z)-exact, then the assertion follows from Lemma 3.5. From now on, we assume that r is not (Δ_y, Δ_z)-exact, i.e., for each i with $1 \leq i \leq I$, either d_i does not satisfy the condition (4.12) or $\sigma_y^{m_i}(d_i) = \sigma_z^{n_i}(d_i)$ for some $m_i, n_i \in \mathbb{Z}$ with $m_i > 0$ and $a_{i,j} \neq \sigma_y^{m_i}\sigma_z^{-n_i}(b_{i,j}) - b_{i,j}$ for any $b_{i,j} \in \mathbb{K}(x,y)[z]$ with $\deg_z(b_{i,j}) < \deg_z(d_i)$.

We first show that $D_x(d_i) = 0$, i.e., $d_i \in \mathbb{K}[y,z]$ for all i with $1 \leq i \leq I$. Over the field $\overline{\mathbb{K}(x,y)}$, we decompose r as

$$r = \sum_{i=1}^{I}\sum_{j=1}^{J_i} \frac{\alpha_{i,j}}{(z-\beta_i)^j},$$

where $\alpha_{i,j}, \beta_i \in \overline{\mathbb{K}(x,y)}$ with $\alpha_{i,J_i} \neq 0$ and for all i,j with $1 \leq i \neq j \leq I$, we have $\beta_i - \sigma_y^n(\beta_j) \notin \mathbb{Z}$ for any $n \in \mathbb{Z}$. It suffices to show that $D_x(\beta_i) = 0$ for all i with $1 \leq i \leq I$. Applying L to r yields

$$L(r) = \sum_{i=1}^{I}\left(\frac{J_i^{\bar{\rho}}\alpha_{i,J_i}D_x(\beta_i)^\rho}{(z-\beta_i)^{J_i+\rho}} + \sum_{j=1}^{J_i+\rho-1}\frac{\tilde{\alpha}_{i,j}}{(z-\beta_i)^j}\right), \tag{4.13}$$

where $J_i^{\bar{\rho}} = J_i(J_i+1)\cdots(J_i+\rho-1)$ and $\tilde{\alpha}_{i,j} \in \overline{\mathbb{K}(x,y)}$. Since L is a telescoper for r, $L(r)$ is (Δ_y, Δ_z)-exact. By Lemma 3.5, either $D_x(\beta_i) = 0$ or $\sigma_y^{m_i}(\beta_i) - \beta_i = n_i$ for some $m_i, n_i \in \mathbb{Z}$ with $m_i > 0$ and

$$J_i^{\bar{\rho}}\alpha_{i,J_i}D_x(\beta_i)^\rho = \sigma_y^{m_i}(\gamma_i) - \gamma_i$$

for some $\gamma_i \in \overline{\mathbb{K}(x,y)}$. Since D_x commutes with σ_y, we have $D_x(\sigma_y^{m_i}(\beta_i) - \beta_i) = \sigma_y^{m_i}(D_x(\beta_i)) - D_x(\beta_i) = 0$. Suppose that $D_x(\beta_i) \neq 0$. Then

$$\alpha_{i,J_i} = \sigma_y^{m_i}\left(\frac{\gamma_i}{J_i^{\bar{\rho}}D_x(\beta_i)^\rho}\right) - \frac{\gamma_i}{J_i^{\bar{\rho}}D_x(\beta_i)^\rho},$$

which contradicts with the assumption that r is not exact.

Since $d_i \in \mathbb{K}[y,z]$ and L is a telescoper for f, we have

$$L(r) = \sum_{i=1}^{I}\sum_{j=1}^{J_i}\frac{L(a_{i,j})}{d_i^j} = \Delta_y(g) + \Delta_z(h),$$

where $g, h \in \mathbb{K}(x,y,z)$. By Lemma 3.5, we have either $L(a_{i,j}) = 0$ or $\sigma_y^{m_i}(d_i) = \sigma_z^{n_i}(d_i)$ for some $m_i, n_i \in \mathbb{Z}$ with $m_i > 0$ and

$$L(a_{i,j}) = \sigma_y^{m_i}\sigma_z^{-n_i}(b_{i,j}) - b_{i,j}$$

for some $b_{i,j} \in \mathbb{K}(x,y)[z]$ with $\deg_z(b_{i,j}) < \deg_z(d_i)$. This implies that

$$L\left(\frac{a_{i,j}}{d_i^j}\right) = \Delta_y(g_{i,j}) + \Delta_z(h_{i,j})$$

for some $g_{i,j}, h_{i,j} \in \mathbb{K}(x,y,z)$. So L is a telescoper for the fraction $a_{i,j}/d_i^j$ all i,j with $1 \leq i \leq I$ and $1 \leq j \leq J_i$. ∎

Problem 4.24 now has been reduced to that for simple fractions of the form

$$f = \frac{a}{d^m}, \tag{4.14}$$

where $a \in \mathbb{K}(x,y)[z]$ and $d \in \mathbb{K}[x,y,z]$ with $\deg_z(a) < \deg_z(d)$ and d being irreducible in z over $\mathbb{K}(x,y)$. Assume that $L \in \mathbb{K}(x)\langle D_x \rangle$ is a telescoper of type $(D_x, \Delta_y, \Delta_z)$ for f. Then $d \in \mathbb{K}[y,z]$ by the same argument as in the proof of Theorem 4.25, which implies that $L(a/d^m) = L(a)/d^m$. Since L is a telescoper, we have $L(a)/d^m$ is (Δ_y, Δ_z)-exact. Now we proceed by a case distinction according to whether or not d satisfies the condition (4.12).

Case 1. If d does not satisfy the condition (4.12), then $L(a) = 0$ by Lemma 3.5. In this case, Problem 4.24 is reduced to the separation problem for $a \in \mathbb{K}(x,y)[z]$. Write $a = \sum_{i=0}^{s} a_i z^i$ with $a_i \in \mathbb{K}(x,y)$. Then $L(a) = 0$ if and only if $L(a_i) = 0$ for all i with $0 \leq i \leq s$. As a special case of [10, Proposition 10], a rational function in $\mathbb{K}(x,y)$ is separable in x and y if and only if its denominator is split in x and y. So the existence problem is then reduced to checking whether some polynomial is split or not, which can be done via GCD computations.

Case 2. If $\sigma_y^m(d) = \sigma_z^n(d)$ for some $m, n \in \mathbb{Z}$ with $m > 0$, then $L(a) = \sigma_y^m \sigma_z^{-n}(b) - b$ for some $b \in \mathbb{K}(x,y)[z]$ with $\deg_z(b) < \deg_z(d)$ by Lemma 3.5. Note that $\mathbb{K}(x,y,z) = \mathbb{K}(x,\bar{y},\bar{z})$ with $\bar{y} = y/m$ and $\bar{z} = ny + mz$. Then for all $p \in \mathbb{K}(x,y)[z]$, $\sigma_y^m \sigma_z^{-n}(p(x,y,z)) = \sigma_{\bar{y}}(\bar{p}(x,\bar{y},\bar{z}))$ and $L(p) = L(\bar{p})$. So the equalities $L(a) = \sigma_y^m \sigma_z^{-n}(b) - b$ and $L(\bar{a}) = \sigma_{\bar{y}}(\bar{b}) - \bar{b}$ are equivalent. This reduces Problem 4.24 to the existence problem of telescopers for rational functions in $\mathbb{K}(x,y)$ of type (D_x, Δ_y), which has been dealt with in [14, Theorem 4.9].

Example 4.26. *Let f be as in Example 4.18. Since $d = z^2 - x - y$ does not satisfies the condition (4.12), we are now in the first case. Note that $1/(x+y)$ is not separable since $x+y$ is not split in x and y. Then f has no telescoper of type $(D_x, \Delta_y, \Delta_z)$.*

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