A Note on Lipshitz's Lemma 3

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Abstract

In this note, we give a remark on the proof of Lemma 3 by Lipshitz in [1]. This remark is motivated by the observation that the statement from line -8 to -3 on page 375 of [1] seems not completely correct.

1 An algebraic description of Lipshitz's Lemma

Let K be a field of characteristic zero, and $K(x, y)$ be the field of rational functions in x and y over K. Denote by $\mathcal{R}_2$ the ring $K(x, y)\langle D_x, D_y\rangle$ of linear differential operators generated by D_x and D_y over $K(x, y)$, whose commutative rules are given by

$$D_x f = f D_x + \frac{\partial f}{\partial x} \quad \text{and} \quad D_y f = f D_y + \frac{\partial f}{\partial y} \quad \text{for all } f \in K(x, y).$$

Lemma 3 in [1] is an easy consequence of the following proposition.

Proposition 1.1 *Let I be a left ideal of $\mathcal{R}_2$. If $\mathcal{R}_2/I$ is a finite-dimensional (left) vector space over $K(x, y)$, then there exists a nonzero element in the intersection of I and $K(x)\langle D_x, D_y\rangle$.*

Before proving Lemma 1.1, we recall some basic facts about differential operators. Let $\mathcal{A}_2$ be the Weyl algebra $k[x, y]\langle D_x, D_y\rangle$, which is a subring

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of $\mathcal{R}_2$. Assume that A and B are two nonzero differential operators of the form

$$A = LD_x^m + A_{m-1}D_x^{m-1} + \cdots + A_0 \quad \text{and} \quad B = LD_y^n + B_{n-1}D_x^{n-1} + \cdots + B_0 \tag{1}$$

where L, A_i, B_j are in $K[x, y]$ with $L \neq 0$, and m, n are positive integers. So A and B are in $k[x, y]\langle D_x\rangle$ and $k[x, y]\langle D_y\rangle$, respectively. These two subrings are both contained in $\mathcal{A}_2$.

Leibniz's formula for differentiation is translated into the language of differential operators as: for all $f \in K(x, y)$

$$D_x^k f = \sum_{\ell=0}^{k} \binom{k}{\ell} \frac{\partial^\ell f}{\partial x^\ell} D_x^{k-\ell} \tag{2}$$

and

$$f D_x^k = \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} D_x^{k-\ell} \frac{\partial^\ell f}{\partial x^\ell}. \tag{3}$$

The relation (2) can be proved by a straightforward induction, while (3) can be proved by applying the adjoint map to (2). Of course, both (2) and (3) hold when x is replaced by y. In the sequel, we merely use the facts that, for all $f \in K[x, y]$,

$$D_x^k f = f D_x^k + P \qquad \text{and} \quad f D_x^k = D_x^k f - P \tag{4}$$

where $P \in K[x, y]\langle D_x\rangle$ is of degree in D_x less than k and total degree in x, y less than that of f.

Let $D = D_x^\beta D_y^\gamma$. If $\beta > m$, Lipshitz claimed that one can always obtain

$$LD \equiv \sum P_\delta D_\delta \mod \langle A\rangle, \tag{5}$$

where the sum on the right hand side is over $D_\delta = D_x^{\delta_1} D_y^{\delta_2}$ with $\delta_1 < \beta$ and $\delta_2 \leq \gamma$. This claim seems not completely correct. In fact, when $\deg_x(L) > 0$ and both β and γ are positive, multiplying one L is not sufficient to obtain (5). For example, let $D = D_x^m D_y$. Write $A = LD_x^m - R_0$, where R_0 is sum of lower order terms in D_x. Then

$$LD_x^m \equiv R_0 \mod \langle A\rangle. \tag{6}$$

Multiplying both sides of (6) by D_y yields

$$\begin{aligned} D_y L D_x^m &\equiv D_y R_0 \mod \langle A\rangle, &(7)\\ LD_y D_x^m - L_y D_x^m &\equiv D_y R_0 \mod \langle A\rangle, &(8)\\ LD_y D_x^m &\equiv L_y D_x^m + D_y R_0 \mod \langle A\rangle. &(9)\end{aligned}$$

In order to reduce the order of $L_y D_x^m$ in (9), we multiply both sides of (9) by L, then

$$L^2 D_x^m D_y \equiv \sum P_\delta D_\delta \mod \langle A \rangle,$$

where the sum on the right hand side is over $D_\delta = D_x^{\delta_1} D_y^{\delta_2}$ with $\delta_1 < m$ and $\delta_2 \leq 1$. In contrast to the statement from line -8 to -3 on page 375 of [1], we have

Lemma 1.2 *Let A and B be given by (1), and J the left ideal generated by A and B in $\mathcal{A}_2$. Assume that d is an upper bound for the total degrees of L, A_i and B_j for all i, j with $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$. Then, for all α, β in $\mathbb{N}$, we have*

$$L D_x^\alpha D_y^\beta \equiv \sum_{i,j} R_{i,j}^{(\alpha,\beta)} D_x^i D_y^j \mod J,$$

where $R_{ij} \in K[x,y]$, $\deg R_{i,j}^{(\alpha,\beta)} \leq d$, either $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$ or $i + j \leq \alpha + \beta - 1$.

Proof. If $\alpha < m$ and $\beta < n$, the claim holds. Assume that $\beta \geq n$. We compute

$$\begin{aligned} L D_x^\alpha D_y^\beta &= (L D_x^\alpha) D_y^\beta = (D_x^\alpha L + P_\alpha) D_y^\beta \quad \text{(by (4))} \\ &= D_x^\alpha L D_y^\beta + P_\alpha D_y^\beta = D_x^\alpha \left(L D_y^{\beta-n} \right) D_y^n + P_\alpha D_y^\beta \\ &= D_x^\alpha \left(D_y^{\beta-n} L + Q_\beta \right) D_y^n + P_\alpha D_y^\beta \quad \text{(by (4))} \\ &= D_x^\alpha D_y^{\beta-n} \left(L D_y^n \right) + D_x^\alpha Q_\beta D_y^n + P_\alpha D_y^\beta \\ &= D_x^\alpha D_y^{\beta-n} \left(B - \textstyle\sum_{j=0}^{n-1} B_j D_y^j \right) + D_x^\alpha Q_\beta D_y^n + P_\alpha D_y^\beta \\ &\equiv -D_x^\alpha D_y^{\beta-n} \left(\textstyle\sum_{j=0}^{n-1} B_j D_y^j \right) + D_x^\alpha Q_\beta D_y^n + P_\alpha D_y^\beta \qquad \mod J. \end{aligned}$$

It follows from the degree constraints on P_α and Q_β that the lemma holds for $\beta \geq n$. Likewise, the lemma holds $\alpha \geq m$. □

Similar to the statement made in line 2 on page 376 in [1], we have

Lemma 1.3 *Let A and B be given by (1), and J the left ideal generated by A and B in $\mathcal{A}_2$. Assume that d is an upper bound for the total degrees of L, A_i and B_j for all i, j with $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$. Then, for all α, β in $\mathbb{N}$, we have*

$$L^{\alpha+\beta+1-\min(m,n)} D_x^\alpha D_y^\beta \equiv \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} R_{i,j}^{(\alpha,\beta)} D_x^i D_y^j \mod J,$$

where $R_{ij} \in K[x,y]$ and $\deg R_{i,j}^{(\alpha,\beta)} \leq (\alpha + \beta + 1 - \min(m,n))\, d$.

Proof. By Lemma 1.2, there are P_{ij}, Q_{ij} and R_{ij} in $K[x, y]$ with total degree no more than d such that

$$\begin{aligned} LD_x^{\alpha}D_y^{\beta} \equiv & \textstyle\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} P_{ij}D_x^iD_y^j \\ & + \textstyle\sum_{i\geq m,\, 0\leq i+j\leq \alpha+\beta-1} Q_{ij}D_x^iD_y^j \\ & + \textstyle\sum_{j\geq n,\, 0\leq i+j\leq \alpha+\beta-1} R_{ij}D_x^iD_y^j \qquad \bmod J. \end{aligned}$$

It follows that

$$\begin{aligned} L^2D_x^{\alpha}D_y^{\beta} \equiv & \textstyle\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} LP_{ij}D_x^iD_y^j \\ & + \textstyle\sum_{i\geq m,\, 0\leq i+j\leq \alpha+\beta-1} Q_{ij}\left(LD_x^iD_y^j\right) \\ & + \textstyle\sum_{j\geq n,\, 0\leq i+j\leq \alpha+\beta-1} R_{ij}\left(LD_x^iD_y^j\right) \qquad \bmod J. \end{aligned}$$

Applying Lemma 1.2 to each $LD_x^iD_y^j$ appearing in the second and third summations yields that

$$\begin{aligned} L^2D_x^{\alpha}D_y^{\beta} \equiv & \textstyle\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} P'_{ij}D_x^iD_y^j \\ & + \textstyle\sum_{i\geq m,\, 0\leq i+j\leq \alpha+\beta-2} Q'_{ij}D_x^iD_y^j \\ & + \textstyle\sum_{j\geq n,\, 0\leq i+j\leq \alpha+\beta-2} R'_{ij}D_x^iD_y^j \qquad \bmod J \end{aligned}$$

for some P'_{ij}, Q'_{ij} and R'_{ij} in $K[x, y]$ with total degrees no more than $2d$. A straightforward induction shows that

$$\begin{aligned} L^kD_x^{\alpha}D_y^{\beta} \equiv & \textstyle\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} P^*_{ij}D_x^iD_y^j \\ & + \textstyle\sum_{i\geq m,\, 0\leq i+j\leq \alpha+\beta-k} Q^*_{ij}D_x^iD_y^j \\ & + \textstyle\sum_{j\geq n,\, 0\leq i+j\leq \alpha+\beta-k} R^*_{ij}D_x^iD_y^j \qquad \bmod J \end{aligned}$$

for some P^*_{ij}, Q^*_{ij} and R^*_{ij} in $K[x, y]$ with total degrees no more than kd.

Setting $k = \alpha + \beta + 1 - \min(m, n)$ yields the lemma. □

We are ready to prove Proposition 1.1. Assume further that I is nontrivial. Then I contains two differential operators A and B given by (1). Assume that J is the left ideal generated by A and B in $\mathcal{A}_2$, It suffices to show that there is a nonzero element in the intersection of J and $K[x]\langle D_x, D_y\rangle$.

We apply the same counting argument used in [1]. Assume that d is an upper bound for all coefficients A_i and B_j Let N a positive integer, and let

$$V_N = \left\{L^N x^{\gamma} D_x^{\alpha} D_y^{\beta} \mid \gamma, \alpha, \beta \in \mathbb{N},\, \gamma + \alpha + \beta \leq N\right\}$$

and

$$W_N = \left\{x^s y^t D_x^i D_y^j \mid s, t, i, j \in \mathbb{N}, s + t \leq N(d + 1), i < m, j < n\right\}.$$

By Lemma 1.3, $L^N x^\gamma D_x^\alpha D_y^\beta$ is congruent a K-linear combination of the elements in W_N modulo J. Since $|V_N| = O(N^3)$ and $|W_N| = O(N^2)$. there must be a nontrivial K-linear combination of the elements in V_N lying in J when N is sufficiently large. □

References

[1] L. Lipshitz, The diagonal of a D-Finite power series is D-Finite, Journal of Algebra, Vol. 113(1988), pp. 373-378.