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A Note on Ostrogradsky and Horowitz's Method

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Abstract. Ostrogradsky and Horowitz's method performs the additive decomposition of rational functions by solving linear systems. We show that there are extraneous factors when the systems in their method are solved by Cramer's rule. The results help us to obtain better bounds on the degrees of the coefficients in the additive decomposition.

1. Introduction

Let k be a field of characteristic zero, and $k(y)$ be the field of rational functions in y over k. For a nonzero rational function $f \in k(y)$, there exist g and r in $k(y)$ such that

$$f = D_y(g) + r \tag{1}$$

with r being proper and having a squarefree denominator. The pair (g, r) above is called an *additive decomposition* of f with respect to y. Up to adding a constant in k to g, a rational function f has a unique additive decomposition. There are several methods for computing the additive decomposition of a rational function (see e.g. [1, Chapter 2]). The most popular one is Hermite reduction [2], which reduces the multiplicity of denominators by the extended Euclidean algorithm recursively. Another method by Ostrogradsky and Horowitz [4, 3] merely needs to compute the squarefree part of the denominator of f and solve a linear system. This feature enables us to derive coefficient or degree bounds for the denominators and numerators of g and r when $k = \mathbb{Q}$ or $\mathbb{Q}(x)$ or the quotient field of a unique factorization domain. Experiments illustrate that extraneous factors arise when we solve linear systems appearing in Ostrogradsky and Horowitz's method. In this note, we describe these extraneous factors in order to derive tighter bounds.

Throughout the note, let R be a unique factorization domain and K be its quotient field. Assume that K is of characteristic zero. Let P and Q be two nonzero polynomials in $R[y]$ with $\deg_y P < \deg_y Q$, $\gcd(P, Q) = 1$ and $\deg_y Q > 0$.

Assume that the squarefree factorization of Q is $Q_1 Q_2^2 \cdots Q_m^m$, where $Q_1, Q_2, \ldots, Q_m$ are squarefree in $R[y]$, pairwise coprime in $K[y]$, and $\deg_y Q_m > 0$. Put $Q^* = Q_1 Q_2 \cdots Q_m$ and $Q^- = Q_2 Q_3^2 \cdots Q_m^{m-1}$. Then $Q = Q^* Q^-$. For later convenience, we set

$$d_y^* = \deg_y Q^* \quad \text{and} \quad d_y^- = \deg_y Q^-.$$

Let Q^{-2} denote the second deflation of Q, which equals to $Q_3 Q_4^2 \cdots Q_m^{m-2}$. For definiteness, we agree that $Q^{-2} = 1$ if $m \leq 2$.

Ostrogradsky and Horowitz's method takes Q^- and Q^* to be the denominators of g and r, respectively. According to (1), there exist two unique polynomials A and a in $K[y]$ with $\deg_y A < d_y^-$ and $\deg_y a < d_y^*$ such that

$$\frac{P}{Q} = D_y\left(\frac{A}{Q^-}\right) + \frac{a}{Q^*}. \tag{2}$$

Note that A and a satisfy (2) if and only if they satisfy the equation

$$P = D_y(A)Q^* - A\widetilde{Q} + aQ^-, \tag{3}$$

where $\widetilde{Q} = Q^* D_y(Q^-)/Q^-$ is a polynomial in $R[y]$ of degree $d_y^* - 1$.

Remark 1.1 *Among the three products in* (3), $D_y(A)Q^*$ *and* $A\widetilde{Q}$ *have degrees (in y) less than* $d_y^* + d_y^- - 1$, *and* aQ^- *has degree less than or equal to* $d_y^* + d_y^- - 1$.

Let $A = \sum_{i=0}^{d_y^- -1} A_i y^i, a = \sum_{j=0}^{d_y^* -1} a_j y^j$ and $P = \sum_{l=0}^{d_y^* + d_y^- -1} P_l y^l$ with undetermined coefficients. Then (3) holds if and only if

$$\left(A_{d_y^- -1}, \ldots, A_0, a_{d_y^* -1}, \ldots, a_0\right) M = \left(P_{d_y^* + d_y^- -1}, \ldots, P_0\right), \tag{4}$$

where M is a $(d_y^* + d_y^-) \times (d_y^* + d_y^-)$ matrix over R obtained by equating the like powers of y in (3). The uniqueness of A and a implies that M is invertible. We call M in (6) the matrix associated with Q, and denoted it by $M(Q)$.

Remark 1.2 *By Remark 1.1, the first column of $M(Q)$ consisting of zeros except the $(d_y^- + 1)$ entry, which is filled with $\mathrm{lc}_y(Q^-)$. The first d_y^- rows are of the form*

$$\begin{pmatrix} 0 & e_1 \mathrm{lc}_y(Q^*) & \dagger & \dagger & \cdots & \cdots & \dagger \\ 0 & 0 & e_2 \mathrm{lc}(Q^*) & \dagger & \cdots & \cdots & \dagger \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdots & \dagger \\ 0 & 0 & 0 & e_{d_y^-} \mathrm{lc}_y(Q^*) & \cdots & \cdots & \dagger \end{pmatrix},$$

where the e_j's are integers, and $\dagger$ stands for some linear combinations of the coefficients of Q^* over $\mathbb{Z}$. The last d_y^* rows consist of the coefficients of $y^{d_y^* -1}Q^-$, ..., yQ^-, Q^-.

2. A factor of the determinant of $M(Q)$ over R

By definitions of Q^*, Q^- and Q^{-2}, we have $\mathrm{lc}_y(Q^*) = \prod_{i=1}^m \mathrm{lc}_y(Q_i)$,

$$\mathrm{lc}_y(Q^-) = \prod_{j=2}^m \mathrm{lc}_y(Q_j)^{j-1} \quad \text{and} \quad \mathrm{lc}_y(Q^-) = \prod_{j=3}^m \mathrm{lc}_y(Q_j)^{j-2}.$$

So $\mathrm{lc}_y(Q^*)^{m-1}$ and $\mathrm{lc}_y(Q^*)^{m-2}$ are divisible by $\mathrm{lc}_y(Q^-)$ and $\mathrm{lc}_y(Q^{-2})$ in R, respectively. The following two technical lemmas will help us to factor $M(Q)$.

Lemma 2.1 *Let $q^* = \mathrm{lc}_y(Q^*)$, $q^- = \mathrm{lc}_y(Q^-)$, $q^{-2} = \mathrm{lc}_y(Q^{-2})$ and*

$$Q^- = q^- y^{d_y^-} + \sum_{j=1}^{d_y^-} q^-_{d_y^- - j} y^{d_y^- - j}.$$

Then $q^{*^j} q^-_{d_y^- - j}$ and $q^{*^{j-1}} q^-_{d_y^- - j}$ are divisible by q^- and q^{-2} for all j with $1 \le j \le d_y^-$, respectively.

Proof. If $j \ge m-1$, then q^{*^j} is divisible by q^-, and so is $q^{*^j} q^-_{d_y^- - j}$. We now consider the case in which $1 \le j < m-1$. Since Q^- is a product of $m(m-1)/2$ factors, $q^-_{d_y^- - j}$ is either zero or the sum of the coefficients that are divisible by at least the power product of $m(m-1)/2 - j$ leading coefficients of these factors. In other words, if $q^-_{d_y^- - j}$ is nonzero, then there are at most j polynomials among the $m(m-1)/2$ factors whose leading terms (with respect to y) do not contribute to form the term $q^-_{d_y^- - j} y^{d_y^- - j}$. Let $n_i = \max(0, i-1-j)$ for all i with $2 \le i \le m-1$, and let $p = q_2^{n_2} \cdots q_m^{n_m}$. Then $q^-_{d_y^- - j}$ is divisible by p. Consequently, the product $q^{*^j} p$ is divisible by q^-, and so is $q^{*^j} q^-_{d_y^- - j}$. The second divisibility follows from the same argument as above. □

Lemma 2.2 *With the notation introduced in Lemma 2.1, let Δ be a determinant of size $(d_y^* + d_y^- - 1)$ in the form*

$$\begin{vmatrix}
q^* & \dagger & \dagger & \cdots & \dagger & \dagger & \dagger & \cdots & \dagger \\
0 & q^* & \dagger & \dagger & \cdots & \dagger & \dagger & \cdots & \dagger \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\
0 & 0 & \cdots 0 & q^* & \dagger & \dagger & \dagger & \cdots & \dagger \\
q^- & q^-_{d_y^- - 1} & q^-_{d_y^- - 2} & \cdots & q_0^- & 0 & 0 & \cdots & 0 \\
0 & q^- & q^-_{d_y^- - 1} & q^-_{d_y^- - 2} & \cdots & q_0^- & 0 & \cdots & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\
0 & \cdots & 0 & q^- & q^-_{d_y^- - 1} & q^-_{d_y^- - 2} & \cdots & q_1^- & q_0^-
\end{vmatrix}
\begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} d_y^- \\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} d_y^* - 1 \end{matrix}$$

where $\dagger$ stands for any element of R and the last $d_y^* - 1$ rows consist of the coefficients of $y^{d_y^* - 2} Q^-$, $y^{d_y^* - 3} Q^-$, …, Q^-. Then Δ is divisible by q^-.

Proof. Let I be the ideal generated by q^- in $k[x]$. Then

$$\Delta \equiv \begin{vmatrix}
q^* & \dagger & \dagger & \cdots & \dagger & \dagger & \dagger & \cdots & \dagger \\
0 & q^* & \dagger & \dagger & \cdots & \dagger & \dagger & \cdots & \dagger \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\
0 & 0 & \cdots 0 & q^* & \dagger & \dagger & \dagger & \cdots & \dagger \\
0 & q^-_{d_y^- - 1} & q^-_{d_y^- - 2} & \cdots & q_0^- & 0 & 0 & \cdots & 0 \\
0 & 0 & q^-_{d_y^- - 1} & q^-_{d_y^- - 2} & \cdots & q_0^- & 0 & \cdots & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\
0 & 0 & \cdots & 0 & 0 & q^-_{d_y^- - 1} & q^-_{d_y^- - 2} & \cdots & q_0^-
\end{vmatrix}
\begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} d_y^- \\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} d_y^* - 1 \end{matrix}$$

modulo I. Expanding the above determinant according the first column, we get

$$\Delta \equiv \left| \begin{array}{cccccccc} {q^*}^2 & \dagger & \dagger & \cdots & \dagger & \dagger & \cdots & \dagger \\ . & . & . & \cdots & . & . & \cdots & . \\ 0 & \cdots & 0 & q^* & \dagger & \dagger & \cdots & \dagger \\ q^* q^-_{d^-_y-1} & q^-_{d^-_y-2} & \cdots & q^-_0 & 0 & 0 & \dots & 0 \\ 0 & q^-_{d^-_y-1} & q^-_{d^-_y-2} & \cdots & q^-_0 & 0 & \dots & 0 \\ . & . & . & \cdots & . & . & \cdots & . \\ 0 & \cdots & 0 & 0 & q^-_{d^-_y-1} & q^-_{d^-_y-2} & \cdots & q^-_0 \end{array} \right| \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} d^-_y - 1 \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} d^*_y - 1 \end{array}$$

modulo I. Since $q^* q^-_{d^-_y-1} \equiv 0 \mod I$ by Lemma 2.1, expanding the above determinant according to the first column modulo I yields a determinant whose first columns has three nonzero elements

$${q^*}^3, \quad {q^*}^2 q^-_{d^-_y-2}, \quad {q^*}^2 q^-_{d^-_y-1}.$$

The last two elements are congruent to zero modulo I by Lemma 2.1. Then expanding the determinant in the same manner yields a determinant whose first column has four nonzero elements, three of which are congruent to zero modulo I, and the other is ${q^*}^4$. we can expand Δ in the above-described manner d^-_y times, which concludes that Δ is in the sum of I and the ideal generated by ${q^*}^{d^-_y}$ in R. Since $d^-_y \geq (m-1)\deg_y Q_m \geq (m-1)$, ${q^*}^{d^-_y}$ is in I, and so is Δ. □

Proposition 2.3 *The determinant of $M(Q)$ is divisible by $\mathrm{lc}(Q^-)^2$ over R.*

Proof. By Remark 1.2, expanding $\det(M(Q))$ according to the first column and moving some integers out of the minor, we see that $\det(M(Q))$ is divisible by the product of $\mathrm{lc}_y(Q^-)$ and the determinant as given in Lemma 2.2. Hence, $\det(M(Q))$ is divisible by $\mathrm{lc}_y(Q^-)^2$. □

3. An extraneous factor

First, we consider the generic case. For all i with $1 \leq i \leq m$, let U_i be a polynomial of degree $\deg_y Q_i$ in y whose coefficients are distinct indeterminates. Put $U^* = U_1 \cdots U_m$ and $U^- = U_2 U_3^2 \cdots U_m^{m-1}$. Then $U = U^* U^-$. The second deflation U^{-2} of U is equal to $U_3 U_4^2 \cdots U_m^{m-2}$. Let S be the ring of polynomials generated by the indeterminate coefficients of the U_i's over R. Then the above-defined generic polynomials are all in $S[y]$. For a polynomial f in $S[y]$, $\mathrm{coeff}(f, y^i)$ stands for the coefficient of y^i in f.

Recall the usual convention that $f \in S[y]$ is zero if and only if $\deg_y f < 0$, and that a fraction in $S(y)$ is proper if the degree of the numerator is less than that of denominator.

Lemma 3.1 *With the notation just introduced, let N be the matrix associated with U. Then there exist T and t in $S[y]$ with $\deg_y T < d^-_y$ and $\deg_y t < d^*_y$ such that the following hold:*

(i)

$$\det(N)\,\frac{P}{U} = D_y\left(\frac{T}{U^-}\right) + \frac{t}{U^*}. \tag{5}$$

(ii) *T is divisible by $\mathrm{lc}_y(U^{-2})$ in $S[y]$, and, moreover, if $\deg_y P < d_y^* + d_y^- - 1$, then T is divisible by $\mathrm{lc}_y(U^-) \cdot \mathrm{lc}_y(U^{-2})$ in $S[y]$.*

(iii) *t is divisible by $\mathrm{lc}_y(U^-)$ in $S[y]$, and, moreover, if $\deg_y P < d_y^* + d_y^- - 1$, then t is divisible by $\mathrm{lc}_y(U^-)^2$ in $S[y]$.*

Proof. Using Cramer's rule to solve a linear system of the form (6), we obtain $A_i = T_i/\det(N)$ for all i with $0 \le i \le d_y^- - 1$, and $a_j = t_j/\det(N)$ for all j with $0 \le j \le d_y^* - 1$, where T_i, t_j are in S. Putting $T = \sum_{i=0}^{d_y^- -1} T_i y^i$ and $t = \sum_{j=0}^{d_y^* -1} t_j y^j$, we prove the first assertion.

Now, we are going to show the second assertion. The case in which $m \le 2$ is trivial. So we may assume that $m > 2$ in the sequel. Set $u_i = \mathrm{lc}_y(U_i)$, $u^* = u_1 u_2 \cdots u_m$, and $u^- = u_2 u_3^2 \cdots u_m^{m-1}$. Let u^{-2} denote the second deflation of u, which equals to $u_3 u_4^2 \cdots u_m^{m-2}$. Recall the form of the associated matrix N of U as follows,

$$\begin{pmatrix}
0 & u^* & \dagger & \dagger & \cdots & \dagger & \dagger & \dagger & \ldots & \dagger \\
0 & 0 & u^* & \dagger & \dagger & \cdots & \dagger & \dagger & \ldots & \dagger \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\
0 & 0 & 0 & \cdots 0 & u^* & \dagger & \dagger & \dagger & \cdots & \dagger \\
u^- & u^-_{d_y^- -1} & u^-_{d_y^- -2} & \cdots & u_0^- & 0 & 0 & 0 & \ldots & 0 \\
0 & u^- & u^-_{d_y^- -1} & u^-_{d_y^- -2} & \cdots & u_0^- & 0 & 0 & \ldots & 0 \\
0 & 0 & u^- & u^-_{d_y^- -1} & u^-_{d_y^- -2} & \cdots & u_0^- & 0 & \ldots & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\
0 & 0 & \cdots & 0 & u^- & u^-_{d_y^- -1} & u^-_{d_y^- -2} & \cdots & u_1^- & u_0^-
\end{pmatrix}
\begin{array}{l}
\left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} d_y^- \\
\left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} d_y^*
\end{array}$$

where $\dagger$ stands for any element of S and the last d_y^* rows consist of the coefficients of $y^{d_y^* -1}U^-$, $y^{d_y^* -2}U^-$, …, U^-. Let $T = \sum_{i=0}^{d_y^- -1} T_i y^i$. Using Cramer's rule to solve the linear system

$$\left(A_{d_y^- -1}, \ldots, A_0, a_{d_y^* -1}, \ldots, a_0\right) M = \left(P_{d_y^* + d_y^- -1}, \ldots, P_0\right), \tag{6}$$

we have $A_i = T_i/\det(N)$ for all i with $0 \le i \le d_y^- - 1$, and T_i is the determinant of the form

$$i\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right.
\begin{vmatrix}
0 & u^* & \dagger & \dagger & \cdots & \dagger & \dagger & \dagger & \ldots & \dagger \\
0 & 0 & u^* & \dagger & \dagger & \cdots & \dagger & \dagger & \ldots & \dagger \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\
P_{d_y^* + d_y^- -1} & P_{d_y^* + d_y^- -2} & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & P_0 \\
0 & 0 & 0 & \cdots & u^* & \dagger & \dagger & \dagger & \cdots & \dagger \\
0 & 0 & 0 & \cdots & 0 & u^* & \dagger & \dagger & \cdots & \dagger \\
u^- & u^-_{d_y^- -1} & u^-_{d_y^- -2} & \cdots & u_0^- & 0 & 0 & 0 & \ldots & 0 \\
0 & u^- & u^-_{d_y^- -1} & u^-_{d_y^- -2} & \cdots & u_0^- & 0 & 0 & \ldots & 0 \\
0 & 0 & u^- & u^-_{d_y^- -1} & u^-_{d_y^- -2} & \cdots & u_0^- & 0 & \ldots & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\
0 & 0 & \cdots & 0 & u^- & u^-_{d_y^- -1} & u^-_{d_y^- -2} & \cdots & u_1^- & u_0^-
\end{vmatrix}
\begin{array}{l}
\left.\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}\right\} d_y^- \\
\left.\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\right\} d_y^*
\end{array}$$

which is obtained from replacing the ith row of $\det(N)$ by $(P_{d_y^*+d_y^- -1}, \dots, P_0)$. It suffices to show that T_i is divisible by u^{-2}. Let I be the ideal generated by u^{-2} in S. Then,

$$
T_i \equiv \begin{vmatrix}
0 & u^* & \dagger & \dagger & \cdots & \dagger & \dagger & \dagger & \dots & \dagger \\
0 & 0 & u^* & \dagger & \dagger & \cdots & \dagger & \dagger & \dots & \dagger \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\
P_{d_y^*+d_y^- -1} & P_{d_y^*+d_y^- -2} & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & P_0 \\
0 & 0 & 0 & \cdots & u^* & \dagger & \dagger & \dagger & \cdots & \dagger \\
0 & 0 & 0 & \cdots & 0 & u^* & \dagger & \dagger & \cdots & \dagger \\
0 & 0 & u^-_{d_y^- -2} & \cdots & u_0^- & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & u^-_{d_y^- -2} & \cdots & u_0^- & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & u^-_{d_y^- -2} & \cdots & u_0^- & 0 & \dots & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\
0 & 0 & \cdots & 0 & 0 & 0 & u^-_{d_y^- -2} & \cdots & u_1^- & u_0^-
\end{vmatrix}
\begin{matrix}
\left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} i \\
\left.\begin{matrix} \\ \\ \end{matrix}\right\} d_y^- - i \\
\left.\begin{matrix} \\ \\ \\ \\ \\ \end{matrix}\right\} d_y^*
\end{matrix}
$$

modulo I. Expanding the determinant according the first column, we get

$$
T_i \equiv P_{d_y^*+d_y^- -1} \begin{vmatrix}
u^* & \dagger & \dagger & \cdots & \dagger & \dagger & \dagger & \dagger & \dots & \dagger & \dagger & \dagger \\
0 & u^* & \dagger & \dagger & \cdots & \dagger & \dagger & \dagger & \dots & \dagger & \dagger & \dagger \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\
0 & 0 & \cdots & u^* & \dagger & \dagger & \dagger & \dagger & \cdots & \dagger & \dagger & \dagger \\
0 & 0 & \cdots & 0 & 0 & u^* & \dagger & \dagger & \cdots & \dagger & \dagger & \dagger \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & u^* & \dagger & \cdots & \dagger & \dagger & \dagger \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & u^* & \cdots & \dagger & \dagger & \dagger \\
0 & u^-_{d_y^- -2} & \cdots & \cdot & \cdot & u_0^- & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & u^-_{d_y^- -2} & \cdots & \cdot & \cdot & u_0^- & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & 0 & u^-_{d_y^- -2} & \cdots & \cdot & \cdot & u_0^- & \dots & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & 0 \\
0 & \cdots & 0 & 0 & 0 & u^-_{d_y^- -2} & \cdot & \cdot & \cdot & \cdot & \cdots & u_0^-
\end{vmatrix}
\begin{matrix}
\left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} i-1 \\
\left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} d_y^- - i \\
\left.\begin{matrix} \\ \\ \\ \\ \\ \end{matrix}\right\} d_y^*
\end{matrix}
$$

modulo I. Expanding the determinant according the first column again and noting that u^{-2}

divides $u^* u^-_{d_y^- -2}$ by Lemma 2.1, we get,

$$
T_i \equiv P_{d_y^* + d_y^- -1}
\begin{vmatrix}
u^{*2} & \dagger & \dagger & \cdots & \dagger & \dagger & \dagger & \dagger & \cdots & \dagger & \dagger & \dagger \\
0 & u^* & \dagger & \dagger & \cdots & \dagger & \dagger & \dagger & \cdots & \dagger & \dagger & \dagger \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\
0 & 0 & \cdots & u^* & \dagger & \dagger & \dagger & \dagger & \cdots & \dagger & \dagger & \dagger \\
0 & 0 & \cdots & 0 & 0 & u^* & \dagger & \dagger & \cdots & \dagger & \dagger & \dagger \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\
0 & 0 & \cdots & 0 & 0 & 0 & u^* & \dagger & \cdots & \dagger & \dagger & \dagger \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & u^* & \cdots & \dagger & \dagger & \dagger \\
0 & u^-_{d_y^- -3} & \cdots & \cdot & u_0^- & 0 & 0 & 0 & \cdots & \dagger & \dagger & \dagger \\
0 & u^-_{d_y^- -2} & \cdots & \cdot & \cdot & u_0^- & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & u^-_{d_y^- -2} & \cdots & \cdot & \cdot & u_0^- & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & u^-_{d_y^- -2} & \cdots & \cdot & \cdot & u_0^- & \cdots & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & 0 \\
0 & \cdots & 0 & 0 & 0 & u^-_{d_y^- -2} & \cdot & \cdot & \cdot & \cdot & \cdots & u_0^-
\end{vmatrix}
\begin{array}{l}
\\
\left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} i-2 \\
\left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} d_y^- - i \\
\left.\begin{array}{l} \\ \\ \\ \\ \\ \end{array}\right\} d_y^*
\end{array}
$$

modulo I. Repeating the above process for $i-2$ times and using Lemma 2.1, we get

$$
T_i \equiv P_{d_y^* + d_y^- -1}
\begin{vmatrix}
u^{*i-1} & \dagger & \dagger & \cdots & \dagger & \dagger & \dagger & \dagger & \cdots & \dagger & \dagger & \dagger \\
0 & 0 & u^* & \dagger & \cdots & \dagger & \dagger & \dagger & \cdots & \dagger & \dagger & \dagger \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\
0 & 0 & \cdots & 0 & u^* & \dagger & \dagger & \dagger & \cdots & \dagger & \dagger & \dagger \\
0 & 0 & \cdots & 0 & 0 & u^* & \dagger & \dagger & \cdots & \dagger & \dagger & \dagger \\
0 & u^-_{d_y^- -i} & \cdots & \cdot & u_0^- & 0 & 0 & 0 & \cdots & \dagger & \dagger & \dagger \\
0 & u^-_{d_y^- -i+1} & \cdots & \cdot & \cdot & u_0^- & 0 & 0 & \cdots & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & 0 \\
0 & u^-_{d_y^- -2} & \cdots & \cdot & \cdot & \cdot & u_0^- & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & u^-_{d_y^- -2} & \cdots & \cdot & \cdot & \cdot & u_0^- & \cdots & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & 0 \\
0 & \cdots & 0 & 0 & 0 & u^-_{d_y^- -2} & \cdot & \cdot & \cdot & \cdot & \cdots & u_0^-
\end{vmatrix}
\begin{array}{l}
\\
\left.\begin{array}{l} \\ \end{array}\right\} 1 \\
\left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} d_y^- - i \\
\left.\begin{array}{l} \\ \\ \\ \\ \\ \\ \end{array}\right\} d_y^*
\end{array}
$$

modulo I. Expanding the above determinant according the first column and using Lemma 2.1, we get

$$T_i \equiv P_{d_y^*+d_y^- -1} \left| \begin{array}{ccccccccccc} 0 & u^* & \dagger & \cdots & \dagger & \dagger & \dagger & \ldots & \dagger & \dagger & \dagger \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\ 0 & \cdots & 0 & u^* & \dagger & \dagger & \dagger & \cdots & \dagger & \dagger & \dagger \\ 0 & \cdots & 0 & 0 & u^* & \dagger & \dagger & \cdots & \dagger & \dagger & \dagger \\ 0 & \cdots & \cdot & u_0^- & 0 & 0 & 0 & \cdots & \dagger & \dagger & \dagger \\ 0 & \cdots & \cdot & \cdot & u_0^- & 0 & 0 & \ldots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & 0 \\ 0 & \cdots & \cdot & \cdot & \cdot & u_0^- & 0 & \ldots & 0 & 0 & 0 \\ 0 & u^-_{d_y^- -2} & \cdots & \cdot & \cdot & \cdot & u_0^- & \ldots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & 0 \\ \cdots & 0 & 0 & 0 & u^-_{d_y^- -2} & \cdot & \cdot & \cdot & \cdot & \cdots & u_0^- \end{array} \right| \quad \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} d_y^- - i \\ \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\end{array}}\right\} d_y^* \end{array}$$

modulo I, which implies that $T_i \equiv 0$ modulo I. The first part of second assertion holds.

If $\deg_y P < d_y^* + d_y^- - 1$, then $P_{d_y^*+d_y^- -1} = 0$ and $T_i = u^- \tilde{T}_i$, where $\tilde{T}_i$ is the determinant of the form

$$\begin{array}{r} i\left\{\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right. \\ \vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\end{array}} \end{array} \left| \begin{array}{ccccccccc} u^* & \dagger & \dagger & \cdots & \dagger & \dagger & \dagger & \ldots & \dagger \\ 0 & u^* & \dagger & \dagger & \cdots & \dagger & \dagger & \ldots & \dagger \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ P_{d_y^*+d_y^- -2} & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & P_0 \\ 0 & 0 & \cdots & u^* & \dagger & \dagger & \dagger & \cdots & \dagger \\ 0 & 0 & \cdots & 0 & u^* & \dagger & \dagger & \cdots & \dagger \\ u^- & u^-_{d_y^- -1} & u^-_{d_y^- -2} & \cdots & u_0^- & 0 & 0 & \ldots & 0 \\ 0 & u^- & u^-_{d_y^- -1} & u^-_{d_y^- -2} & \cdots & u_0^- & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ 0 & \cdots & 0 & u^- & u^-_{d_y^- -1} & u^-_{d_y^- -2} & \cdots & u_1^- & u_0^- \end{array} \right| \quad \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\end{array}}\right\} d_y^- \\ \left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} d_y^* - 1 \end{array}$$

which is obtained by removing the first column and the $(d^- + 1)$th row of the determinant T_i. The same argument as above shows that $\tilde{T}_i$ is divisible by u^{-2}. Thus T is divisible by $\mathrm{lc}_y(U^-) \cdot \mathrm{lc}_y(U^{-2})$ in $S[y]$.

For proving the last assertion, we need two intermediate results.

Claim 1. $\mathrm{lc}_y(U^-) | \operatorname{coeff}(t, y^{d_y^* - 1})$.

Proof of Claim 1. Let $v = \operatorname{coeff}(t, y^{d_y^* - 1})$ in t. By (5),

$$\det(N)\, P = U^* D_y(T) - \widetilde{U}T + U^- t,$$

where $\widetilde{U} \in S[y]$ with degree less than d_y^*. The coefficient of $y^{d_y^* + d_y^- - 1}$ in the right-hand side of the above equation is equal to $v\mathrm{lc}_y(U^-)$, because both $\deg_y(U^* D_y(T))$ and $\deg_y(\widetilde{U}T)$ are less than $d_y^* + d_y^- - 1$. Hence, $v = 0$ if $\deg_y P < d_y^* + d_y^- - 1$; and $v\mathrm{lc}_y(U^-) = \det(N)\, \mathrm{lc}_y(P)$ if $\deg_y P = d_y^* + d_y^- - 1$. In the latter case, $\mathrm{lc}_y(U^-)$ divides v by Proposition 2.3. Claim 1 is proved.

If $d_y^* = 1$, then $\deg_y t \leq 0$. Hence, $\mathrm{lc}_y(U^-)$ divides t by Claim 1. In the rest of the proof, we assume that $d_y^* > 1$. Then there are at least two rows in M consisting of either zero or coefficients of U^-. Hence t is divisible by u_i^{i-1} if $\deg_y Q_i = 0$.

Since $u_2, \ldots, u_m$ are independent indeterminates, it suffices to prove that t is divisible by u_i^{i-1} for all i with $2 \leq i \leq m$ and $\deg_y Q_i > 0$. Without loss of generality, we prove that t is divisible by u_m^{m-1}. To this end, let ϕ_m be the R-homomorphism from $S[y]$ to $S[y]$ that maps u_m to zero and fixes other indeterminates.

Claim 2. If $w \in S[y]$ with $\deg_y w < d_y^*$ and $u_m | \operatorname{coeff}(w, y^{d_y^*-1})$, then $\phi_m(w)/\phi_m(U^*)$ is a proper fraction in y whose denominator is squarefree.

Proof of Claim 2. Since $d_y^* > 1$ and U^* is a product of generic polynomials, $\phi_m(U^*)$ is a squarefree polynomial of degree $d_y^* - 1$, which is postive. But $\phi_m(w)$ has degree less than $d_y^* - 1$. Claim 2 is proved.

For simplicity, we denote by F the fraction T/U^-. Since ϕ_m and D_y commute and $\phi_m(U^-)$ is nonzero, applying ϕ_m to (5) yields

$$0 = D_y \circ \phi_m(F) + \phi_m(t)/\phi_m(U^*)$$

by Proposition 2.3. It follows from Claims 1 and 2 that $\phi_m(t) = 0$, because a derivative of a fraction is unequal to a proper fraction whose denominator is squarefree. Hence, u_m divides t in $S[y]$. Assume that u_m^ℓ divides w for some ℓ with $1 \leq \ell < m - 1$. It suffices to show that $u_m^{\ell+1}$ divides t. Assume $t = u_m^\ell w$ for some $w \in S[y]$. Rewrite (5) as

$$\det(N)\frac{P}{U} = D_y(F) + \frac{u_m^\ell w}{U^*}. \tag{7}$$

Let D_m be the usual partial differential operator with respect to u_m. Applying $\phi_m \circ D_m^\ell$ to (7) yields

$$0 = D_y \circ \phi_m \circ D_m^\ell(F) + \frac{\ell!\phi_m(w)}{\phi_m(U^*)} \tag{8}$$

by Proposition 2.3. By Claim 1 and since $\ell < m - 1$, $\operatorname{coeff}(w, y^{d_y^*-1})$ is divisible by u_m. By Claim 2 and the above equality, $\phi_m(w) = 0$, that is, t is divisible by $u_m^{\ell+1}$. The first part of the last assertion holds.

If $\deg_y P < d_y^* + d_y^- - 1$, then $\deg_y t \leq d_y^* - 2$ by the proof of Claim 1. The first part of the last assertion implies that u_m divides t. Assume that u_m^ℓ divides w for some $\ell < 2(m-1)$ and let $t = u_m^\ell w$. Then both (7) and (8) hold because $\mathrm{lc}_y(Q^-)^2 | \det(N)$ by Proposition 2.3. Moreover, $\deg_y w \leq d_y^* - 2$ since $\deg_y t \leq d_y^* - 2$. We have that $\phi_m(w) = 0$ because $\deg_y(\phi_m(w)) \leq d_y^* - 2$ and $\deg_y(\phi_m(U)) = d_y^* - 1$. □

We now specialize Lemma 3.1 to $R[y]$.

Proposition 3.2 *Let δ be the determinant of the matrix associated with Q. Then the following statements hold:*

(i) There exist $B, b \in R[y]$ such that $\deg_y B < d_y^-$, $\deg_y b < d_y^*$, *and*

$$\frac{P}{Q} = D_y\left(\frac{B}{\delta Q^-}\right) + \frac{b}{\delta Q^*}. \tag{9}$$

(ii) *B is divisible by* $\mathrm{lc}_y(Q^{-2})$*, and, moreover,* B *is divisible by* $\mathrm{lc}_y(Q^-)\cdot\mathrm{lc}_y(Q^{-2})$ *if* $\deg_y P < d_y^* + d_y^- - 1$.

(iii) b *is divisible by* $\mathrm{lc}_y(Q^-)$*, and, moreover,* b *is divisible by* $\mathrm{lc}_y(Q^-)^2$ *if* $\deg_y P < d_y^* + d_y^- - 1$.

Proof. Let ψ be the R-homomorphism from $S[y]$ to $R[y]$ that maps y to y and U_i to Q_i for all i with $1 \le i \le m$. Then

$$\psi(U^*) = Q^*, \quad \psi(U^-) = Q^-, \psi(U^{-2}) = Q^{-2}, \quad \text{and} \quad \psi(U) = Q.$$

The homomorphism also maps the determinant of $M(U)$ to δ. Moreover, D_y and ψ commute. The first assertion is proved by applying ψ to (5). The others hold by an easy application of ψ to Lemma 3.1 (ii) and (iii). □

4. Applications to the additive decomposition of bivariate rational functions

In this section we assume that $R = k[x]$, where k is a field of characteristic zero. Then $K = k(x)$. For an element $f \in k(x,y)$, the denominator and numerator of f are denoted by $\mathrm{den}(f)$ and $\mathrm{num}(f)$, which are assumed to be coprime.

Proposition 4.1 *Let* $P, Q \in k[x,y]$ *with* $\deg_y Q^* = d_y^*$, $\deg_x Q^* = d_x^*$, $\deg_x Q^- = d_x^-$ *and* $\deg_y Q^- = d_x^-$. *Assume that* $d_y^* > 0$ *and set* $\mu = d_x^* d_y^- + d_x^- d_y^*$. *If (2) holds, where* $A, a \in k(x)[y]$, $\deg_y A < d_y^-$ *and* $\deg_y a < d_y^-$*, then the following statements hold:*

1. *If* $\deg_y P = d_y^* + d_y^- - 1$*, then the degrees of* $\mathrm{den}(A)$ *and* $\mathrm{den}(a)$ *are respectively bounded by*
$$\mu - \deg_x \mathrm{lc}_y(Q^{-2}) \quad \textit{and} \quad \mu - \deg_x \mathrm{lc}_y(Q^-).$$
The degrees of $\mathrm{num}(A)$ *and* $\mathrm{num}(a)$ *are respectively bounded by*
$$\mu - d_x^- + \deg_x P - \deg_x \mathrm{lc}_y(Q^{-2}) \quad \textit{and} \quad \mu - d_x^- + \deg_x P - \deg_x \mathrm{lc}_y(Q^-).$$

2. *If* $\deg_y P < d_y^* + d_y^- - 1$*, then the degrees of* $\mathrm{den}(A)$ *and* $\mathrm{den}(a)$ *are respectively bounded by*
$$\mu - \deg_x \mathrm{lc}_y(Q^-) - \deg_x \mathrm{lc}_y(Q^{-2}) \quad \textit{and} \quad \mu - 2\deg_x \mathrm{lc}_y(Q^-).$$
The degrees of $\mathrm{num}(A)$ *and* $\mathrm{num}(a)$ *are respectively bounded by*
$$\mu - d_x^- + \deg_x P - \deg_x \mathrm{lc}_y(Q^-) - \deg_x \mathrm{lc}_y(Q^{-2})$$
and
$$\mu - d_x^- + \deg_x P - 2\deg_x \mathrm{lc}_y(Q^-).$$

Proof. Let δ be the determinant of $M(Q)$. Then $\deg_x \delta \le d_x^* d_y^- + d_x^- d_y^*$. Solving system (3) by Cramer's rule, we find that $a = b/\delta$ where $b \in k[x,y]$ is given in Proposition 3.2. Note that $\deg_x b \le d_x^* d_y^- + d_x^- d_y^* - d_x^- + \deg_x P$ by Cramer's rule. The two assertions then follow from divisibility described in Proposition 3.2 (ii) and (iii), respectively. □

Corollary 4.2 *With the notation introduced in Proposition 4.1, we further assume that* $Q^- = Q^*$. *Then the following hold:*

1. *If* $\deg_y P = 2d_y^* - 1$, *then*

$$\deg_x(\mathrm{den}(a)) \leq (2d_y^* - 1)d_x^*,$$

and

$$\deg_x(\mathrm{num}(a)) \leq 2(d_y^* - 1)d_x^* + \deg_x P.$$

2. *If* $\deg_y P < 2d_y^* - 1$, *then*

$$\deg_x(\mathrm{den}(a)) \leq 2(d_y^* - 1)d_x^*,$$

and

$$\deg_x(\mathrm{num}(a)) \leq (2d_y^* - 3)d_x^* + \deg_x P.$$

Proof. Let U^* be a polynomial in y with $\deg_y U^* = \deg_y Q^*$ whose coefficients are distinct indeterminates. Let R be the ring generated by the coefficients of U^* over $k[x]$. Set $U = {U^*}^2$ and $N = M(U)$. By Proposition 3.2(i), there are $T, t \in R[y]$ with $\deg_y T < d_y^*$ and $\deg_y t < d_y^*$ such that

$$\frac{P}{U} = D_y\left(\frac{T}{\det(N)U^*}\right) + \frac{t}{\det(N)U^*}.$$

By Proposition 3.2, $\det(N) = \mathrm{lc}(U^*)^2 H$, where H is a homogeneous polynomials of degree $2d_y^* - 2$ in the coefficients of U^* over $k[x]$. By Cramer's rule, the coefficients of T is a product of $\mathrm{lc}(Q^*)$ and a homogenous polynomials of degree $2d_y^* - 2$ (resp. of degree $2d_y^* - 3$) if $\deg_y P = 2d_y^* - 1$ (resp. $\deg_y P < 2d_y^* - 1$). Moreover, $\deg_x T \leq \deg_x P$. The corollary is then proved by specializing U^* to Q^* together with Proposition 3.2(iii). □

Example 4.3 *Assume that* $Q = {Q^*}^2$. *Then* $\deg_y Q = 2d_y^*$. *Performing Hermite reduction on* $D_x(Q^*)/Q$ *with respect to* y *yields*

$$\frac{D_x(Q^*)}{Q} = D_y\left(\frac{H}{Q^*}\right) + \frac{h}{Q^*}$$

with $H, h \in k[x, y]$. *If* $d_y^* = 1$, *then Corollary 4.2 (i) implies*

$$\deg_x(\mathrm{den}(h)) \leq d_x^* \quad \text{and} \quad \deg_x(\mathrm{num}(h)) \leq d_x^* - 1.$$

If $d_y^* > 1$, *then* $\deg_y D_x(Q^*) \leq d_y^* < 2d_y^* - 1$. *By Corollary 4.2 (ii),*

$$\deg_x(\mathrm{den}(h)) \leq 2d_x^*(d_y^* - 1) \quad \text{and} \quad \deg_x(\mathrm{num}(h)) \leq 2d_x^*(d_y^* - 1) - 1.$$

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