Integral Bases for P-Recursive Sequences

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Chinese Academy of Sciences
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ISSAC 2020, Kalamata, Greece
July 20-23, 2020

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joint work with S. Chen, M. Kauers, and T. Verron

Integral functions: algebraic case

Notation.

- C: a field of characteristic zero (e.g. $\mathbb{Q}$, $\mathbb{C}$).
- $C(x)$: the field of rational functions in x.
- $M \in C[x,y]$: irreducible over $C(x)$ with $r = \deg_y(M)$.
- $K = C(x)[y]/\langle M \rangle \cong C(x)(\beta)$: the algebraic function field.

$$K = \left\{ a_0 + a_1\beta + \cdots + a_{r-1}\beta^{r-1} \mid a_i \in C(x) \right\}$$

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Definition. A function $f \in K$ is called integral over $C[x]$ if

$$f^d + p_{d-1}\, f^{d-1} + \cdots + p_0 = 0 \quad \text{with } p_i \in C[x].$$

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Definition. The integral closure of $C[x]$ in K is defined as

$$\mathcal{O}_K := \{\, f \in K \mid f \text{ is integral over } C[x] \,\}.$$

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Example. $K = C(x)(\beta)$ with $\beta = \sqrt[3]{x^2}$.

$$\left\{1, \beta, \tfrac{1}{x}\beta^2\right\} \text{ is an integral basis.}$$

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Answer. van Hoeij's algorithm, etc.

Computation of integral bases: van Hoeij's algorithm

Input. $M \in C[x,y]$ monic irreducible over $C(x)$ with $r = \deg_y(M)$;
output. an integral basis $\{B_0, \cdots, B_{r-1}\}$.

1. Start with $(B_0, \ldots, B_{r-1}) := (1, \beta, \ldots, \beta^{r-1})$.
2. For $d \in \{0, 1, \ldots, r-1\}$
3. While there exist $a_0, \ldots, a_{d-1} \in C[x]$ such that

$$A = \frac{a_0 B_0 + \cdots + a_{d-1} B_{d-1} + B_d}{p(x)}$$

   is integral and $p(x) \in C[x] \setminus C$; replace B_d by A.
4. Return $B_0, \ldots, B_{r-1}$.

Computation of integral bases: van Hoeij's algorithm

Example. $M = (\frac{25}{16}x^3 + 2x^4) - x^3 y - (2x+1)y^2 + y^3.$

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$\mathcal{O}_0$

$C[x] + C[x]\beta + C[x]\beta^2$

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$\alpha = 0, \quad B_2 := \dfrac{-\beta+\beta^2}{x}$ $\qquad \mathcal{O}_1$

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$\alpha \in C, \quad B_d := \frac{a_0 B_0 + \cdots a_{d-1} B_{d-1} + B_d}{x - \alpha}$ $\quad \mathcal{O}_n$

$\cdots$

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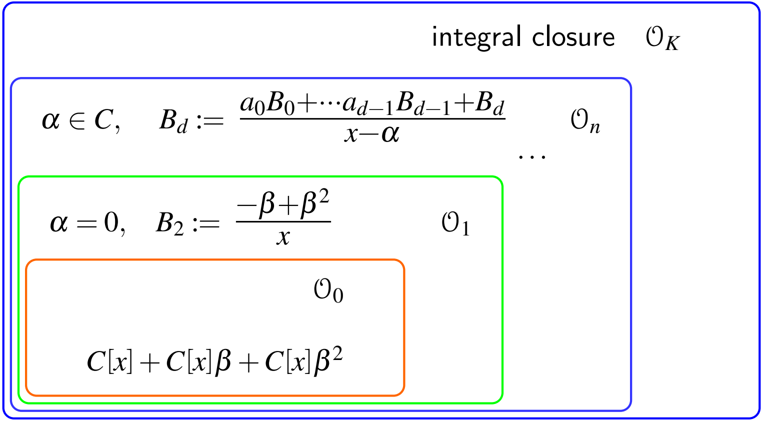

$\{1, \beta, \frac{1}{x}(-\beta + \beta^2)\}$ is an integral basis of $C(x)[y]/\langle M \rangle$.

Integral bases: general framework

Definition. Let k be a field. The map $\nu : k \to \mathbb{Z} \cup \{\infty\}$ is called a valuation if for all $a, b \in k$

- $\nu(a) = \infty$ iff $a = 0$;
- $\nu(ab) = \nu(a) + \nu(b)$;
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Example. For a nonzero $f \in C(x)$, define $\nu_z(f) = m$ if

$$f = (x-z)^m \, \frac{a}{b} \quad \text{where } (x-z) \nmid a, b.$$

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The corresponding valuation ring is

$$C[x]_{x-z} = \left\{ \frac{a}{b} \in C(x) \,\middle|\, (x-z) \nmid b \right\}.$$

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Example. $(C(x), \nu_z)$ is a valued field.

Fact. The valuation ring is integrally closed.

Integral Bases: general framework

Definition. Let V be a vector space over (k, ν). The map $\mathrm{val}: V \to \mathbb{Z} \cup \{\infty\}$ is called a value function if for all $B, B_1, B_2 \in V$ and $u \in k$

- $\mathrm{val}(B) = \infty$ iff $B = 0$;
- $\mathrm{val}(u \cdot B) = \nu(u) + \mathrm{val}(B)$;
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An element B of V is called integral if $\mathrm{val}(B) \geq 0$.

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Problem 2. How to compute a basis if it exists?

Computation of integral bases: general case

Input. a vector space basis $\{B_1, \ldots, B_r\}$ of (V, val) over $(C(x), \nu_z)$.
output. an integral basis.

1. For $d \in \{1, \ldots, r\}$ do:
2. Replace B_d by $(x-z)^{-\mathrm{val}(B_d)} B_d$.
3. While there exist $a_1, \ldots, a_{d-1} \in C[x]$ such that

$$A = \frac{a_1 B_1 + \cdots + a_{d-1} B_{d-1} + B_d}{x-z}$$

is integral; replace B_d by A.

4. Return $B_1, \ldots, B_r$.

Existence of integral bases: general case

Theorem. Let (V, val) be a valued vector space over $(C(x), \nu)$. TFAE.

(a) There is an integral basis of (V, val).

(b) There is a discriminant function $\mathrm{Disc} : \mathbb{B}_V \to \mathbb{Z}$, where $\mathbb{B}_V$ is the set of all bases of V.

(c) The algorithm terminates.

(d) The completion of V w.r.t ν is of dimension $r = \dim_k(V)$.

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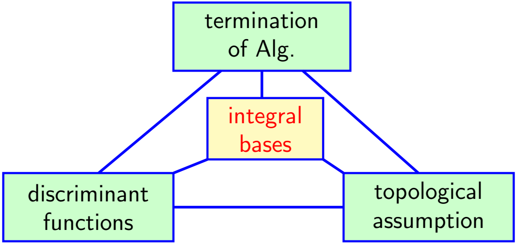

Integral Bases: three cases

Algebraic case

- $K = C(x)[y]/\langle M \rangle$, where $M \in C(x)[y]$ irreducible
- The integral elements of K form a free $C[x]$-module.
- Computation: van Hoeij's algorithm 1994, etc.

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D-finite case

- $V = C(x)[D]/\langle L \rangle$, $Dx = xD + 1$, where $L \in C(x)[D]$ admits a fundamental system of solutions in $C[[[x - z]]]$.
- The integral elements of V form a free $C[x]$-left module.
- Computation: Kauers-Koutschan's algorithm 2015.

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P-recursive case

- Question. What are integral elements?

P-recursive sequences

Definition. A sequence $f : \mathbb{Z} \to C$ is called P-recursive if

$$p_0(n)f(n) + p_1(n)f(n+1) + \cdots + p_r(n)f(n+r) = 0 \quad \text{for } p_i \in C[x].$$

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Example. The Harmonic sequence $f(n) = \sum_{k=1}^{n} \frac{1}{k}$ satisfies

$$(n+1)f(n) - (2n+3)f(n+1) + (n+2)f(n+2) = 0.$$

P-recursive sequences: solution space

Setting.

- $L = p_0(n) + p_1(n)S + \cdots + p_r(n)S^r \in C[n][S]$ with $p_0, p_r \neq 0$.
- $V = C(n)[S]/\langle L \rangle$, $Sn = (n+1)S$

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Let $\alpha \in C$. For a sequence $f \in C^{\alpha+\mathbb{Z}} := \{ u : \alpha + \mathbb{Z} \to C \}$,

- Operator action:

$$L \cdot f = p_0(n)f(n) + p_1(n)f(n+1) + \cdots + p_r(n)f(n+r).$$

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- Solution space:

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Question. How to decide the solution space $\mathrm{Sol}(L)$?

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Example 1. $f(n+1)+f(n)-f(n+2)=0$

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$\alpha=0$	$\cdots$	-2	-1					
1st sol	$\cdots$	1	0					
2nd sol	$\cdots$	0	1					

P-recursive sequences: solution space

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1st sol	$\cdots$	1	0	1	1	2	3	$\cdots$
2nd sol	$\cdots$	0	1	1	2	3	5	$\cdots$

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Example 2. $nf(n)+2n^2f(n+1)+(n+1)^2f(n+3)=0$

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$$f(-1)+2\cdot f(0)+0\cdot f(2)=0$$

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sol	$\cdots$	0	1	0	0	?	$\cdots$

$$f(-1)+2\cdot f(0)+0\cdot f(2)=0$$

Contradiction!

Deformed P-recursive sequences

Setting [van Hoeij1999].

- $L = p_0(n) + p_1(n)S + \cdots + p_r(n)S^r \in C[n][S]$ with $p_0, p_r \neq 0$.

Let q be a new parameter. For a sequence

$$f \in C^{\alpha+\mathbb{Z}} := \{u : \alpha + \mathbb{Z} \to C((q))\},$$

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$$L \cdot f = p_0(n+q)f(n) + p_1(n+q)f(n+1) + \cdots + p_r(n+q)f(n+r).$$

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$$L \cdot f = p_0(n+q)f(n) + p_1(n+q)f(n+1) + \cdots + p_r(n+q)f(n+r).$$

- Solution space:

$$\mathrm{Sol}(L) := \{ f : \alpha + \mathbb{Z} \to C((q)) \mid L \cdot f = 0\}.$$

Deformed P-recursive sequences: solution space

Example. $L = n + 2n^2S + (n+1)^2S^3$.

Deformed P-recursive sequences: solution space

Example. $L = n + 2n^2 S + (n+1)^2 S^3$.

$\alpha = 0$	$\cdots$	-2	-1	0				
1st sol	$\cdots$	1	0	0				
2nd sol	$\cdots$	0	1	0				
3rd sol	$\cdots$	0	0	1				

Deformed P-recursive sequences: solution space

Example. $L = n + 2n^2 S + (n+1)^2 S^3$.

$\alpha = 0$	$\cdots$	-2	-1	0	1	2	3	$\cdots$
1st sol	$\cdots$	1	0	0	$\frac{-q+2}{(q-1)^2}$	$\frac{2q-4}{q^2}$	$\frac{-4q+8}{(q+1)^2}$	$\cdots$
2nd sol	$\cdots$	0	1	0	0	$\frac{-q+1}{q^2}$	$\frac{2q-2}{(q+1)^2}$	$\cdots$
3rd sol	$\cdots$	0	0	1	$\frac{-2q^2+8q-8}{(q-1)^2}$	$\frac{4q^2-16q+16}{q^2}$	$\frac{-8q^2+31q-32}{(q+1)^2}$	$\cdots$

Fact. $\mathrm{Sol}(L)$ is a $C((q))$-vector space of dimension $r = \mathrm{ord}(L)$.

Deformed P-recursive sequences: solution space

Example. $L = n + 2n^2 S + (n+1)^2 S^3$.

$\alpha = 0$	$\cdots$	-2	-1	0	1	2	3	$\cdots$
1st sol	$\cdots$	1	0	0	$2 + \cdots$	$-4q^{-2} + 2q^{-1}$	$8 + 20q + \cdots$	$\cdots$
2nd sol	$\cdots$	0	1	0	0	$q^{-2} - q^{-1}$	$-1 + 6q + \cdots$	$\cdots$
3rd sol	$\cdots$	0	0	1	$-8 + \cdots$	$16q^{-2} - 16q^{-1} + \cdots$	$-32 + 95q + \cdots$	$\cdots$

Question. How to define a value function on $V = C(x)[S]/\langle L \rangle$?

Value functions: P-recursive case

Definition. For an operator $B \in V = C(n)[S]/\langle L \rangle$, we define $\mathrm{val}_z \colon V \to \mathbb{Z} \cup \{\infty\}$ by

$$\mathrm{val}_z(B) := \min_{b \in \mathrm{Sol}(L)} \Big(\nu_q((B \cdot b)(z)) - \liminf_{n \to \infty} \nu_q(b(z-n)) \Big)$$

for any $z \in \alpha + \mathbb{Z}$.

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Remark. For a normalized basis $\{b_1, \cdots, b_r\}$ of $\mathrm{Sol}(L)$, we have

$$\mathrm{val}_z(B) = \min_{j=1}^{r} \Big(\nu_q\big((B \cdot b_j)(z)\big) \Big).$$

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Theorem. The integral elements of V form a free $C[n]_{n-z}$-module.

$$\mathrm{Disc}_z(B_1, \cdots, B_r) := \nu_q\big(\det(((B_i \cdot b_j)(z))_{i,j=1}^r)\big)$$

Integral bases: P-recursive case

Example. $L = n + 2n^2 S + (n+1)^2 S^3$

Integral bases: P-recursive case

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$\alpha = 0$	$\cdots$	-2	-1	0	1	2	3	$\cdots$
1st sol	$\cdots$	1	0	0	$2+\cdots$	$-4q^{-2}+2q^{-1}$	$8+20q+\cdots$	$\cdots$
2nd sol	$\cdots$	0	1	0	0	$q^{-2}-q^{-1}$	$-1+6q+\cdots$	$\cdots$
3rd sol	$\cdots$	0	0	1	$-8+\cdots$	$16q^{-2}-16q^{-1}+\cdots$	$-32+95q+\cdots$	$\cdots$

Integral bases: P-recursive case

Example. $L = n + 2n^2 S + (n+1)^2 S^3$

$n = 1$	1			
1st sol	$2 + \cdots$			
2nd sol	0			
3rd sol	$-8 + \cdots$			

Integral bases: P-recursive case

Example. $L = n + 2n^2 S + (n+1)^2 S^3$

$n=1$	1	S		
1st sol	$2+\cdots$	$-4q^{-2}+2q^{-1}$		
2nd sol	0	$q^{-2}-q^{-1}$		
3rd sol	$-8+\cdots$	$16q^{-2}-16q^{-1}+\cdots$		

1 is an integral element of $C(n)[S]/\langle L\rangle$, but S not.

Integral bases: P-recursive case

Example. $L = n + 2n^2 S + (n+1)^2 S^3$

$n=1$	1	$(n-1)^2 S$		
1st sol	$2+\cdots$	$-4+2q$		
2nd sol	0	$1-q$		
3rd sol	$-8+\cdots$	$16-16q+\cdots$		

Integral bases: P-recursive case

Example. $L = n + 2n^2 S + (n+1)^2 S^3$

$n=1$	1	$(n-1)^2 S$	S^2	
1st sol	$2+\cdots$	$-4+2q$	$8+20q+\cdots$	
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Integral bases: P-recursive case

Example. $L = n + 2n^2 S + (n+1)^2 S^3$

$n=1$	1	$(n-1)^2 S$	S^2	$S^2 - 2(n-1)^2 S$
1st sol	$2+\cdots$	$-4+2q$	$8+20q+\cdots$	$24q+\cdots$
2nd sol	0	$1-q$	$-2+6q+\cdots$	$4q+\cdots$
3rd sol	$-8+\cdots$	$16-16q+\cdots$	$-32+95q+\cdots$	$63q+\cdots$

Integral bases: P-recursive case

Example. $L = n + 2n^2 S + (n+1)^2 S^3$

$n=1$	1	$(n-1)^2 S$	S^2	$-2(n-1)S + \frac{1}{n-1}S^2$
1st sol	$2+\cdots$	$-4+2q$	$8+20q+\cdots$	$24+\cdots$
2nd sol	0	$1-q$	$-2+6q+\cdots$	$4+\cdots$
3rd sol	$-8+\cdots$	$16-16q+\cdots$	$-32+95q+\cdots$	$63+\cdots$

integral closure

$$B_2 := \frac{1}{n-1}((n-1)^2 S + S^2)$$

$$C[n]_{n-1} + C[n]_{n-1}(n-1)^2 S + C[n]_{n-1} S^2$$

Integral bases: P-recursive case

Example. $L = n + 2n^2 S + (n+1)^2 S^3$

$n=1$	1	$(n-1)^2 S$	S^2	$-2(n-1)S + \frac{1}{n-1}S^2$
1st sol	$2+\cdots$	$-4+2q$	$8+20q+\cdots$	$24+\cdots$
2nd sol	0	$1-q$	$-2+6q+\cdots$	$4+\cdots$
3rd sol	$-8+\cdots$	$16-16q+\cdots$	$-32+95q+\cdots$	$63+\cdots$

$\{\ 1,\ (n-1)^2 S,\ -2(n-1)S + \frac{1}{n-1}S^2\ \}$ is an integral basis of $C(n)[S]/\langle L\rangle$ at $z=1$.

Summary

Main results.

- Extend van Hoeij's algorithm to valued vector spaces.
- Construct integral bases for P-recursive sequences.

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Thank you!

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Future work.

- Applications to symbolic summation.

Thank you!

See you on Monday, July 20, 2020, during the on-line session for questions and discussions!