

Symbolic Summation of Multivariate Rational Functions *

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Abstract

Symbolic summation as an active research topic of symbolic computation provides efficient algorithmic tools for evaluating and simplifying different types of sums arising from mathematics, computer science, physics and other areas. Most of existing algorithms in symbolic summation are mainly applicable to the problem with univariate inputs. A long-term project in symbolic computation is to develop theories, algorithms and software for the symbolic summation of multivariate functions. This paper will give complete solutions to two challenging problems in symbolic summation of multivariate rational functions, namely the rational summability problem and the existence problem of telescopers for multivariate rational functions. Our approach is based on the structure of Sato's isotropy groups of polynomials, which enables us to reduce the problems to testing the shift equivalence of polynomials. Our results provide a complete solution to the discrete analogue of Picard's problem on differential forms and can be used to detect the applicability of the Wilf-Zeilberger method to multivariate rational functions.

Keywords: Isotropy group; shift equivalence; symbolic summation; telescoper

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1 Introduction

Symbolic summation is a classical and active research topic in symbolic computation [78], whose central problem is evaluating and simplifying different types of sums arising from combinatorics [53, 55, 65], computer science [38], theoretical physics [12, 72] and other areas. Similar to the integration case, there are two types of summation problems: one is the indefinite summation problem, and the other is the definite summation problem. The two summation problems are connected by the discrete Leibniz–Newton formula. In the early days, methods for symbolic summation were mainly summarized in calculus of finite differences (for instance one can see the books [13, 49]). Since the early 1970s, efficient symbolic algorithms have been developed for symbolic summation [78, Chapter 23]. Abramov’s algorithm [1–3] solves the indefinite summation problem for univariate rational functions. The classical Hermite reduction for symbolic integration of rational functions was extended to the summation case by Paule via greatest factorial factorizations in [64] with more developments in [10, 61, 69]. The indefinite summation problem for hypergeometric terms is handled by Gosper’s algorithm [37]. For sequences in a general difference field, the corresponding problem was studied by Karr in [51, 52] with significant improvements by Schneider [71] and recent fruitful applications in quantum field theory [12, 74]. Most of the existing complete algorithms are mainly applicable to the problem with univariate inputs. A long-term project in symbolic computation proposed by Andrews and Paule in [8] is to develop theories, algorithms and software for symbolic summation of multivariate functions. Along this way, some algorithms have been developed to deal with a special class of double sums [29] and binomial multiple sums [15]. One of the long standing open problems in this project is developing the multivariate version of Gosper’s algorithm (see Problem 5.1 in [24]).

In the multivariate setting, a more intrinsic way for formulating the summation problem is to use the language of difference forms, which is a discrete analogue of differential forms. The combinatorial theory of difference forms was first developed by Zeilberger in [84] with applications in computer-generated proofs of combinatorial identities and series convergence acceleration [85]. Motivated by problems in geometric integration, the geometric theory of difference forms was properly developed by Mansfield and Hydon in [60] with a discrete analogue of de Rham cohomology. In both theories, it is a fundamental problem to decide whether a given closed difference form is exact or not. The corresponding problem in the differential case was first proposed by Picard in 1889 (see Note III in [67, pp. 475–479]), which is deciding whether a given rational function $F(x, y, z) \in \mathbb{C}(x, y, z)$ can be written as $F(x, y, z) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$ for some rational functions $P, Q, R \in \mathbb{C}(x, y, z)$. In the language of differential forms, Picard’s problem is equivalent to deciding whether a differential 3-form $\omega = F(x, y, z)dx dy dz$ is exact or not in the field of rational functions. Picard’s problem can be naturally extended to the case of multivariate rational functions. Picard himself solved the problem for bivariate rational functions but it is still not completely solved in the general multivariate case. Some fundamental and significant work had been done by Griffiths in the 1960s in his papers on periods of rational integrals [39, 40] and later by Dimca [31] and more recently by Lairez [57]. As the first main result of our paper, we will solve the discrete analogue of Picard’s problem by generalizing Gosper’s algorithm to the case of multivariate rational functions.

From the indefinite summation problem to the definite one, creative telescoping [83] is a crucial tool for computing definite sums with applications in computer-generated proofs of combinatorial identities [65, 80, 81]. One of fundamental problems related to creative telescoping is the existence problem of telescopers for special functions which is equivalent to the termination of Zeilberger’s algorithm [82, 83] and can be used to detect the hypertranscendence and algebraic independence of functions defined by indefinite sums or integrals [45, 73]. A sufficient condition, namely holonomicity, on the existence of telescopers was first given by Zeilberger in 1990 using Bernstein’s

theory of holonomic D-modules [11]. Wilf and Zeilberger in [81] proved that telescopers exist for proper hypergeometric terms. However, holonomicity and properness are only sufficient conditions. Abramov and Le [6] solved the existence problem of telescopers for rational functions in two discrete variables. This work was soon extended to the hypergeometric case by Abramov [5], the q -hypergeometric case in [28], the mixed rational and hypergeometric case in [18, 26], and most recently the P-recursive case in [33]. The criteria on the existence of telescopers for rational functions in three variables were given in [19, 20, 22]. In arbitrary number of variables, there is no available algorithm for deciding the existence of telescopers for rational functions. As the second main result of our paper, we will solve the existence problem of telescopers for multivariate rational functions by reducing the problem to the computation of Sato's isotropy groups and the testing of the shift equivalence of multivariate polynomials.

The rest of this paper is organized as follows. In Section 2, we define the existence problem of telescopers and the summability problem precisely and recall some basic complexity estimates for later use. We present a general scheme for designing algorithms to solve the shift equivalence testing problem in Section 3, and compare our new algorithms with the other known algorithms in Section 7. In Section 4, we first recall the notion of isotropy groups of polynomials and their basic properties, and then introduce orbital decompositions for rational functions. We apply orbital decompositions in Section 5 to reduce the rational summability problem for general rational functions to that for simple fractions. After this, we present a criterion on the summability of such simple fractions. We not only decide the summability of a rational function but also construct the indefinite sums explicitly if it is summable. In Section 6, we again use the structure of isotropy groups and orbital decompositions to derive a criterion for the existence of telescopers for rational functions in variables t and \mathbf{x} . Moreover, we present an algorithm for computing a telescoper if it exists.

2 Preliminaries

In this section, we will recall some basic terminologies of symbolic summation and creative telescoping and overview some complexity results for later use.

2.1 Telescopers and summability of rational functions

Through out the paper, let \mathbb{K} be a field of characteristic zero and $\mathbb{K}(t, \mathbf{x})$ be the field of rational functions in t and $\mathbf{x} = \{x_1, \dots, x_m\}$ over \mathbb{K} . For each $v \in \mathbf{v} = \{t, x_1, \dots, x_m\}$, the shift operator σ_v with respect to v is defined as the \mathbb{K} -automorphism of $\mathbb{K}(\mathbf{v})$ such that

$$\sigma_v(v) = v + 1 \text{ and } \sigma_v(w) = w \text{ for all } w \in \mathbf{v} \setminus \{v\}.$$

Let $\mathcal{R} := \mathbb{K}(\mathbf{v})\langle S_t, S_{x_1}, \dots, S_{x_m} \rangle$ denote the ring of linear recurrence operators over $\mathbb{K}(\mathbf{v})$, in which $S_{v_i} S_{v_j} = S_{v_j} S_{v_i}$ for all $v_i, v_j \in \mathbf{v}$ and $S_v f = \sigma_v(f) S_v$ for any $f \in \mathbb{K}(\mathbf{v})$ and $v \in \mathbf{v}$. The action of an operator $L = \sum_{i_0, i_1, \dots, i_m \geq 0} a_{i_0, i_1, \dots, i_m} S_t^{i_0} S_{x_1}^{i_1} \cdots S_{x_m}^{i_m} \in \mathcal{R}$ on a rational function $f \in \mathbb{K}(\mathbf{v})$ is defined as

$$L(f) = \sum_{i_0, i_1, \dots, i_m \geq 0} a_{i_0, i_1, \dots, i_m} \sigma_t^{i_0} \sigma_{x_1}^{i_1} \cdots \sigma_{x_m}^{i_m} (f).$$

For each $v \in \mathbf{v}$, the difference operator Δ_v with respect to v is defined by $\Delta_v = S_v - \mathbf{1}$, where $\mathbf{1}$ stands for the identity map on $\mathbb{K}(\mathbf{v})$.

We now introduce the notion of telescopers for rational functions in $\mathbb{K}(t, \mathbf{x})$.

Definition 2.1 (Telescoper). *Let n be a positive integer such that $1 \leq n \leq m$ and let $f \in \mathbb{K}(t, \mathbf{x})$ be a rational function. A nonzero linear recurrence operator $L \in \mathbb{K}(t)\langle S_t \rangle$ is called a telescoper of type $(\sigma_t; \sigma_{x_1}, \dots, \sigma_{x_n})$ for f if there exist $g_1, \dots, g_n \in \mathbb{K}(t, \mathbf{x})$ such that*

$$L(f) = \Delta_{x_1}(g_1) + \dots + \Delta_{x_n}(g_n).$$

The rational functions g_1, \dots, g_n are called the certificates of L .

Problem 2.2 (Existence Problem of Telescopers). *Given a rational function $f \in \mathbb{K}(t, \mathbf{x})$ and an integer n with $1 \leq n \leq m$, decide whether or not f has a telescoper of type $(\sigma_t; \sigma_{x_1}, \dots, \sigma_{x_n})$. If so, find a telescoper L and its certificates g_1, \dots, g_n .*

In order to decide the existence of telescopers for $f \in \mathbb{K}(t, \mathbf{x})$, one may first use the shortcut to decide whether $L = \mathbf{1}$ is a telescoper for f . This is equivalent to the following summability problem of f in $\mathbb{F}(\mathbf{x})$ with $\mathbb{F} = \mathbb{K}(t)$.

Definition 2.3 (Summability). *Let \mathbb{F} be a field of characteristic zero and n be a positive integer such that $1 \leq n \leq m$. A rational function $f \in \mathbb{F}(\mathbf{x})$ is called $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -summable in $\mathbb{F}(\mathbf{x})$ if there exist $g_1, \dots, g_n \in \mathbb{F}(\mathbf{x})$ such that*

$$f = \Delta_{x_1}(g_1) + \dots + \Delta_{x_n}(g_n).$$

The rational functions g_1, \dots, g_n , if they exist, are called the certificates of f .

Problem 2.4 (Rational Summability Problem). *Given a rational function $f \in \mathbb{F}(\mathbf{x})$ and an integer n with $1 \leq n \leq m$, decide whether or not f is $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -summable in $\mathbb{F}(\mathbf{x})$. If so, find a tuple (g_1, \dots, g_n) such that the g_i 's are the certificates of f .*

In terms of arithmetic size, the certificate tends to be much larger than the telescoper but certificates are often not needed in applications. Similar to the case of univariate rational summation [36], we also output the certificate as a sum of the products of several rational functions applied by shift operations and difference isomorphisms (see the definitions in the specific algorithms). Such a certificate is called an *unnormalised certificate*.

The main idea of solving the summability problem is using mathematical induction to reduce the number of difference operators in this problem. To say explicitly, we shall reduce the $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -summability problem for $f \in \mathbb{F}(\mathbf{x})$ to the $(\sigma_{x_1}, \dots, \sigma_{x_r})$ -summability problem for another rational function $a \in \mathbb{F}(\mathbf{x})$, where r is smaller than n and the base field $\mathbb{F}(\mathbf{x})$ in the summability problem is unchanged. Similarly, we shall reduce the existence problem of telescopers of type $(\sigma_t; \sigma_{x_1}, \dots, \sigma_{x_n})$ for $f \in \mathbb{K}(t, \mathbf{x})$ to the existence problem of telescopers of type $(\sigma_t; \sigma_{x_1}, \dots, \sigma_{x_r})$ for some rational function $a \in \mathbb{K}(t, \mathbf{x})$.

We introduce below a general definition of the summability problem and existence problem of telescopers, which plays a central role to set up the reduction process for solving Problems 2.4 and 2.2. Let $G_t = \langle \sigma_t, \sigma_{x_1}, \dots, \sigma_{x_n} \rangle$ be the group generated by the shift operators $\sigma_t, \sigma_{x_1}, \dots, \sigma_{x_n}$ under the operation of composition of functions. Then G_t is a free abelian group. For any $\tau \in G_t$, the difference operator Δ_τ is defined by

$$\Delta_\tau = S_t^{i_0} S_{x_1}^{i_1} \dots S_{x_n}^{i_n} - \mathbf{1} \quad \text{if } \tau = \sigma_t^{i_0} \sigma_{x_1}^{i_1} \dots \sigma_{x_n}^{i_n}.$$

For short, we use Δ_v to denote Δ_{σ_v} for each $v \in \mathbf{v}$. A finite subset $\{\tau_1, \dots, \tau_r\}$ of G_t is said to be \mathbb{Z} -linearly independent if for all $a_1, \dots, a_r \in \mathbb{Z}$, we have

$$\tau_1^{a_1} \dots \tau_r^{a_r} = \mathbf{1} \quad \text{if and only if} \quad a_1 = a_2 = \dots = a_r = 0.$$

Let $G = \langle \sigma_{x_1}, \dots, \sigma_{x_n} \rangle$ be the subgroup of G_t generated by shift operators $\sigma_{x_1}, \dots, \sigma_{x_n}$. Let $\{\tau_1, \dots, \tau_r\} (1 \leq r \leq n)$ be a family of \mathbb{Z} -linearly independent elements in G . In general, a rational function $f \in \mathbb{F}(\mathbf{x})$ is called (τ_1, \dots, τ_r) -*summable* in $\mathbb{F}(\mathbf{x})$ if

$$f = \Delta_{\tau_1}(g_1) + \dots + \Delta_{\tau_r}(g_r)$$

for some $g_1, \dots, g_r \in \mathbb{F}(\mathbf{x})$. Choose an element $\tau_0 = \sigma_t^{k_0} \sigma_{x_1}^{k_1} \dots \sigma_{x_n}^{k_n} \in G_t$ such that k_0 is nonzero. Then $\tau_0, \tau_1, \dots, \tau_r$ are \mathbb{Z} -linearly independent in G_t . Let $T_0 = S_t^{k_0} S_{x_1}^{k_1} \dots S_{x_n}^{k_n} \in \mathcal{R}$ be the operator corresponding to τ_0 . We say that a nonzero operator $L \in \mathbb{K}(t)\langle T_0 \rangle$ is a *telescoper of type* $(\tau_0; \tau_1, \dots, \tau_r)$ for $f \in \mathbb{K}(t, \mathbf{x})$ if $L(f)$ is (τ_1, \dots, τ_r) -summable in $\mathbb{K}(t, \mathbf{x})$.

Let R be a ring and $\sigma: R \rightarrow R$ be a ring automorphism of R . The pair (R, σ) is called a *difference ring*. If R is a field, we call the pair (R, σ) a *difference field*. Let (R_1, σ_1) and (R_2, σ_2) be two difference rings and $\phi: R_1 \rightarrow R_2$ be a ring homomorphism. If ϕ satisfies the property that $\phi \circ \sigma_1 = \sigma_2 \circ \phi$, that means the following diagram

$$\begin{array}{ccc} R_1 & \xrightarrow{\phi} & R_2 \\ \sigma_1 \downarrow & & \downarrow \sigma_2 \\ R_1 & \xrightarrow{\phi} & R_2 \end{array}$$

commutes, then ϕ is called a *difference homomorphism*. If in addition ϕ is a bijection, then its inverse ϕ^{-1} is also a difference homomorphism. In this case, we call ϕ a *difference isomorphism*. The notion of difference isomorphisms will be used to state our summability criteria and the existence criteria of telescopers.

An operator $L \in \mathbb{K}(t)\langle S_t \rangle$ is called a *common left multiple* of operators $L_1, \dots, L_r \in \mathbb{K}(t)\langle S_t \rangle$ if there exist $R_1, \dots, R_r \in \mathbb{K}(t)\langle S_t \rangle$ such that

$$L = R_1 L_1 = \dots = R_r L_r.$$

Since $\mathbb{K}(t)\langle S_t \rangle$ is a left Euclidean domain, such an operator L always exists. Among all of such multiples, the monic one of smallest degree in S_t is called the *least common left multiple* (LCLM). Efficient algorithms for computing LCLM have been developed in [7, 14, 16].

Remark 2.5. Let $f = f_1 + \dots + f_r$ with $f_i \in \mathbb{K}(t, \mathbf{x})$. If each f_i has a telescoper L_i of type $(\sigma_t; \sigma_{x_1}, \dots, \sigma_{x_n})$ for $i = 1, \dots, r$, then the LCLM of L_i 's is a telescoper of the same type for f . This fact follows from the commutativity between operators in $\mathbb{K}(t)\langle S_t \rangle$ and the difference operators Δ_{x_i} 's.

2.2 Complexity estimates

All complexity estimates of the algorithms in this paper are in terms of arithmetical operations in \mathbb{K} , denoted by “ops”. The notation \bar{O} indicates the complexity estimates with hidden polylogarithmic factors.

Let $\mathbf{y} = \{y_1, y_2, \dots, y_r\}$ be a subset of $\mathbf{v} = \{t, x_1, \dots, x_m\}$ and $\mathbf{d} = (d_1, d_2, \dots, d_r)$ be a vector in \mathbb{N}^r . Let $\mathbb{K}[\mathbf{y}]_{\mathbf{d}}$ denote the set of polynomials in $\mathbb{K}[\mathbf{y}]$ whose degrees in y_i are no more than d_i for $i = 1, 2, \dots, r$. Let $\mathbb{K}(\mathbf{y})_{\mathbf{d}}$ denote the set of rational functions in $\mathbb{K}(\mathbf{y})$ with numerators and denominators in $\mathbb{K}[\mathbf{y}]_{\mathbf{d}}$. In particular, we denote $\mathbb{K}[\mathbf{y}]_{\mathbf{d}}$ (resp. $\mathbb{K}(\mathbf{y})_{\mathbf{d}}$) by $\mathbb{K}[\mathbf{y}]_d$ (resp. $\mathbb{K}(\mathbf{y})_d$) for simplicity if $d_1 = d_2 = \dots = d_r = d$. For a rational function $f(\mathbf{y}) = p(\mathbf{y})/q(\mathbf{y}) \in \mathbb{K}(\mathbf{y})$, where $p(\mathbf{y})$ and $q(\mathbf{y})$ are coprime polynomials, the degree of $f(\mathbf{y})$ in y_i is defined as $\max\{\deg_{y_i}(p(\mathbf{y})), \deg_{y_i}(q(\mathbf{y}))\}$. In particular, for $f(\mathbf{y}) \in \mathbb{Q}(\mathbf{y})$, let $\|f\|$ denote the *max-norm* of f , i.e., the maximal absolute value of the integer coefficients appearing in the numerator and denominator of f with respect to \mathbf{y} .

Similarly, for a matrix or vector \mathcal{M} over \mathbb{Z} , the *max-norm* of \mathcal{M} , denoted by $\|\mathcal{M}\|$, is defined as the maximal absolute value of its entries.

We first recall some complexity estimates of the basic operations on univariate polynomials and rational functions (see the books [17, 78] for their proofs).

Fact 2.6. *Let d be an integer in \mathbb{N} . The following operations can be performed in $\tilde{O}(d)$ ops in \mathbb{K} :*

- (1) *addition, multiplication and differentiation of elements in $\mathbb{K}[x]_d$ and $\mathbb{K}(x)_d$;*
- (2) *computing the greatest common divisor of two elements in $\mathbb{K}[x]_d$;*
- (3) *partial fraction decomposition of an element in $\mathbb{K}(x)_d$ with a given factorization of its denominator.*

Efficient algorithms for basic operations on multivariate polynomials have been developed in [63, 77]. We summarize the needed results as follows.

Fact 2.7. *For a vector $\mathbf{d} = (d_1, \dots, d_m) \in \mathbb{N}^m$, it takes $\tilde{O}(md_1 \cdots d_m)$ ops in \mathbb{K} for multipoint evaluation or interpolation in $\mathbb{K}[\mathbf{x}]_{\mathbf{d}}$ from the given values on $O(d_1 \cdots d_m)$ points which form an m -dimensional tensor product grid.*

We recall the complexity estimates of factoring multivariate polynomials from [58, Theorem 3.26] and [59, Equation (II)].

Fact 2.8. *For a vector $\mathbf{d} = (d_1, \dots, d_m) \in \mathbb{N}^m$ and an integer $M \in \mathbb{N}^+$, let f be a polynomial in $\mathbb{Z}[\mathbf{x}]_{\mathbf{d}}$ with $\|f\| = M$. Then it takes $\tilde{O}((\min(\mathbf{d}))^{m-1}(d_1 \cdots d_m)^5(d_1 \cdots d_m + \log(M)))$ ops in \mathbb{Q} to factor f into the product of irreducible factors and the max-norms of these factors are $2^{d_1 + \cdots + d_n} \sqrt{(d_1 + 1) \cdots (d_n + 1)} M$.*

Let $\omega \in (2, 3]$ be a feasible exponent of matrix multiplication in \mathbb{K} , i.e., two square matrices of order r can be multiplied using $O(r^\omega)$ ops. Solving a system of linear equations is almost as hard as multiplying two matrices, see [17] for more details. In particular, one approach to solve a linear system over \mathbb{Q} for integer solutions is computing the Hermite normal form of its corresponding matrix, see [30]. Combining the algorithms for computing Hermite normal forms developed in [30, 32, 43, 44, 48, 50, 75, 76], we obtain the following result.

Fact 2.9. *(See [17, Theorem (16.18)], [44, Proposition 2.3 and its proof]) A \mathbb{K} -linear system L of size r can be solved in $O(r^\omega)$ ops in \mathbb{K} . Furthermore, if $K = \mathbb{Q}$ and the absolute values of the coefficients of L are integers bounded by M , then one can find a special solution of L with a \mathbb{Q} -basis for the corresponding homogeneous linear system over \mathbb{Z}^r using $\tilde{O}(r^{\omega+1} \log(M))$ bit ops and the max-norms of these vectors are $O(r^{r/2} M^r)$.*

3 Shift equivalence testing of polynomials

In this section, we first state the problem of Shift Equivalence Testing (SET) and give an overview of our algorithm for solving the SET problem in Section 3.1. The idea of our algorithm is inspired by the DOS algorithm [34, 35]. Then we develop a general scheme for designing algorithms to solve the SET problem, whose proof is given in Section 3.2. More precisely, we introduce admissible covers of the associated polynomial system with the SET problem and prove that every admissible cover corresponds to an algorithm for solving the SET problem. In Section 3.3, we give two special admissible covers in practice, one of which corresponds to the DOS algorithm.

3.1 Overview of the general algorithm

Let \mathbb{F} be a field of characteristic zero and let $\mathbb{F}[\mathbf{x}]$ be the ring of polynomials in $\mathbf{x} = \{x_1, \dots, x_n\}$ over \mathbb{F} . Two polynomials $p, q \in \mathbb{F}[\mathbf{x}]$ are said to be *shift equivalent* if there exist $s_1, \dots, s_n \in \mathbb{F}$ such that

$$p(x_1 + s_1, \dots, x_n + s_n) = q(x_1, \dots, x_n).$$

The set $\{\mathbf{s} \in \mathbb{F}^n \mid p(\mathbf{x} + \mathbf{s}) = q(\mathbf{x})\}$ is called the *dispersion set* of p and q over \mathbb{F} , denoted by $F_{p,q}$. Recall basic properties of the dispersion set in [35].

Lemma 3.1. (See [35, Observation 4.2 and Lemma 4.4]) *Let $p, q \in \mathbb{F}[\mathbf{x}]$. Then*

- (1) $F_{p,p}$ is a linear subspace of \mathbb{F}^n over \mathbb{F} .
- (2) $F_{p,q} = \mathbf{s} + F_{p,p}$ for any $\mathbf{s} \in F_{p,q}$ if $F_{p,q} \neq \emptyset$.

The problem of Shift Equivalence Testing can be stated as follows.

Problem 3.2 (Shift Equivalence Testing Problem). *Given $p, q \in \mathbb{F}[x_1, \dots, x_n]$, decide whether there exists $\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{F}^n$ such that*

$$p(\mathbf{x} + \mathbf{s}) = q(\mathbf{x}).$$

If such a vector \mathbf{s} exists, compute the dispersion set $F_{p,q}$ of p and q over \mathbb{F} . In this case, by Lemma 3.1, it suffices to find a special solution \mathbf{s} in $F_{p,q}$ and a basis of $F_{p,p}$ over \mathbb{F} .

A related problem is testing the shift equivalence over integers, i.e. deciding whether there exists a vector $\mathbf{s} \in \mathbb{Z}^n$ such that $p(\mathbf{x} + \mathbf{s}) = q(\mathbf{x})$. We denote the set $\{\mathbf{s} \in \mathbb{Z}^n \mid p(\mathbf{x} + \mathbf{s}) = q(\mathbf{x})\}$ by $Z_{p,q}$. The computation of $Z_{p,q}$ will play an important role in the next sections where we study the rational summability problem and the existence problem of telescopers. By Lemma 3.1, we know $F_{p,q}$ is a linear variety over \mathbb{F} . Once the computation of $F_{p,q}$ boils down to solving linear systems, we can also compute $Z_{p,q}$ by combining the same methods for the SET problem over \mathbb{F} and any algorithm for computing integer solutions of linear systems.

In the univariate case, the SET problem was solved by computing the resultant of two polynomials [1]. In the multivariate case, there are three different methods for solving the SET problem in the literature. In 1996, Grigoriev first gave a recursive algorithm (G) for the SET problem in [41, 42]. In 2010, motivated by solving linear partial difference equations, another algorithm (KS) for computing $Z_{p,q}$ via the Gröbner basis method was given by Kauers and Schneider in [54]. In 2014, a new algorithm with better complexity was given by Dvir, Oliveira and Shpilka (DOS) in [34, 35]. We have implemented all of the three algorithms in Maple and the experimental comparison is tabulated in Section 7. The timings indicate that the DOS algorithm is the most efficient one among the three methods in practice.

In this section, we introduce n new variables $\mathbf{a} = \{a_1, \dots, a_n\}$. The SET problem is equivalent to finding the zeros of the polynomial $p(\mathbf{x} + \mathbf{a}) - q(\mathbf{x}) \in \mathbb{F}[\mathbf{a}, \mathbf{x}]$ with respect to \mathbf{a} . Collecting its coefficients in \mathbf{x} , we obtain a polynomial system. A direct approach to the SET problem is solving this polynomial system. Without exploring the hidden structure of the polynomial system, this naive approach could be very in-efficient. The common idea of the above three methods is to find the defining linear system of $F_{p,q}$, which avoids solving the polynomial system directly. To do this, the DOS algorithm finds an appropriate finite cover of the polynomial system. Then it reduces the SET problem to solving several linear systems successively by evaluating the non-linear part of polynomials. This kind of evaluation is called the *linearization* of polynomials, whose definition will be strictly stated below.

We first introduce some notations for later use. For any two vectors $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n), \boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_n) \in \mathbb{N}^n$, we say $\boldsymbol{\alpha} \geq \boldsymbol{\beta}$ if and only if $\alpha_i \geq \beta_i$ for all $1 \leq i \leq n$. This defines a partial order on \mathbb{N}^n . For a subset $\mathbf{y} = \{y_1, y_2, \dots, y_m\}$ of $\mathbf{x} = \{x_1, x_2, \dots, x_n\}$ with $\hat{\mathbf{y}} := \mathbf{x} \setminus \mathbf{y}$, let $f(\mathbf{x}) = \sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}}(\hat{\mathbf{y}}) \mathbf{y}^{\boldsymbol{\alpha}} \in \mathbb{F}[\hat{\mathbf{y}}][\mathbf{y}]$. Let $H_{\mathbf{y}}^d(f(\mathbf{x}))$ denote the homogeneous component of $f(\mathbf{x})$ of degree d in \mathbf{y} and let $\text{Supp}_{\mathbf{y}}(f)$ denote the set $\{\mathbf{y}^{\boldsymbol{\alpha}} \mid c_{\boldsymbol{\alpha}}(\hat{\mathbf{y}}) \neq 0\}$. For simplicity, when $\mathbf{y} = \mathbf{x}$, we write $H_{\mathbf{y}}^{\ell}(f(\mathbf{x}))$ as $H^{\ell}(f(\mathbf{x}))$ and $\text{Supp}_{\mathbf{y}}(f)$ as $\text{Supp}(f)$. For a subset $S \subseteq \mathbb{F}[\mathbf{x}]$, let $\mathbb{V}_{\mathbb{F}}(S)$ be the zero set $\{\mathbf{s} \in \mathbb{F}^n \mid f(\mathbf{s}) = 0, \forall f \in S\}$.

Definition 3.3 (Linearization). *Let $f(\mathbf{x}) = H_{\mathbf{y}}^0(f)(\mathbf{y}) + H_{\mathbf{y}}^1(f)(\mathbf{y}) + \dots + H_{\mathbf{y}}^d(f)(\mathbf{y})$ be the homogeneous decomposition of $f \in \mathbb{F}[\mathbf{x}] = \mathbb{F}[\hat{\mathbf{y}}][\mathbf{y}]$. For a vector $\mathbf{s} \in \mathbb{F}^m$, we call the linear polynomial $H_{\mathbf{y}}^0(f)(\mathbf{y}) + H_{\mathbf{y}}^1(f)(\mathbf{y}) + \sum_{i=2}^d H_{\mathbf{y}}^i(f)(\mathbf{s})$ the linearization of f at \mathbf{s} with respect to \mathbf{y} , denoted by $L_{\mathbf{y}=\mathbf{s}}(f)$. Note that $L_{\mathbf{y}=\mathbf{s}}(f) = f$ if $d \leq 1$. For a polynomial set $S \subseteq \mathbb{F}[\mathbf{x}]$, let $L_{\mathbf{y}=\mathbf{s}}(S) := \{L_{\mathbf{y}=\mathbf{s}}(f) \mid f \in S\}$ be the linearization of S .*

In the following we present the main idea of algorithm DOS [34, 35] and our new algorithm ADC in a general framework.

In order to compute $F_{p,q}$, we first write

$$p(\mathbf{x} + \mathbf{a}) - q(\mathbf{x}) = \sum_{\boldsymbol{\alpha} \in \Lambda} c_{\boldsymbol{\alpha}}(\mathbf{a}) \mathbf{x}^{\boldsymbol{\alpha}},$$

where $c_{\boldsymbol{\alpha}}(\mathbf{a}) \in \mathbb{F}[\mathbf{a}]$ and Λ is a finite subset of \mathbb{N}^n . Let

$$S := \{c_{\boldsymbol{\alpha}}(\mathbf{a}) \in \mathbb{F}[\mathbf{a}] \mid c_{\boldsymbol{\alpha}}(\mathbf{a}) \text{ is a nonzero coefficient of } \mathbf{x}^{\boldsymbol{\alpha}} \text{ in } p(\mathbf{x} + \mathbf{a}) - q(\mathbf{x})\}. \quad (3.1)$$

Then $F_{p,q} = \mathbb{V}_{\mathbb{F}}(S)$ is the zero set of S in \mathbb{F}^n . First, we classify all polynomials in S according to their total degrees in \mathbf{a} and write $S = S_0^D \cup \dots \cup S_{d'}^D$, where $d' = \deg_{\mathbf{a}}(p(\mathbf{x} + \mathbf{a}) - q(\mathbf{x}))$ and

$$S_i^D = \{c_{\boldsymbol{\alpha}}(\mathbf{a}) \in S \mid \deg_{\mathbf{a}}(c_{\boldsymbol{\alpha}}(\mathbf{a})) = i\}$$

for $i = 0, \dots, d'$. Then $\mathbb{V}_{\mathbb{F}}(S) = \mathbb{V}_{\mathbb{F}}(\cup_{i=0}^{d'} S_i^D)$. We may assume that $S_0^D = \emptyset$, otherwise p, q are not shift equivalent and return $F_{p,q} = \emptyset$. If $S_0^D \cup S_1^D$ has no solution in \mathbb{F}^n , return $F_{p,q} = \emptyset$. Otherwise take an arbitrary solution $\mathbf{s}^{(0)} \in \mathbb{V}_{\mathbb{F}}(S_0^D \cup S_1^D)$. Note that all polynomials in $S_0^D \cup S_1^D$ are linear and thus such an element $\mathbf{s}^{(0)}$ can be computed straightforwardly. We shall prove that the nonlinear system $S_0^D \cup S_1^D \cup S_2^D$ has the same solutions as its linearization $S_0^D \cup S_1^D \cup L_{\mathbf{a}=\mathbf{s}^{(0)}}(S_2^D)$ at the point $\mathbf{s}^{(0)}$. If the latter linear system has no solution, return $F_{p,q} = \emptyset$. Otherwise, take an arbitrary solution $\mathbf{s}^{(1)} \in \mathbb{V}_{\mathbb{F}}(S_0^D \cup S_1^D \cup L_{\mathbf{a}=\mathbf{s}^{(0)}}(S_2^D))$ by solving the linear system. Then consider the linearization of $\cup_{i=0}^3 S_i^D$ at $\mathbf{s}^{(1)}$ and we shall prove that $\mathbb{V}_{\mathbb{F}}(\cup_{i=0}^3 S_i^D) = \mathbb{V}_{\mathbb{F}}(\cup_{i=0}^3 L_{\mathbf{a}=\mathbf{s}^{(1)}}(S_i^D))$. Continuing the above process, we will finally find an equivalent linear system of the polynomial system $S = \cup_{i=0}^{d'} S_i^D$ by linearization.

Example 3.4. *Let $p = x^2 + 2xy + y^2 + 2x + 6y$ and $q = x^2 + 2xy + y^2 + 4x + 8y + 11$ be two polynomials in $\mathbb{Q}[x, y]$. Decide whether p, q are shift equivalent with respect to x, y . Since*

$$p(x + a, y + b) - q(x, y) = (2a + 2b - 2) \cdot x + (2a + 2b - 2) \cdot y + (a^2 + 2ab + b^2 + 2a + 6b - 11),$$

we have $S = S_1^D \cup S_2^D$, where $S_1^D = \{2a + 2b - 2\}$ and $S_2^D = \{a^2 + 2ab + b^2 + 2a + 6b - 11\}$. Take an arbitrary solution $(a, b) = (1, 0)$ of S_1^D . The linearization of S_2^D at $(1, 0)$ is

$$L_{(a,b)=(1,0)}(S_2^D) = \{1^2 + 2 \cdot 1 \cdot 0 + 0^2 + 2a + 6b - 11\} = \{2a + 6b - 10\}.$$

In this example, the linear system $S_1^D \cup L_{(a,b)=(1,0)}(S_2^D)$ is indeed equivalent to the polynomial system $S_1^D \cup S_2^D$. So $F_{p,q} = \mathbb{V}_{\mathbb{F}}(S_1^D \cup L_{(a,b)=(1,0)}(S_2^D)) = \{(-1, 2)\}$.

Since shift operations do not change the total degree in \mathbf{x} , the homogeneous components of both sides of $p(\mathbf{x} + \mathbf{a}) = q(\mathbf{x})$ with respect to \mathbf{x} must be equal. The homogeneous decomposition of $p(\mathbf{x} + \mathbf{a}) - q(\mathbf{x})$ yields another cover $\{S_0^H, S_1^H, \dots, S_d^H\}$ of S , where $d = \max\{\deg_{\mathbf{x}}(p(\mathbf{x})), \deg_{\mathbf{x}}(q(\mathbf{x}))\}$ and

$$S_i^H := \{c_{\alpha}(\mathbf{a}) \in S \mid c_{\alpha}(\mathbf{a}) \text{ is the coefficient of } \mathbf{x}^{\alpha} \text{ in } H_{\mathbf{x}}^{d-i}(p(\mathbf{x} + \mathbf{a})) - H_{\mathbf{x}}^{d-i}(q(\mathbf{x}))\}$$

for $i = 0, 1, \dots, d$. In the DOS algorithm, they first introduced the above method of linearization to solve the polynomial system $S = S_0^H \cup S_1^H \cup \dots \cup S_d^H$ and proved the correctness of their algorithm by using formal partial derivatives. In Example 3.4, $S = S_1^D \cup S_2^D = S_1^H \cup S_2^H$, where $S_i^H = S_i^D$ for $i = 1, 2$. In general, these two covers are different. A natural question is for which cover, we can use the method of linearization to compute the dispersion set. One answer is the admissible cover defined below. In fact, the above two covers $\{S_0^D, S_1^D, \dots, S_d^D\}$ and $\{S_0^H, S_1^H, \dots, S_d^H\}$, called by **a-degree cover** and **x-homogeneous cover** respectively, are both admissible, which will be proved in Section 3.3.

Definition 3.5 (Admissible cover). *Let $S \subseteq \mathbb{F}[\mathbf{a}]$ be as in (3.1). A collection $\{S_0, S_1, \dots, S_m\}$ of subsets is called a cover of S if S is the union of S_0, S_1, \dots, S_m . Such a cover $\{S_0, S_1, \dots, S_m\}$ is called an admissible cover of S if it satisfies the following two conditions:*

- (1) *The degree of any polynomial in S_0 is at most one.*
- (2) *For every $\ell = 1, 2, \dots, m$, if $c_{\alpha}(\mathbf{a}) \in S_{\ell}$, then $c_{\beta}(\mathbf{a}) \in \cup_{i=0}^{\ell-1} S_i$ for all $\beta \in \mathbb{N}^n$ with $\beta > \alpha$ and $\mathbf{x}^{\beta} \in \text{Supp}_{\mathbf{x}}(p(\mathbf{x} + \mathbf{a}) - q(\mathbf{x}))$.*

A general algorithm for solving the SET problem via the method of linearization is as follows. This algorithm inherits one feature of the DOS algorithm: it could be early terminated when p, q are not shift equivalent. If two nonzero polynomials $p(\mathbf{x})$ and $q(\mathbf{x})$ are shift equivalent, then they have the same degree d in x and $H^d(p(\mathbf{x})) = H^d(q(\mathbf{x}))$, which means $\deg(p(\mathbf{x}) - q(\mathbf{x})) < \deg(p(\mathbf{x}))$. Therefore, we can check the degree condition at the beginning of the algorithm for better efficiency.

Algorithm 3.6 (Shift Equivalence Testing). **ShiftEquivalent**($p, q, [x_1, \dots, x_n]$).

INPUT: two multivariate polynomials $p, q \in \mathbb{F}[\mathbf{x}]$;

OUTPUT: a special solution of $F_{p,q}$ and an \mathbb{F} -basis of $F_{p,p}$ if p and q are shift equivalent; $\{\}$ otherwise.

- 1 *if $p(\mathbf{x}) = q(\mathbf{x}) = 0$, return $\mathbf{0}$ and the standard basis of \mathbb{F}^n .*
- 2 *if $\deg(p(\mathbf{x}) - q(\mathbf{x})) \geq \deg(p(\mathbf{x}))$, return $\{\}$.*
- 3 *set $S := \text{Coefficients}(p(\mathbf{x} + \mathbf{a}) - q(\mathbf{x}), \mathbf{x}) \subseteq \mathbb{F}[\mathbf{a}]$.*
- 4 *let $\{S_0, S_1, \dots, S_m\}$ be an admissible cover of S .*
- 5 *set $\mathbf{s}^{(0)} := \mathbf{0}$.*
- 6 **for** $\ell = 0, \dots, m$ **do**
- 7 *set $L^{(\ell)} := \cup_{i=0}^{\ell} L_{\mathbf{a}=\mathbf{s}^{(\ell)}}(S_i)$.*
- 8 *solve the linear system in \mathbf{a} defined by $L^{(\ell)}$.*
- 9 *if the linear system $L^{(\ell)}$ has no solution, return $\{\}$.*
- 10 *else there is a special solution $\mathbf{s}' \in \mathbb{F}^n$ by evaluating each free variable at 0, set $\mathbf{s}^{(\ell+1)} := \mathbf{s}'$.*
- 11 **return** $\mathbf{s}^{(m+1)}$ *and an \mathbb{F} -basis of the solution space of the homogeneous linear equations induced by $L^{(m)}$.*

The correctness of Algorithm 3.6 is guaranteed by the following theorem.

Theorem 3.7. *If the cover $\{S_0, S_1, \dots, S_m\}$ of S is admissible, then for all $\ell = 0, 1, \dots, m$, we have either $\mathbb{V}_{\mathbb{F}}\left(\bigcup_{i=0}^{\ell-1} S_i\right) = \emptyset$ or*

$$\mathbb{V}_{\mathbb{F}}\left(\bigcup_{i=0}^{\ell} S_i\right) = \mathbb{V}_{\mathbb{F}}\left(\bigcup_{i=0}^{\ell} L_{\mathbf{a}=\mathbf{s}^{(\ell)}}(S_i)\right) \quad \text{for any } \mathbf{s}^{(\ell)} \in \mathbb{V}_{\mathbb{F}}\left(\bigcup_{i=0}^{\ell-1} S_i\right).$$

The proof of Theorem 3.7 will be given in the next subsection.

Theorem 3.8. *For a vector $\mathbf{d} \in \mathbb{N}^n$, let $p(\mathbf{x})$ and $q(\mathbf{x})$ be two multivariate polynomials in $\mathbb{F}[\mathbf{x}]_{\mathbf{d}}$. Then Algorithm 3.6 can test whether p and q are shift equivalent and output a special solution of $F_{p,q}$ with an \mathbb{F} -basis of $F_{p,p}$ if $F_{p,q} \neq \emptyset$ using $\tilde{O}(d_1^\omega \cdots d_n^\omega)$ ops in \mathbb{F} . Furthermore, combining the algorithms for computing the integer solutions of linear systems, Algorithm 3.6 can test whether p and q are shift equivalent over \mathbb{Z} and output a special solution of $Z_{p,q}$ with an \mathbb{Q} -basis of $Z_{p,p}$ in the case of $Z_{p,q} \neq \emptyset$; if n is fixed and $p(\mathbf{x}), q(\mathbf{x}) \in \mathbb{Z}[\mathbf{x}]_\delta$ with max-norms bounded by the positive integer M , then the max-norms of the output vectors are $O(M^{\delta^{O(1)}})$ and the algorithm costs $\tilde{O}(\delta^{n(\omega+1)} \log(M))$ ops.*

Proof. The first three steps take $\tilde{O}(2nd_1^2 \cdots d_n^2)$ ops by Fact 2.7. Since both \mathbf{a} -degree cover and \mathbf{x} -homogeneous cover can be obtained by traversing elements in S , Step 4 can be performed in $\tilde{O}(d_1^2 \cdots d_n^2)$ ops. By the definitions of \mathbf{a} -degree covers and \mathbf{x} -homogeneous covers, we have that m is of the size $\tilde{O}(d_1 + \cdots + d_n)$. Note that by Lemma 3.16 below, Step 7 in the loop can be replaced by setting $L^{(\ell)}$ to be the union of $L^{(\ell-1)}$ and $L_{\mathbf{a}=\mathbf{s}^{(\ell)}}(S_\ell)$ if $\ell \geq 1$, and hence the size of the linear system in Step 8 is no more than $|S_\ell| + n$. As a result, the cost of Step 7 is $\tilde{O}(|S_\ell|nd_1 \cdots d_n)$ ops and that of Step 8 is $\tilde{O}((|S_\ell| + n)^\omega)$ ops in each iteration. This implies that the loop takes no more than $\tilde{O}(C)$ ops, where

$$\begin{aligned} C &= \sum_{\ell=0}^m ((|S_\ell| + n)^\omega + |S_\ell|nd_1 \cdots d_n) \leq \left(\sum_{\ell=0}^m (|S_\ell| + n)\right)^\omega + \left(\sum_{\ell=0}^m |S_\ell|\right) nd_1 \cdots d_n \\ &= (|S| + mn)^\omega + n|S|d_1 \cdots d_n. \end{aligned}$$

Since $|S|$ is no more than $d_1 d_2 \cdots d_n$, the loop needs $\tilde{O}(d_1^\omega \cdots d_n^\omega)$ ops that dominates the whole costs. This completes the proof of the first claim.

Now we turn to the case of computing $Z_{p,q}$ by Algorithm 3.6. The cost of the algorithm follows from Fact 2.9 since the timing is dominated by Step 8 in the loop. For the max-norms of the output vectors, note that the solution sets of $L^{(\ell)}$ would be updated at most $n + 1$ times, so the unavoidable coefficient explosions would occur in at most $n + 1$ iterations of the loop. Then we obtain the desired estimate of these max-norms by Fact 2.9. \blacksquare

From the above complexity analysis, we can not distinguish the algorithms with \mathbf{a} -degree cover and \mathbf{x} -homogeneous cover. In Section 7, we have implemented both algorithms to compare the practical efficiency. The experiments show that our ADC algorithm is more efficient than the DOS algorithm for sparse polynomials.

3.2 Proof of correctness of Theorem 3.7

Before proving Theorem 3.7, we need several lemmas to explore the inner structure of polynomials $c_\alpha(\mathbf{a})$ in S . First we give an explicit expression of the non-constant homogeneous components of $c_\alpha(\mathbf{a})$ and find a recurrence relation among the homogeneous components. Then we explain the

role of the admissible cover and the magic of linearization in Algorithm 3.6. Finally, we prove Theorem 3.7 by induction on ℓ .

For a vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n$, let $|\boldsymbol{\alpha}| := \sum_{i=1}^n \alpha_i$ and $\binom{|\boldsymbol{\alpha}|}{\boldsymbol{\alpha}} := \frac{|\boldsymbol{\alpha}|!}{\alpha_1! \alpha_2! \dots \alpha_n!}$. Let ∂_{x_i} denote the partial derivative with respect to x_i and $\boldsymbol{\partial}^\alpha$ denote $(\partial_{x_1})^{\alpha_1} (\partial_{x_2})^{\alpha_2} \dots (\partial_{x_n})^{\alpha_n}$. For n variables $\mathbf{a} = \{a_1, a_2, \dots, a_n\}$, we use $D_{\mathbf{a}}$ to denote the directional derivative in the direction of \mathbf{a} , i.e., $D_{\mathbf{a}} := \sum_{i=1}^n a_i \partial_{x_i}$. For $\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{F}^n$, the notation $D_{\mathbf{s}}$ means $D_{\mathbf{a}}|_{\mathbf{a}=\mathbf{s}}$. Then for any $k \in \mathbb{N}^+$,

$$D_{\mathbf{a}}^k := (D_{\mathbf{a}})^k = \sum_{|\boldsymbol{\alpha}|=k} \binom{k}{\boldsymbol{\alpha}} \mathbf{a}^\alpha \boldsymbol{\partial}^\alpha$$

by the multinomial theorem since ∂_{x_i} and ∂_{x_j} commute.

By the directional derivative and Taylor's expansion, the homogeneous components of polynomials in S_ℓ^H can be expressed as follows.

Lemma 3.9. (See [35, Lemma 3.5]) *Let $d := \max\{\deg_{\mathbf{x}}(p(\mathbf{x})), \deg_{\mathbf{x}}(q(\mathbf{x}))\}$. For any $k \in \mathbb{N}$ and $\ell \in \{0, 1, \dots, d\}$, we have*

$$H_{\mathbf{x}}^{d-\ell}(p(\mathbf{x} + \mathbf{a})) - H_{\mathbf{x}}^{d-\ell}(q(\mathbf{x})) = \sum_{i=0}^{\ell} \frac{1}{i!} D_{\mathbf{a}}^i \left(H_{\mathbf{x}}^{d-\ell+i}(p(\mathbf{x})) \right) - H_{\mathbf{x}}^{d-\ell}(q(\mathbf{x})) \quad (3.2)$$

and

$$H_{\mathbf{a}}^k \left(H_{\mathbf{x}}^{d-\ell}(p(\mathbf{x} + \mathbf{a})) - H_{\mathbf{x}}^{d-\ell}(q(\mathbf{x})) \right) = \begin{cases} \frac{1}{k!} D_{\mathbf{a}}^k \left(H_{\mathbf{x}}^{d-\ell+k}(p(\mathbf{x})) \right), & \text{if } k \geq 1, \\ H_{\mathbf{x}}^{d-\ell}(p(\mathbf{x})) - H_{\mathbf{x}}^{d-\ell}(q(\mathbf{x})), & \text{if } k = 0. \end{cases} \quad (3.3)$$

Moreover, for any $c_{\boldsymbol{\alpha}}(\mathbf{a}) \in S$ and $k \geq 1$, $H_{\mathbf{a}}^k(c_{\boldsymbol{\alpha}}(\mathbf{a}))$ is the coefficient of \mathbf{x}^α in $\frac{1}{k!} D_{\mathbf{a}}^k \left(H_{\mathbf{x}}^{|\alpha|+k}(p) \right)$.

Proof. Note that $c_{\boldsymbol{\alpha}}(\mathbf{a})$ is exactly the coefficient of \mathbf{x}^α in $H_{\mathbf{x}}^{|\alpha|}(p(\mathbf{x} + \mathbf{a})) - H_{\mathbf{x}}^{|\alpha|}(q(\mathbf{x}))$, so it is sufficient to prove Equations (3.2) and (3.3). By Taylor's expansion, we have

$$p(\mathbf{x} + \mathbf{a}) = \sum_{i=0}^d \frac{1}{i!} D_{\mathbf{a}}^i(p)(\mathbf{x}) = \sum_{i=0}^d \sum_{j=0}^d \frac{1}{i!} D_{\mathbf{a}}^i(H_{\mathbf{x}}^j(p))(\mathbf{x}).$$

Note that if $D_{\mathbf{a}}^i(H_{\mathbf{x}}^j(p))$ is not equal to zero, then it is homogeneous of degree $j-i$ in \mathbf{x} . Consequently, we obtain Equation (3.2). Moreover, note that $D_{\mathbf{a}}^i(H_{\mathbf{x}}^{d-\ell+i}(p))(\mathbf{x})$ is homogeneous of degree i with respect to \mathbf{a} , so we get Equation (3.3), which completes the proof. \blacksquare

Since

$$\frac{1}{k!} D_{\mathbf{a}}^k \left(H_{\mathbf{x}}^{|\alpha|+k}(p(\mathbf{x})) \right) = \frac{1}{k!} \sum_{|\boldsymbol{\beta}|=k} \binom{k}{\boldsymbol{\beta}} \mathbf{a}^\beta \boldsymbol{\partial}^\beta \left(H_{\mathbf{x}}^{|\alpha|+k}(p(\mathbf{x})) \right),$$

extracting the terms $H^k(c_{\boldsymbol{\alpha}}(\mathbf{a})) \cdot \mathbf{x}^\alpha$ in the both sides of the equality, we have

$$H^k(c_{\boldsymbol{\alpha}}(\mathbf{a})) \cdot \mathbf{x}^\alpha = \frac{1}{k!} \sum_{|\boldsymbol{\beta}|=k} \binom{k}{\boldsymbol{\beta}} \mathbf{a}^\beta \boldsymbol{\partial}^\beta \left([\mathbf{x}^{\alpha+\beta}](p(\mathbf{x})) \cdot \mathbf{x}^{\alpha+\beta} \right), \quad (3.4)$$

where $[\mathbf{x}^{\alpha+\beta}](p(\mathbf{x}))$ denotes the coefficient of $\mathbf{x}^{\alpha+\beta}$ in $p(\mathbf{x})$. Therefore, for any $f(\mathbf{x}) \in \mathbb{F}[\mathbf{x}]$, we can write $D_{\mathbf{a}, \boldsymbol{\alpha}}^k(f(\mathbf{x})) := \sum_{|\boldsymbol{\beta}|=k} \binom{k}{\boldsymbol{\beta}} \mathbf{a}^\beta \boldsymbol{\partial}^\beta ([\mathbf{x}^{\alpha+\beta}](f(\mathbf{x})) \cdot \mathbf{x}^{\alpha+\beta})$ and use $D_{\mathbf{s}, \boldsymbol{\alpha}}^k$ to denote $D_{\mathbf{a}, \boldsymbol{\alpha}}^k|_{\mathbf{a}=\mathbf{s}}$ for $\mathbf{s} \in \mathbb{F}^n$. The following lemma is derived straightforwardly.

Lemma 3.10. Let $k \in \mathbb{N}^+$ and $c_\alpha(\mathbf{a}) \in S$. Then we have $H^k(c_\alpha(\mathbf{a})) \cdot \mathbf{x}^\alpha = \frac{1}{k!} D_{\mathbf{a}, \alpha}^k(p(\mathbf{x}))$.

For the directional derivative, we know $D_{\mathbf{a}}^k(f(\mathbf{x})) = (D_{\mathbf{a}}^1)^k(f(\mathbf{x}))$. However $D_{\mathbf{a}, \alpha}^k(f(\mathbf{x}))$ may be different from $(D_{\mathbf{a}, \alpha}^1)^k(f(\mathbf{x}))$, as the following example shows.

Example 3.11. Let $\mathbb{F} = \mathbb{Q}$, $p(x, y), q(x, y) \in \mathbb{Q}[x, y]$ with $p(x, y) = x^3 + y^3$ and $q(x, y) = p(x, y) + 1$. Expanding $p(x + a, y + b) - q(x, y)$, we have $p(x + a, y + b) - q(x, y) = 3a \cdot x^2 + 3b \cdot y^2 + 3a^2 \cdot x + 3b^2 \cdot y + (a^3 + b^3 - 1)$. Then we have $c_{(1,0)}(a, b) = 3a^2$,

$$\begin{aligned} D_{(a,b),(1,0)}^1(p(\mathbf{x})) &= \sum_{i+j=1} \binom{1}{(i,j)} a^i b^j \cdot \partial_x^i \partial_y^j ([x^{1+i} y^{0+j}] (p(x, y)) \cdot x^{1+i} y^{0+j}) \\ &= \binom{1}{(1,0)} a \cdot \partial_x ([x^2] (p(x, y)) \cdot x^2) + \binom{1}{(0,1)} b \cdot \partial_y ([xy] (p(x, y)) \cdot xy) = 0, \\ D_{(a,b),(1,0)}^2(p(\mathbf{x})) &= \sum_{i+j=2} \binom{2}{(i,j)} a^i b^j \cdot \partial_x^i \partial_y^j ([x^{1+i} y^{0+j}] (p(x, y)) \cdot x^{1+i} y^{0+j}) \\ &= \binom{2}{(2,0)} a^2 \cdot \partial_x^2 ([x^3] (p(x, y)) \cdot x^3) + \binom{2}{(1,1)} ab \cdot \partial_x \partial_y ([x^2 y] (p(x, y)) \cdot x^2 y) \\ &\quad + \binom{2}{(0,2)} b^2 \cdot \partial_y^2 ([xy^2] (p(x, y)) \cdot xy^2) \\ &= \frac{2!}{0!2!} a^2 \cdot \partial_x^2 (x^3) = 6a^2 x \end{aligned}$$

and $\left(D_{(a,b),(1,0)}^1 \right)^2 (p(\mathbf{x})) = D_{(a,b),(1,0)}^1(0) = 0$. Therefore, we can check that $H^k(c_{(1,0)}(a, b)) \cdot x$ is equal to $\frac{1}{k!} D_{(a,b),(1,0)}^k(p(x, y))$ for $k = 1, 2$, but $D_{(a,b),(1,0)}^2(p(\mathbf{x}))$ is not equal to $\left(D_{(a,b),(1,0)}^1 \right)^2 (p(\mathbf{x}))$.

Now we rewrite the expression of $D_{\mathbf{a}, \alpha}^k(f)$ and derive a recurrence relation for $D_{\mathbf{a}, \alpha}^k(f)$.

Lemma 3.12. Let $\alpha \in \mathbb{N}^n$, $k, \ell \in \mathbb{N}^+$ and $f \in \mathbb{F}[\mathbf{x}]$. Let $\mathbf{e}_i \in \mathbb{N}^n$ denote a unit vector with the i -th component being one and others being zero. Then we have:

$$\begin{aligned} (1) \quad D_{\mathbf{a}, \alpha}^k(f(\mathbf{x})) &= \sum_{j_1=1}^n \cdots \sum_{j_k=1}^n \mathbf{a}^{\sum_{i=1}^k \mathbf{e}_{j_i}} \boldsymbol{\partial}^{\sum_{i=1}^k \mathbf{e}_{j_i}} \left([\mathbf{x}^{\alpha + \sum_{i=1}^k \mathbf{e}_{j_i}}] (f(\mathbf{x})) \cdot \mathbf{x}^{\alpha + \sum_{i=1}^k \mathbf{e}_{j_i}} \right). \\ (2) \quad D_{\mathbf{a}, \alpha}^{k+\ell}(f(\mathbf{x})) &= \sum_{|\beta|=\ell} \binom{\ell}{\beta} \mathbf{a}^\beta \boldsymbol{\partial}^\beta \left(D_{\mathbf{a}, \alpha+\beta}^k(f(\mathbf{x})) \right). \end{aligned}$$

Proof. (1) Note that for any $\beta \in \mathbb{N}^n$ with $|\beta| = k$, β can be expressed as a sum of k unit vectors. Moreover, there are $\binom{k}{\beta}$ different k -tuples (j_1, j_2, \dots, j_k) such that $\beta = \sum_{i=1}^k \mathbf{e}_{j_i}$. Then the claim follows from the definition of $D_{\mathbf{a}, \alpha}^k(f(\mathbf{x}))$.

(2) Applying (1) twice, we have

$$\begin{aligned}
& D_{\mathbf{a}, \boldsymbol{\alpha}}^{k+\ell}(f(\mathbf{x})) \\
&= \sum_{j_1=1}^n \cdots \sum_{j_k=1}^n \sum_{j_{k+1}=1}^n \cdots \sum_{j_{k+\ell}=1}^n \mathbf{a}^{\sum_{i=1}^k \mathbf{e}_{j_i} + \sum_{i=k+1}^{k+\ell} \mathbf{e}_{j_i}} \boldsymbol{\partial}^{\sum_{i=1}^k \mathbf{e}_{j_i} + \sum_{i=k+1}^{k+\ell} \mathbf{e}_{j_i}} \\
&\quad \left(\left[\mathbf{x}^{\boldsymbol{\alpha} + \sum_{i=1}^k \mathbf{e}_{j_i} + \sum_{i=k+1}^{k+\ell} \mathbf{e}_{j_i}} \right] (f(\mathbf{x})) \cdot \mathbf{x}^{\boldsymbol{\alpha} + \sum_{i=1}^k \mathbf{e}_{j_i} + \sum_{i=k+1}^{k+\ell} \mathbf{e}_{j_i}} \right) \\
&= \sum_{j_{k+1}=1}^n \cdots \sum_{j_{k+\ell}=1}^n \mathbf{a}^{\sum_{i=k+1}^{k+\ell} \mathbf{e}_{j_i}} \boldsymbol{\partial}^{\sum_{i=k+1}^{k+\ell} \mathbf{e}_{j_i}} \\
&\quad \left(\sum_{j_1=1}^n \cdots \sum_{j_k=1}^n \mathbf{a}^{\sum_{i=1}^k \mathbf{e}_{j_i}} \boldsymbol{\partial}^{\sum_{i=1}^k \mathbf{e}_{j_i}} \left(\left[\mathbf{x}^{\boldsymbol{\alpha} + \sum_{i=1}^k \mathbf{e}_{j_i}} \right] (f(\mathbf{x})) \cdot \mathbf{x}^{\boldsymbol{\alpha} + \sum_{i=1}^k \mathbf{e}_{j_i}} \right) \right) \\
&= \sum_{j_{k+1}=1}^n \cdots \sum_{j_{k+\ell}=1}^n \mathbf{a}^{\sum_{i=k+1}^{k+\ell} \mathbf{e}_{j_i}} \boldsymbol{\partial}^{\sum_{i=k+1}^{k+\ell} \mathbf{e}_{j_i}} \left(D_{\mathbf{a}, \boldsymbol{\alpha} + \sum_{i=k+1}^{k+\ell} \mathbf{e}_{j_i}}^k (f(\mathbf{x})) \right).
\end{aligned}$$

Then as the proof of (1), we can finally obtain (2) by setting $\boldsymbol{\beta} = \sum_{i=k+1}^{k+\ell} \mathbf{e}_{j_i}$. ■

Example 3.13. Let $\mathbb{F} = \mathbb{Q}$ and $p = x^4 + x^2y + y^3 \in \mathbb{Q}[x, y]$. After expanding, we get $p(x+a, y+b) = x^4 + 4a \cdot x^3 + x^2y + y^3 + (6a^2 + b) \cdot x^2 + 2a \cdot xy + 3b \cdot y^2 + (4a^3 + 2ab) \cdot x + (a^2 + 3b^2) \cdot y + (a^4 + a^2b + b^3)$. All terms of $p(x+a, y+b)$ are listed in Figure 3.1.

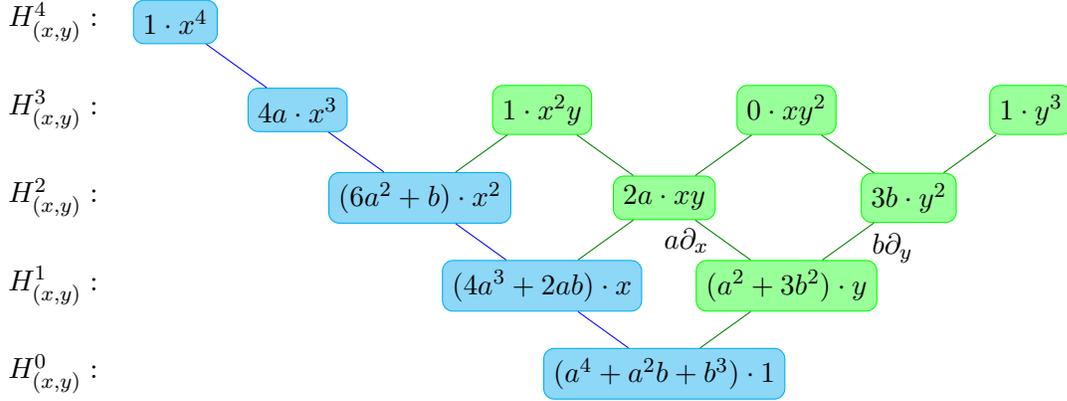


Figure 3.1: Terms of the polynomial $p(x+a, y+b)$

Taking $q(x, y) = 0$ in Lemma 3.10, we get $D_{(a,b),(i,j)}^k(p(x, y)) = k! H_{(a,b)}^k([x^i y^j](p(x+a, y+b))) \cdot x^i y^j$ for all $k \geq 1$. So we can read off $D_{(a,b),(i,j)}^k(p(x, y))$ from Figure 3.1. For instance,

$$D_{(a,b),(0,1)}^2(p(x, y)) = 2! \cdot (a^2 + 3b^2) \cdot y, \quad D_{(a,b),(1,1)}^1(p(x, y)) = 2a \cdot xy \quad \text{and} \quad D_{(a,b),(0,2)}^1(p(x, y)) = 3b \cdot y^2.$$

Taking $k = \ell = 1$ in Lemma 3.12 (2), we obtain a recurrence relation among these three terms:

$$\begin{aligned}
D_{(a,b),(0,1)}^2(p(x, y)) &= \sum_{i+j=1} \binom{1}{(i,j)} a^i b^j \partial_x^i \partial_y^j \left(D_{(a,b),(i,1+j)}^1(p(x, y)) \right) \\
&= \binom{1}{(1,0)} a \partial_x \left(D_{(a,b),(1,1)}^1(p(x, y)) \right) + \binom{1}{(0,1)} b \partial_y \left(D_{(a,b),(0,2)}^1(p(x, y)) \right).
\end{aligned}$$

This implies

$$2(a^2 + 3b^2)y = a\partial_x(2axy) + b\partial_y(3by^2). \quad (3.5)$$

By the definition of $D_{\mathbf{a},\alpha}^k(p)$ and Lemma 3.10, we get

$$2(a^2 + 3b^2)y = a^2\partial_x^2(x^2y) + 2ab\partial_x\partial_y(0 \cdot xy^2) + b^2\partial_y^2(y^3). \quad (3.6)$$

Note that the term x^4 does not involve in the above two equations (3.5) and (3.6) because $y \nmid x^4$. In this example, the term x^4 only affects all terms in the blue branch, such as $3! \cdot 4a^3x = a^3\partial_x^3(x^4)$.

Without introducing the notation $D_{\mathbf{a},\alpha}^k$, by Lemma 3.9 (or Lemma 3.5 in [35]) we only get “global” relations, such as

$$2! \cdot H_{(a,b)}^2 \left(H_{(x,y)}^1(p(x+a, y+b)) \right) = D_{(a,b)}^2 \left(H_{(x,y)}^{1+2}(p(x, y)) \right).$$

This implies two relations among the rows (instead of the points) in the figure:

$$\begin{aligned} 2(2abx + a^2y + 3b^2y) &= (a\partial_x + b\partial_y)^2(x^2y + 0 \cdot xy^2 + y^3) \\ &= (a\partial_x + b\partial_y)(bx^2 + 2axy + 3by^2). \end{aligned}$$

From Observation 3.4 in [35], we know if $D_{\mathbf{a}}^1(f(\mathbf{x})) = D_{\mathbf{b}}^1(f(\mathbf{x}))$, then $D_{\mathbf{a}}^k(f(\mathbf{x})) = D_{\mathbf{b}}^k(f(\mathbf{x}))$ for all $k \geq 1$. Now we show that for any $\mathbf{s} \in \mathbb{F}^n$, $D_{\mathbf{s},\alpha}^k(f(\mathbf{x}))$ can be determined by $D_{\mathbf{s},\beta}^1(f(\mathbf{x}))$ for all $\beta \in \mathbb{N}^n$ with $\beta \geq \alpha$ and $|\beta| = |\alpha| + k - 1$. This is why we introduce the second condition in the definition of an admissible cover.

Lemma 3.14. *Let $\mathbf{r}, \mathbf{s} \in \mathbb{F}^n$, $\alpha \in \mathbb{N}^n$, $k \in \mathbb{N}^+$ and $f(\mathbf{x}) \in \mathbb{F}[\mathbf{x}]$. If $D_{\mathbf{r},\beta}^1(f(\mathbf{x})) = D_{\mathbf{s},\beta}^1(f(\mathbf{x}))$ for all $\beta \in \mathbb{N}^n$ with $\beta \geq \alpha$ and $|\beta| = |\alpha| + k - 1$, then we have $D_{\mathbf{r},\alpha}^k(f(\mathbf{x})) = D_{\mathbf{s},\alpha}^k(f(\mathbf{x}))$.*

Proof. The proof is by induction on k . It is clear to see that the lemma is true for $k = 1$. Now assume the equality holds for k . For $k + 1$, assume that $D_{\mathbf{r},\beta}^1(f(\mathbf{x})) = D_{\mathbf{s},\beta}^1(f(\mathbf{x}))$ for all $\beta \in \mathbb{N}^n$ with $\beta \geq \alpha$ and $|\beta| = |\alpha| + (k + 1) - 1$. We have $D_{\mathbf{r},\alpha}^{k+1}(f(\mathbf{x})) = \sum_{i=1}^n \mathbf{r}^{e_i} \partial^{e_i} (D_{\mathbf{r},\alpha+e_i}^k(f(\mathbf{x})))$ by Lemma 3.12 (2). Note that for all $\gamma \in \mathbb{N}^n$ with $\gamma \geq \alpha + e_i$ and $|\gamma| = |\alpha + e_i| + k - 1$, we have $\gamma \geq \alpha$ and $|\gamma| = |\alpha| + (k + 1) - 1$. Thus by assumption we have $D_{\mathbf{r},\gamma}^1(f(\mathbf{x})) = D_{\mathbf{s},\gamma}^1(f(\mathbf{x}))$. It follows from the inductive hypothesis that $D_{\mathbf{r},\alpha+e_i}^k(f(\mathbf{x})) = D_{\mathbf{s},\alpha+e_i}^k(f(\mathbf{x}))$. So

$$\begin{aligned} D_{\mathbf{r},\alpha}^{k+1}(f(\mathbf{x})) &= \sum_{i=1}^n \mathbf{r}^{e_i} \partial^{e_i} \left(D_{\mathbf{s},\alpha+e_i}^k(f(\mathbf{x})) \right) \\ &= \sum_{i=1}^n \mathbf{r}^{e_i} \partial^{e_i} \left(\sum_{|\gamma|=k} \binom{k}{\gamma} \mathbf{s}^\gamma \partial^\gamma ([\mathbf{x}^{\alpha+e_i+\gamma}](f(\mathbf{x})) \cdot \mathbf{x}^{\alpha+e_i+\gamma}) \right) \\ &= \sum_{|\gamma|=k} \binom{k}{\gamma} \mathbf{s}^\gamma \partial^\gamma \left(\sum_{i=1}^n \mathbf{r}^{e_i} \partial^{e_i} ([\mathbf{x}^{\alpha+\gamma+e_i}](f(\mathbf{x})) \cdot \mathbf{x}^{\alpha+\gamma+e_i}) \right) \\ &= \sum_{|\gamma|=k} \binom{k}{\gamma} \mathbf{s}^\gamma \partial^\gamma D_{\mathbf{r},\alpha+\gamma}^1(f(\mathbf{x})). \end{aligned}$$

Because $\alpha + \gamma \geq \alpha$ and $|\alpha + \gamma| = |\alpha| + |\gamma| = |\alpha| + (k + 1) - 1$, we have $D_{\mathbf{r},\alpha+\gamma}^1(f(\mathbf{x})) = D_{\mathbf{s},\alpha+\gamma}^1(f(\mathbf{x}))$ by assumption. Applying Lemma 3.12 (2) again, the proof is completed. \blacksquare

Combining Lemma 3.10 and the above lemma, we get the following lemma immediately.

Lemma 3.15. *Let $c_\alpha(\mathbf{a}) \in S$, $k \in \mathbb{N}$ with $k \geq 2$ and $\mathbf{r}, \mathbf{s} \in \mathbb{F}^n$. If*

$$H^1(c_\beta)(\mathbf{r}) = H^1(c_\beta)(\mathbf{s}) \quad \text{for all } \beta \in \mathbb{N}^n \text{ with } \beta \geq \alpha \text{ and } |\beta| = |\alpha| + k - 1,$$

then we have

$$H^k(c_\alpha)(\mathbf{r}) = H^k(c_\alpha)(\mathbf{s}).$$

Now we are ready to show that for an admissible cover, the linearization does not change the zero set of the polynomial system S .

Lemma 3.16. *Let $\{S_0, S_1, \dots, S_m\}$ be an admissible cover of S and $\ell \in \{0, 1, \dots, m\}$. If there exist $\mathbf{r}, \mathbf{s} \in \mathbb{V}_{\mathbb{F}}(\cup_{i=0}^{\ell-1} S_i)$, then for all $c_\alpha(\mathbf{a}) \in \cup_{i=0}^{\ell} S_i$ we have*

$$L_{\mathbf{a}=\mathbf{r}}(c_\alpha)(\mathbf{a}) = L_{\mathbf{a}=\mathbf{s}}(c_\alpha)(\mathbf{a}). \quad (3.7)$$

Furthermore, we have $c_\alpha(\mathbf{r}) = L_{\mathbf{a}=\mathbf{s}}(c_\alpha)(\mathbf{r})$.

Proof. Since $c_\alpha(\mathbf{r}) = L_{\mathbf{a}=\mathbf{r}}(c_\alpha)(\mathbf{r})$, it is sufficient to prove Equation (3.7) by induction on ℓ .

For $\ell = 0$, we have $\deg(c_\alpha(\mathbf{a})) \leq 1$ by Definition 3.5, so $L_{\mathbf{a}=\mathbf{r}}(c_\alpha)(\mathbf{a}) = c_\alpha(\mathbf{a}) = L_{\mathbf{a}=\mathbf{s}}(c_\alpha)(\mathbf{a})$.

For $\ell > 0$, suppose the lemma holds for smaller ℓ . Then it is sufficient to show that $H^k(c_\alpha)(\mathbf{r}) = H^k(c_\alpha)(\mathbf{s})$ for all $k \in \mathbb{N}$ with $k \geq 2$. By Lemma 3.15, we know the proof is completed by showing that $H^1(c_\beta)(\mathbf{r}) = H^1(c_\beta)(\mathbf{s})$ for $\beta \in \mathbb{N}^n$ with $\beta \geq \alpha$ and $|\beta| = |\alpha| + k - 1$. Because $|\beta| \geq |\alpha| + 2 - 1 > |\alpha|$, we have $\beta > \alpha$. Then we get either $c_\beta(\mathbf{a}) \in \cup_{i=0}^{\ell-1} S_i$ or $c_\beta(\mathbf{a}) = 0$ by Definition 3.5. For $\ell - k + 1 \leq \ell - 2 + 1 \leq \ell - 1$, we have $\mathbf{r}, \mathbf{s} \in \mathbb{V}_{\mathbb{F}}(\cup_{i=0}^{\ell-1} S_i) \subseteq \mathbb{V}_{\mathbb{F}}(\cup_{i=0}^{\ell-k+1} S_i)$. This means \mathbf{r} and \mathbf{s} are zeros of all polynomials $c_\beta(\mathbf{a}) \in S_{\ell-k+1}$. Then we have

$$L_{\mathbf{a}=\mathbf{r}}(c_\beta)(\mathbf{r}) = c_\beta(\mathbf{r}) = 0 \quad \text{and} \quad L_{\mathbf{a}=\mathbf{s}}(c_\beta)(\mathbf{s}) = 0. \quad (3.8)$$

On the other hand, $\mathbf{r}, \mathbf{s} \in \mathbb{V}_{\mathbb{F}}(\cup_{i=0}^{\ell-k+1} S_i)$ because $\mathbb{V}_{\mathbb{F}}(\cup_{i=0}^{\ell-k+1} S_i) \subseteq \mathbb{V}_{\mathbb{F}}(\cup_{i=0}^{\ell-k+1} S_i)$. By the inductive hypothesis with $\ell - k + 1$, we get $L_{\mathbf{a}=\mathbf{r}}(c_\beta)(\mathbf{a}) = L_{\mathbf{a}=\mathbf{s}}(c_\beta)(\mathbf{a})$. So

$$H^0(L_{\mathbf{a}=\mathbf{r}}(c_\beta)) = H^0(L_{\mathbf{a}=\mathbf{s}}(c_\beta)). \quad (3.9)$$

Note that $H^1(c_\beta)(\mathbf{a}) = H^1(L_{\mathbf{a}=\mathbf{r}}(c_\beta))(\mathbf{a}) = H^1(L_{\mathbf{a}=\mathbf{s}}(c_\beta))(\mathbf{a})$. Combining the equations (3.8) and (3.9), we have

$$\begin{aligned} H^1(c_\beta)(\mathbf{r}) &= H^1(L_{\mathbf{a}=\mathbf{r}}(c_\beta))(\mathbf{r}) = -H^0(L_{\mathbf{a}=\mathbf{r}}(c_\beta)) \\ &= -H^0(L_{\mathbf{a}=\mathbf{s}}(c_\beta)) = H^1(L_{\mathbf{a}=\mathbf{s}}(c_\beta))(\mathbf{s}) = H^1(c_\beta)(\mathbf{s}), \end{aligned}$$

which completes the proof. ■

Proof of Theorem 3.7. We shall prove the theorem by induction on ℓ .

For $\ell = 0$, we know that any $c_\alpha(\mathbf{a})$ in S_0 satisfies $\deg(c_\alpha(\mathbf{a})) \leq 1$ by Definition 3.5. Thus we have $L_{\mathbf{a}=\mathbf{s}^{(0)}}(S_0) = S_0$ and $\mathbb{V}_{\mathbb{F}}(S_0) = \mathbb{V}_{\mathbb{F}}(L_{\mathbf{a}=\mathbf{s}^{(0)}}(S_0))$.

For $\ell > 0$, assume the theorem holds for $\ell - 1$, i.e., $\mathbb{V}_{\mathbb{F}}(\cup_{i=0}^{\ell-1} S_i) = \mathbb{V}_{\mathbb{F}}(\cup_{i=0}^{\ell-1} L_{\mathbf{a}=\mathbf{s}^{(\ell-1)}}(S_i))$. Taking $\mathbf{r}, \mathbf{s} = \mathbf{s}^{(\ell-1)}, \mathbf{s}^{(\ell)} \in \mathbb{V}_{\mathbb{F}}(\cup_{i=0}^{\ell-2} S_i)$ in Lemma 3.16, we know that the linearizations of $c_\beta(\mathbf{a})$ at $\mathbf{s}^{(\ell)}$ and $\mathbf{s}^{(\ell-1)}$ are equal for all $c_\beta(\mathbf{a}) \in \cup_{i=0}^{\ell-1} S_i$. This means $\cup_{i=0}^{\ell-1} L_{\mathbf{a}=\mathbf{s}^{(\ell)}}(S_i) = \cup_{i=0}^{\ell-1} L_{\mathbf{a}=\mathbf{s}^{(\ell-1)}}(S_i)$. Then we have

$$\mathbb{V}_{\mathbb{F}}\left(\cup_{i=0}^{\ell} L_{\mathbf{a}=\mathbf{s}^{(\ell)}}(S_i)\right) \subseteq \mathbb{V}_{\mathbb{F}}\left(\cup_{i=0}^{\ell-1} L_{\mathbf{a}=\mathbf{s}^{(\ell)}}(S_i)\right) = \mathbb{V}_{\mathbb{F}}\left(\cup_{i=0}^{\ell-1} L_{\mathbf{a}=\mathbf{s}^{(\ell-1)}}(S_i)\right) = \mathbb{V}_{\mathbb{F}}\left(\cup_{i=0}^{\ell-1} S_i\right),$$

where the last equality follows from the inductive hypothesis. Note that $\mathbb{V}_{\mathbb{F}}(\cup_{i=0}^{\ell} S_i)$ is also a subset of $\mathbb{V}_{\mathbb{F}}(\cup_{i=0}^{\ell-1} S_i)$. So we only need to prove that for all $\mathbf{r} \in \mathbb{V}_{\mathbb{F}}(\cup_{i=0}^{\ell-1} S_i)$,

$$\mathbf{r} \in \mathbb{V}_{\mathbb{F}}\left(\cup_{i=0}^{\ell} S_i\right) \quad \text{if and only if} \quad \mathbf{r} \in \mathbb{V}_{\mathbb{F}}\left(\cup_{i=0}^{\ell} L_{\mathbf{a}=\mathbf{s}^{(\ell)}}(S_i)\right). \quad (3.10)$$

Because $\mathbf{s}^{(\ell)} \in \mathbb{V}_{\mathbb{F}}(\cup_{i=0}^{\ell-1} S_i)$, we have $c_{\alpha}(\mathbf{r}) = L_{\mathbf{a}=\mathbf{s}^{(\ell)}}(c_{\alpha})(\mathbf{r})$ for all $c_{\alpha}(\mathbf{a}) \in \cup_{i=0}^{\ell} S_i$ by Lemma 3.16. Then the claim (3.10) follows immediately. \blacksquare

The main distinction between the reasoning of our general scheme and the DOS algorithm can be shown in Lemma 3.15. In [35], Lemma 3.5 proves Equation (3.2) which we extend to Lemmas 3.9 and 3.10. Observation 3.4 in [35] is generalized by Lemma 3.14. The subsequent proof for the correctness of the DOS algorithm can be summarized by the lemma below. Then the remaining steps for proving Theorem 3.7 and the correctness of the DOS algorithm are similar.

Lemma 3.17. *Let $d := \max\{\deg_{\mathbf{x}}(p(\mathbf{x})), \deg_{\mathbf{x}}(q(\mathbf{x}))\}$, $i \in \{0, 1, \dots, d\}$, $k \in \mathbb{N}$ with $k \geq 2$ and $\mathbf{r}, \mathbf{s} \in \mathbb{F}^n$. If*

$$H^1(c_{\beta})(\mathbf{r}) = H^1(c_{\beta})(\mathbf{s}) \quad \text{for all } c_{\beta}(\mathbf{a}) \in S_{i-k+1}^H,$$

then we have

$$H^k(c_{\alpha})(\mathbf{r}) = H^k(c_{\alpha})(\mathbf{s}) \quad \text{for all } c_{\alpha}(\mathbf{a}) \in S_i^H.$$

Given two points \mathbf{r} and \mathbf{s} in \mathbb{F}^n , we use diagrams in Figures 3.2 and 3.3 to explain how Lemma 3.15 makes the statement more precise than this lemma for the case where $d = \deg_{\mathbf{x}}(p) = \deg_{\mathbf{x}}(q) = 4$ and $n = 2$. Let $p(x+a, y+b) - q(x, y) = \sum_{(\alpha, \beta) \in \Lambda} c_{(\alpha, \beta)}(a, b)x^{\alpha}y^{\beta}$. For the k -th homogeneous component of $c_{(\alpha, \beta)}$, if its values at \mathbf{r} and \mathbf{s} are equal, we draw a point at position (α, β, k) in the space. Furthermore, if $H^k(c_{(\alpha, \beta)})(\mathbf{r}) = H^k(c_{(\alpha, \beta)})(\mathbf{s})$ for all $(\alpha, \beta) \in \mathbb{N}^2$ such that the sum of α and β is a fixed constant i , which means there are points (α, β, k) on the same line, then we draw a segment to connect them to each other. Note that the degree of any polynomial in S_{d-i}^H is no more than i , which will be proved exactly in Lemma 3.19. Lemma 3.17 implies that the dark green segment on one triangle face can conclude all the segments on this face with $k \geq 2$ in Figure 3.2. More precisely, Lemma 3.15 tells us that in Figure 3.3, on one triangle face, every point with $k \geq 2$ can be deduced from the part of dark green segment which is cut out by two dotted line from this point. For example, Point A can be inferred from Segment ℓ .

3.3 Two special admissible covers

By Theorem 3.7, we see that any admissible cover of the polynomial system S corresponds to an algorithm for solving the SET problem via linear system solving. We now present two special admissible covers. The first admissible cover defined below classifies the polynomials in the set S according to their degrees in \mathbf{a} .

Theorem 3.18 (a-degree cover). *Let d' be the degree of $p(\mathbf{x} + \mathbf{a}) - q(\mathbf{x})$ with respect to \mathbf{a} . Let $S_i^D := \{c_{\alpha}(\mathbf{a}) \in S \mid \deg(c_{\alpha}(\mathbf{a})) = i\}$. Then the cover $\{S_0^D, S_1^D, \dots, S_{d'}^D\}$ of S is admissible.*

Proof. We will check that this cover satisfies the two conditions mentioned in Definition 3.5. The condition (1) can be checked directly by definition. As for (2), assume that $c_{\alpha}(\mathbf{a}) \in S_{\ell}^D$ and β is an arbitrary vector in \mathbb{N}^n with $\beta > \alpha$ and $\mathbf{x}^{\beta} \in \text{Supp}_{\mathbf{x}}(p(\mathbf{x} + \mathbf{a}) - q(\mathbf{x}))$. We will argue by contradiction that $\deg(c_{\alpha}(\mathbf{a})) > \deg(c_{\beta}(\mathbf{a}))$. By assumption, we know that $\deg(c_{\alpha}(\mathbf{a})) = \ell$. If there is a monomial $\mathbf{a}^{\gamma} \in \text{Supp}(c_{\beta}(\mathbf{a}))$ with $|\gamma| \geq \ell$, then by Equation (3.4), we have $\mathbf{x}^{\gamma+\beta} \in \text{Supp}(p(\mathbf{x}))$.

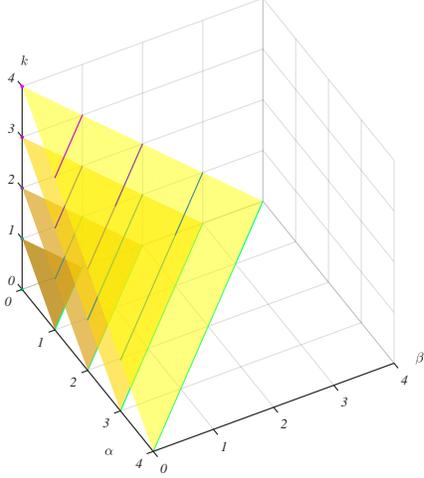


Figure 3.2: Graph for Lemma 3.17

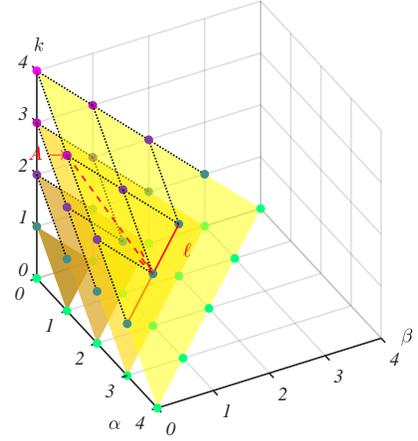


Figure 3.3: Graph for Lemma 3.15

Since $|\gamma + \beta - \alpha| \geq |\beta| - |\alpha| > 0$, we obtain $\mathbf{a}^{\gamma+\beta-\alpha} \in \text{Supp}(H^{|\gamma+\beta-\alpha|}(c_\alpha(\mathbf{a}))) \in \text{Supp}(c_\alpha(\mathbf{a}))$ by Equation (3.4). However, note that $\beta > \alpha$ implies $|\beta| > |\alpha|$, so $|\gamma + \beta - \alpha| = |\gamma| + |\beta| - |\alpha| > \ell$, which leads to a contradiction to the fact that ℓ is the degree of $c_\alpha(\mathbf{a})$. ■

Note that $\beta > \alpha$ implies $|\beta| > |\alpha|$. This inspires the second admissible cover called the *x-homogeneous cover*. Before we prove it, we first present a useful lemma.

Lemma 3.19. *Let $d := \deg_{\mathbf{x}}(p(\mathbf{x}))$. For any $\alpha \in \mathbb{N}^n$ with $c_\alpha(\mathbf{a}) \in S$, we have $\deg_{\mathbf{a}}(c_\alpha(\mathbf{a})) \leq d - |\alpha|$.*

Proof. Note that $[\mathbf{x}^{\alpha+\beta}](p(\mathbf{x})) \neq 0$ yields $|\alpha + \beta| \leq d$, so $k = |\beta| \leq d - |\alpha|$ in Equation (3.4). That is to say, $H^k(c_\alpha(\mathbf{a})) \cdot \mathbf{x}^\alpha = 0$ if $k > d - |\alpha|$. Then the conclusion follows. ■

Theorem 3.20 (x-homogeneous cover). *Let d be the maximal degree of $p(\mathbf{x})$ and $q(\mathbf{x})$. Let $S_i^H := \{c_\alpha(\mathbf{a}) \in S \mid |\alpha| = d - i\}$ for $i = 0, 1, \dots, d$. Then the cover $\{S_0^H, S_1^H, \dots, S_d^H\}$ of S is admissible.*

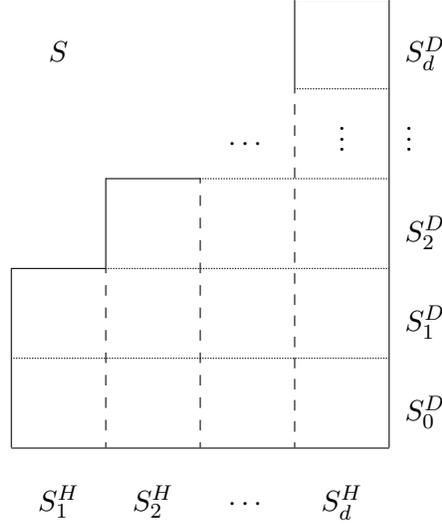
Proof. We first show that $\{S_0^H, S_1^H, \dots, S_d^H\}$ is exactly a cover of S . It is sufficient to show that S is a subset of $\cup_{i=0}^d S_i^H$. This is true because we have $\mathbf{x}^\alpha \in \text{Supp}_{\mathbf{x}}(p(\mathbf{x} + \mathbf{a}) - q(\mathbf{x})) \subseteq \text{Supp}(p(\mathbf{x})) \cup \text{Supp}(q(\mathbf{x}))$ for arbitrary $c_\alpha(\mathbf{a}) \in S$ and then $0 \leq |\alpha| \leq d$.

Then we check this cover is admissible. If there exists $c_\alpha(\mathbf{a}) \in S_0^H$ with a nonlinear monomial \mathbf{a}^β , then we have $\mathbf{x}^{\alpha+\beta} \in \text{Supp}(p(\mathbf{x}))$ by Equation (3.4). For $|\beta| \geq 2$, there is a unit vector $\mathbf{e}_j \in \mathbb{N}^n$ such that $\mathbf{e}_j < \alpha + \beta$. Thus $\mathbf{a}^{\mathbf{e}_j} \mathbf{x}^{\alpha+\beta-\mathbf{e}_j} \in \text{Supp}_{\mathbf{x} \cup \mathbf{a}}(D_{\mathbf{a}, \alpha+\beta-\mathbf{e}_j}^1(p(\mathbf{x})))$. By Lemma 3.10, $\mathbf{x}^{\alpha+\beta-\mathbf{e}_j} \in \text{Supp}_{\mathbf{x}}(p(\mathbf{x} + \mathbf{a}) - q(\mathbf{x}))$. However, $|\alpha + \beta - \mathbf{e}_j| = |\alpha| + |\beta| - |\mathbf{e}_j| \geq d + 2 - 1 > d$, which leads to a contradiction to the fact that $\deg_{\mathbf{x}}(p(\mathbf{x} + \mathbf{a}) - q(\mathbf{x})) \leq \max\{\deg_{\mathbf{x}}(p(\mathbf{x})), \deg_{\mathbf{x}}(q(\mathbf{x}))\} = d$. So the condition (1) in Definition 3.5 is satisfied. Finally, for any $\ell = 1, 2, \dots, d$, let $c_\alpha(\mathbf{a}) \in S_\ell^H$. Then for all $\beta \in \mathbb{N}^n$ with $\beta > \alpha$ and $\mathbf{x}^\beta \in \text{Supp}_{\mathbf{x}}(p(\mathbf{x} + \mathbf{a}) - q(\mathbf{x}))$, we have $|\beta| > |\alpha| = d - \ell$, so $c_\beta(\mathbf{a}) \in S_{d-|\beta|}^H \subseteq \cup_{i=0}^{\ell-1} S_i^H$. Therefore, the cover $\{S_0^H, S_1^H, \dots, S_d^H\}$ also satisfies the condition (2) in Definition 3.5 and the proof is complete. ■

The cover $\{S_0^H, S_1^H, \dots, S_d^H\}$ is exactly the one defined in the DOS algorithm and we call it *x-homogeneous cover*. As a consequence, we reproved the correctness of the DOS algorithm in our general framework of admissible covers.

After introducing two special admissible covers, we would like to compare them and explain their connection. For simplicity, we always assume $H_{\mathbf{x}}^d(p(\mathbf{x} + \mathbf{a})) = H_{\mathbf{x}}^d(q(\mathbf{x}))$ with $\deg(p(\mathbf{x})) =$

$\deg(q(\mathbf{x})) = d$ in the following discussion. For this special case, $S_0^H = \emptyset$. We get $\deg_{\mathbf{a}}(p(\mathbf{x} + \mathbf{a}) - q(\mathbf{x})) = d$ by Equation (3.4). Lemma 3.19 yields $S_\ell^H \subseteq \cup_{i=0}^\ell S_i^D$. Hence the connection between two different covers is like the following figure.



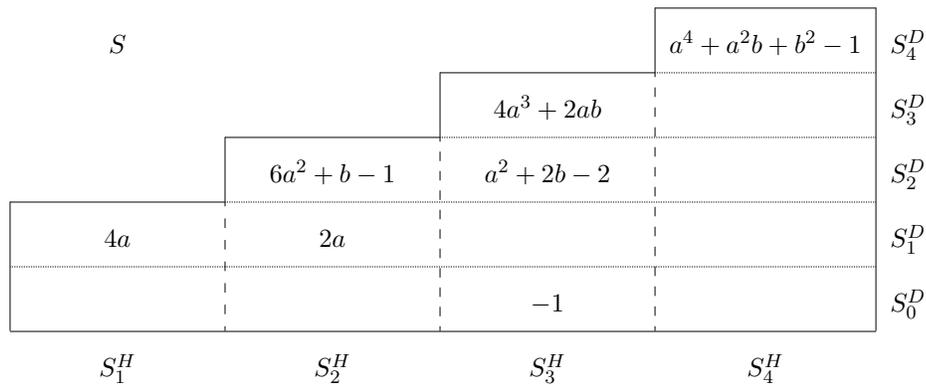
Now we give some examples to illustrate the two algorithms induced by the two elaborated admissible covers.

Example 3.21. Let $\mathbb{F} = \mathbb{Q}$, $p_i(x, y, z), q_i(x, y, z) \in \mathbb{Q}[x, y, z]$ for $i = 1, 2$ with $p_1(x, y, z) = x^4 + x^2y + y^2$, $q_1(x, y, z) = p_1(x, y + 1, z + 2) + z$, $p_2(x, y, z) = x^4 + x^3y + xy^2 + z^2$ and $q_2(x, y, z) = p_2(x, y + 1, z + 2) + xy$.

(1) Compute F_{p_1, q_1} . We expand $p_1(x + a, y + b, z + c) - q_1(x, y, z)$ and get that

$$\begin{aligned} & p_1(x + a, y + b, z + c) - q_1(x, y, z) \\ &= (4a \cdot x^3) + ((6a^2 + b - 1) \cdot x^2 + 2a \cdot xy) \\ & \quad + ((4a^3 + 2ab) \cdot x + (a^2 + 2b - 2) \cdot y - z) + (a^4 + a^2b + b^2 - 1). \end{aligned}$$

Then we can separate the coefficients of $p_1(x + a, y + b, z + c) - q_1(x, y, z)$ with respect to x , y and z in two different methods as following.



So we can get $F_{p_1, q_1} = \emptyset$ at once if we use the \mathbf{a} -degree cover, while by the \mathbf{x} -homogeneous cover, we will calculate until we get S_3^H .

(2) Compute F_{p_2, q_2} . We expand $p_2(x + a, y + b, z + c) - q_2(x, y, z)$ and get that

$$\begin{aligned} & p_2(x + a, y + b, z + c) - q_2(x, y, z) \\ &= ((4a + b - 1) \cdot x^3 + 3a \cdot x^2y) \\ & \quad + ((6a^2 + 3ab) \cdot x^2 + (3a^2 + 2b - 3) \cdot xy + a \cdot y^2) \\ & \quad + ((4a^3 + 3a^2b + b^2 - 1) \cdot x + (a^3 + 2ab) \cdot y + (2c - 4) \cdot z) \\ & \quad + (a^4 + a^3b + ab^2 + c^2 - 4). \end{aligned}$$

Then we can separate the coefficients of $p_2(x + a, y + b, z + c) - q_2(x, y, z)$ with respect to x , y and z in two different methods as following.

			$a^4 + a^3b + ab^2 + c^2 - 4$	S_4^D
		$4a^3 + 3a^2b + b^2 - 1$		S_3^D
	$6a^2 + 3ab$	$a^3 + 2ab$		S_2^D
	$3a^2 + 2b - 3$			
$4a + b - 1$	a	$2c - 4$		S_1^D
$3a$				
				S_0^D
S_1^H	S_2^H	S_3^H	S_4^H	

So we can get $F_{p_2, q_2} = \emptyset$ if we use \mathbf{x} -homogeneous cover and calculate S_2^H , while by \mathbf{a} -degree cover, we have to solve $2c - 4 = 0$ needlessly.

4 Isotropy groups and orbital decompositions

In this section, we first recall the notion of isotropy groups under shifts, which plays a central role in the summability criteria and existence criteria of telescopers. Then we present different types of partial fraction decompositions of $\mathbb{F}(\mathbf{x})$ with respect to different orbital factorizations as in [19]. These decompositions can be computed via algorithms for the SET problem over integers and will be used in the next sections for reducing the rational summability problem and the existence problem of telescopers to simpler cases.

4.1 Isotropy groups

Let $G = \langle \sigma_{x_1}, \dots, \sigma_{x_n} \rangle$ be the free abelian group generated by shift operators $\sigma_{x_1}, \dots, \sigma_{x_n}$ and A be a subgroup of G . Let p be a multivariate polynomial in $\mathbb{F}[\mathbf{x}]$. The set

$$[p]_A := \{\sigma(p) \mid \sigma \in A\}$$

is called the A -orbit of p . Two polynomials $p, q \in \mathbb{F}[\mathbf{x}]$ are said to be A -shift equivalent or A -equivalent if $[p]_A = [q]_A$, denoted by $p \sim_A q$. The relation \sim_A is an equivalence relation.

Definition 4.1 (Sato's Isotropy Group [70]). *Let A and p be defined as above. The set*

$$A_p := \{\sigma \in A \mid \sigma(p) = p\}.$$

is a subgroup of A , called the isotropy group of p in A .

If two polynomials p, q in $\mathbb{F}[\mathbf{x}]$ are A -shift equivalent, then $A_p = A_q$. The following remark says that we can test the A -equivalence of polynomials and compute a basis of A_p by algorithms for the SET problem over integers in Section 3.

Remark 4.2. (1) *Two polynomials $p, q \in \mathbb{F}[\mathbf{x}]$ are G -equivalent if and only if there exists a $\sigma \in G$ such that $\sigma(p) = q$. Therefore, the G -equivalence relation of p, q can be obtained via the computation of $Z_{p,q}$ in Section 3. When $p = q$, the group G_p is isomorphic to $Z_{p,p}$. Both of them are free abelian groups and a basis of G_p can be given by a basis of $Z_{p,p}$.*

(2) *When $A = \langle \sigma_{x_1}, \dots, \sigma_{x_r} \rangle$ with $1 \leq r \leq n$, we can view p, q as polynomials in x_1, \dots, x_r and the other variables are parameters. Then the A -equivalence relation of p, q and a basis of the isotropy group A_p are also available by algorithms in Section 3.*

(3) *In general, let $A = \langle \tau_1, \dots, \tau_r \rangle$, where $\{\tau_1, \dots, \tau_r\}$ ($1 \leq r \leq n$) are \mathbb{Z} -linearly independent. We will utilize Proposition 5.12 below to construct a difference isomorphism between $(\mathbb{F}(\mathbf{x}), \tau_i)$ and $(\mathbb{F}(\mathbf{x}), \sigma_{x_i})$ such that $\phi \circ \tau_i = \sigma_{x_i} \circ \phi$ for $1 \leq i \leq r$. Let $B = \langle \sigma_{x_1}, \dots, \sigma_{x_r} \rangle$. Then p and q are A -equivalent if and only if $\phi(p)$ and $\phi(q)$ are B -equivalent. Furthermore, we have $\tau_1^{a_1} \cdots \tau_r^{a_r} \in A_p$ if and only if $\sigma_{x_1}^{a_1} \cdots \sigma_{x_r}^{a_r} \in B_{\phi(p)}$ for any $a_1, \dots, a_r \in \mathbb{Z}$.*

A structure property of the quotient group G/G_p is given by Sato [70, Lemma A-3] as follows.

Lemma 4.3. G/G_p is a free abelian group.

If $p \in \mathbb{F}[\mathbf{x}] \setminus \mathbb{F}$ is a non-constant polynomial, then G_p is a proper subgroup of G . By Lemma 4.3, we have $\text{rank}(G_p) < \text{rank}(G)$, where $\text{rank}(G)$ denotes the rank of the free abelian group G . This property about the rank of isotropy groups plays a key role in the reduction method of solving rational summability problem and the existence problem of telescopers.

If $n > 1$, let $H = \langle \sigma_{x_1}, \dots, \sigma_{x_{n-1}} \rangle$ be the subgroup of G generated by $\sigma_{x_1}, \dots, \sigma_{x_{n-1}}$. The isotropy group of p in H is $H_p = \{\tau \in H \mid \tau(p) = p\}$. By Lemma 4.3, both G/G_p and H/H_p are free abelian groups. So the ranks of G_p and H_p are strictly less than those of G and H respectively if p has positive degree in x_1 .

Lemma 4.4. G_p/H_p is a free abelian group of $\text{rank}(G_p/H_p) \leq 1$.

Proof. Define a group homomorphism $\varphi : G_p/H_p \rightarrow \mathbb{Z}$ by

$$\sigma_{x_1}^{k_1} \cdots \sigma_{x_n}^{k_n} H_p \mapsto k_n.$$

It can be verified that φ is well-defined. For any $\tau_1, \tau_2 \in G_p$, if they are in the same coset of H_p in G_p , then $\tau_1 \tau_2^{-1} \in H_p$. This implies $\tau_1 \tau_2^{-1} \in H$ and hence $\varphi(\tau_1 H_p) = \varphi(\tau_2 H_p)$. Moreover, the converse is true since $G_p \cap H = H_p$. So φ is injective. Then we have $G_p/H_p \cong \text{im } \varphi = k\mathbb{Z}$ for some integer $k \in \mathbb{Z}$. So G_p/H_p is a free abelian group generated by $\varphi^{-1}(k)$. ■

Remark 4.5. Let p be a polynomial in $\mathbb{F}[\mathbf{x}]$. By Remark 4.2(1), one can compute a basis $\{\tau_1, \tau_2, \dots, \tau_r\}$ of G_p . If $\tau_i \in H$ for all $i = 1, \dots, r$, then $G_p = H_p$ and $G_p/H_p = \{\mathbf{1}\}$. So $\text{rank}(G_p/H_p) = 0$ and $\{\tau_1, \tau_2, \dots, \tau_r\}$ is a basis of H_p . If $\tau_\ell \notin H$ for some $\ell \in \{1, \dots, r\}$, then $\text{rank}(G_p/H_p) = 1$. Write $\tau_i = \sigma_{x_1}^{b_{i,1}} \cdots \sigma_{x_n}^{b_{i,n}}$ with $b_{i,j} \in \mathbb{Z}$ for each $i = 1, \dots, r$. Let $B = (b_{i,j}) \in \mathbb{Z}^{r \times n}$. Since $\tau_\ell \notin H$, we have $b_{\ell,n} \neq 0$. Using unimodular row reduction, one can compute a unimodular matrix $U \in \mathbb{Z}^{r \times r}$ such that $C = UB$, where $C = (c_{i,j}) \in \mathbb{Z}^{r \times n}$ satisfies $c_{1,n} = \gcd(b_{1,n}, b_{2,n}, \dots, b_{r,n}) \neq 0$ and $c_{i,n} = 0$ for all $i = 2, \dots, r$. Let $\sigma_i = \sigma_{x_1}^{c_{i,1}} \cdots \sigma_{x_n}^{c_{i,n}}$ for each $i = 1, \dots, r$. Then $\{\sigma_1, \dots, \sigma_r\}$ is another basis of G_p because U is an invertible matrix over \mathbb{Z} . Moreover, $G_p/H_p = \langle \bar{\sigma}_1 \rangle$ and $\{\sigma_2, \dots, \sigma_r\}$ is a basis of H_p .

Example 4.6. Consider polynomials in $\mathbb{Q}[x, y, z]$. Let $G = \langle \sigma_x, \sigma_y, \sigma_z \rangle$ and $H = \langle \sigma_x, \sigma_y \rangle$.

- (1) For $p = x^2 + 2xy + z^2$, we have $G_p = H_p = \{\mathbf{1}\}$.
- (2) For $p = (x - 3y)^2(y + z) + 1$, we have $G_p = \langle \tau \rangle$ and $H_p = \{\mathbf{1}\}$, where $\tau = \sigma_x^3 \sigma_y \sigma_z^{-1}$. So $G_p/H_p = \langle \bar{\tau} \rangle$, where $\bar{\tau} = \tau H_p$ denotes the coset in G_p/H_p represented by $\tau \in G_p$.
- (3) Let $p = x + 2y + z$, we have $G_p = \langle \tau_1, \tau_2 \rangle$ and $H_p = \langle \tau_2 \rangle$, where $\tau_1 = \sigma_x \sigma_y^{-1} \sigma_z$ and $\tau_2 = \sigma_x^2 \sigma_y^{-1}$. So $G_p/H_p = \langle \bar{\tau}_1 \rangle$.

4.2 Orbital decompositions

A polynomial $p \in \mathbb{F}[\mathbf{x}]$ is said to be *monic* if the leading coefficient of p is 1 under a fixed monomial order. Let $\hat{\mathbf{x}}_1$ denote the $m - 1$ variables x_2, \dots, x_m . For any subgroup A of $G = \langle \sigma_{x_1}, \dots, \sigma_{x_n} \rangle$ and any polynomial Q in $\mathbb{F}(\hat{\mathbf{x}}_1)[x_1]$, one can classify all of the monic irreducible factors in x_1 of Q into distinct A -orbits which leads to a factorization

$$Q = c \cdot \prod_{i=1}^I \prod_{j=1}^{J_i} \tau_{i,j}(d_i)^{e_{i,j}},$$

where $c \in \mathbb{F}(\hat{\mathbf{x}}_1)$, $I, J_i, e_{i,j} \in \mathbb{N}$, $\tau_{i,j} \in A$, $d_i \in \mathbb{F}[\mathbf{x}]$ being monic irreducible polynomials in distinct A -orbits, and for each i , $\tau_{i,j}(d_i) \neq \tau_{i,j'}(d_i)$ if $1 \leq j \neq j' \leq J_i$. With respect to this fixed representation, we have the unique irreducible partial fraction decomposition for a rational function $f = P/Q \in \mathbb{F}(\mathbf{x})$ of the form

$$f = p + \sum_{i=1}^I \sum_{j=1}^{J_i} \sum_{\ell=1}^{e_{i,j}} \frac{a_{i,j,\ell}}{\tau_{i,j}(d_i)^\ell}, \quad (4.1)$$

where $p, a_{i,j,\ell} \in \mathbb{F}(\hat{\mathbf{x}}_1)[x_1]$ with $\deg_{x_1}(a_{i,j,\ell}) < \deg_{x_1}(d_i)$ for all i, j, ℓ . Note that the representation in (4.1) depends on the choice of representatives d_i in distinct A -orbits. However, the sum $\sum_{j=1}^{J_i} \frac{a_{i,j,\ell}}{\tau_{i,j}(d_i)^\ell}$ only depends on the multiplicity ℓ and the orbit $[d_i]_A$ instead of its representative d_i . Based on this fact, we shall formulate a unique decomposition of a rational function with respect to the group A . In this sense, we can decompose $\mathbb{F}(\mathbf{x})$ as a vector space over $\mathbb{E} = \mathbb{F}(\hat{\mathbf{x}}_1)$.

Given an irreducible polynomial $d \in \mathbb{F}[\mathbf{x}]$ with $\deg_{x_1}(d) > 0$ and $j \in \mathbb{N}^+$, we define a subspace of $\mathbb{F}(\mathbf{x})$

$$V_{[d]_A, j} = \text{Span}_{\mathbb{E}} \left\{ \frac{a}{\tau(d)^j} \mid a \in \mathbb{E}[x_1], \tau \in A, \deg_{x_1}(a) < \deg_{x_1}(d) \right\}. \quad (4.2)$$

For any fraction in $V_{[d]_A, j}$, the irreducible factors of its denominator are in the same A -orbit as d . Let $V_0 = \mathbb{E}[x_1]$ denote the set of all polynomials in x_1 . By the irreducible partial fraction

decomposition, any rational function $f \in \mathbb{F}(\mathbf{x})$ can be uniquely written in the form

$$f = f_0 + \sum_j \sum_{[d]_A} f_{[d]_A, j}, \quad (4.3)$$

where $f_0 \in V_0$ and $f_{[d]_A, j}$ are in distinct $V_{[d]_A, j}$ spaces. Let T_A be the set of all distinct A -orbits of monic irreducible polynomials in $\mathbb{F}[\mathbf{x}]$, whose degrees with respect to x_1 are positive. Then $\mathbb{F}(\mathbf{x})$ has the following direct sum decomposition

$$\mathbb{F}(\mathbf{x}) = V_0 \oplus \left(\bigoplus_{j \in \mathbb{N}^+} \bigoplus_{[d]_A \in T_A} V_{[d]_A, j} \right). \quad (4.4)$$

Definition 4.7. *The decomposition (4.4) of $\mathbb{F}(\mathbf{x})$ is called the orbital decomposition of $\mathbb{F}(\mathbf{x})$ with respect to the variable x_1 and the group A . If f is written in the form (4.3), we call f_0 and $f_{[d]_A, j}$ orbital components of f with respect to x_1 and A .*

A key feature of subspaces $V_{[d]_A, j}$ is the A -invariant property. In the field of univariate rational functions, the orbital decomposition of $\mathbb{F}(x_1)$ with respect to the group $A = \langle \sigma_{x_1} \rangle$ was first given in [51] by Karr.

Lemma 4.8. *If $f \in V_{[d]_A, j}$ and $P \in \mathbb{F}(\hat{\mathbf{x}}_1)[A]$, then $P(f) \in V_{[d]_A, j}$.*

Proof. Let $f = \sum a_i / \tau_i(d)^j$ and $P = \sum p_\sigma \sigma$ with $p_\sigma \in \mathbb{F}(\hat{\mathbf{x}}_1)$ and $\sigma \in A$. For any $\sigma \in A$, we have that $\sigma \tau_i$ is still in A , because A is a group. Since the shift operators do not change the degree and multiplicity of a polynomial, we have $\deg_{x_1}(\sigma(a_i)) < \deg_{x_1}(d)$ and then $\frac{p_\sigma \sigma(a_i)}{\sigma(\tau_i(d))^j}$ is in $V_{[d]_A, j}$. So $P(f) \in V_{[d]_A, j}$ by the linearity of the space. \blacksquare

Example 4.9. *Let $\mathbb{F} = \mathbb{Q}$, $\mathbb{E} = \mathbb{Q}(y, z)$ and $G = \langle \sigma_x, \sigma_y, \sigma_z \rangle$. Consider the rational function f_1 in $\mathbb{Q}(x, y, z)$ of the form*

$$f_1 = \underbrace{\frac{x - z^2}{x^2 + 2xy + z^2}}_{d_{1,1}} + \underbrace{\frac{x - y - 2z}{x^2 + 2xy + 2x + z^2}}_{d_{1,2}} + \underbrace{\frac{y + z^2}{x^2 + 2xy + 8x + 2y + z^2 - 2z + 8}}_{d_{1,3}}.$$

If $A = \langle \sigma_x \rangle$, then the orbital partial fraction decomposition of f_1 is

$$f_1 = f_{1,1} + f_{1,2} + f_{1,3} \text{ with } f_{1,1} = \frac{x - z^2}{d_{1,1}}, f_{1,2} = \frac{x - y - 2z}{d_{1,2}} \text{ and } f_{1,3} = \frac{y + z^2}{d_{1,3}},$$

where $f_{1,i} \in V_{[d_{1,i}]_A, 1}$ for $i = 1, 2, 3$ and $d_1, d_{1,2}, d_{1,3}$ are in distinct $\langle \sigma_x \rangle$ -orbits. If $A = \langle \sigma_x, \sigma_y \rangle$, then the orbital partial fraction decomposition of f_1 is

$$f_1 = f_{1,1} + f_{1,2} \text{ with } f_{1,1} = \frac{x - z^2}{d_1} + \frac{x - y - 2z}{\sigma_y(d_1)} \text{ and } f_{1,2} = \frac{y + z^2}{d_{1,3}},$$

where $f_{1,1} \in V_{[d_1]_A, 1}$, $f_{1,2} \in V_{[d_{1,3}]_A, 1}$ and $d_1, d_{1,3}$ are in distinct $\langle \sigma_x, \sigma_y \rangle$ -orbits. If $A = \langle \sigma_x, \sigma_y, \sigma_z \rangle$, then $f_1 \in V_{[d_1]_A, 1}$ is one component in the orbital decomposition because

$$f_1 = \frac{x - z^2}{d_1} + \frac{x - y - 2z}{\sigma_y(d_1)} + \frac{y + z^2}{\sigma_x \sigma_y^3 \sigma_z^{-1}(d_1)}.$$

Example 4.10. Let $\mathbb{F} = \mathbb{Q}$ and $G = \langle \sigma_x, \sigma_y, \sigma_z \rangle$. Consider the rational function $f = f_1 + f_2 + f_3$ in $\mathbb{Q}(x, y, z)$ with f_1 given in Example 4.9,

$$f_2 = \frac{x+z}{\underbrace{(x-3y)^2(y+z)+1}_{d_2}} \quad \text{and} \quad f_3 = \left(y + \frac{z}{y^2+z-1} - \frac{1}{y^2+z} \right) \frac{1}{\underbrace{(x+2y+z)^2}_{d_3}}.$$

If $A = G$, then the orbital partial fraction decomposition of f is

$$f = f_1 + f_2 + f_3 \quad \text{with} \quad f_i \in V_{[d_i]_G, 1} \text{ for } i = 1, 2 \text{ and } f_3 \in V_{[d_3]_G, 2},$$

where d_1, d_2, d_3 are in distinct $\langle \sigma_x, \sigma_y, \sigma_z \rangle$ -orbits.

5 The rational summability problem

In this section, we solve the rational summability problem for multivariate rational functions and design an algorithm for rational summability testing. In Section 5.1 we use a special orbital decomposition in Section 4.2 to reduce the summability problem of a general rational function to its orbital components and then further to simple fractions by Abramov's reduction. In Section 5.2, we use the structure of isotropy groups to reduce the number of variables in the summability problem inductively.

5.1 Orbital reduction for summability

Let f be a rational function in $\mathbb{F}(\mathbf{x})$, where $\mathbf{x} = \{x_1, \dots, x_m\}$. Recall that $\hat{\mathbf{x}}_1 = \{x_2, \dots, x_m\}$. Let n be an integer such that $1 \leq n \leq m$. We consider the $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -summability problem of f in $\mathbb{F}(\mathbf{x})$. Let $G = \langle \sigma_{x_1}, \dots, \sigma_{x_n} \rangle$. Taking $\mathbb{E} = \mathbb{F}(\hat{\mathbf{x}}_1)$ and $A = G$ in equality (4.2), we get the subspace $V_{[d]_G, j}$ of $\mathbb{F}(\mathbf{x})$

$$V_{[d]_G, j} = \text{Span}_{\mathbb{E}} \left\{ \frac{a}{\tau(d)^j} \mid a \in \mathbb{E}[x_1], \tau \in G, \deg_{x_1}(a) < \deg_{x_1}(d) \right\}.$$

where $j \in \mathbb{N}^+$ and $d \in \mathbb{E}[\mathbf{x}]$ is irreducible with $\deg_{x_1}(d) > 0$. According to Equation (4.3), f can be decomposed into the form

$$f = f_0 + \sum_j \sum_{[d]_G} f_{[d]_G, j}, \quad (5.1)$$

where $f_0 \in V_0 = \mathbb{E}[x_1]$ and $f_{[d]_G, j}$ are in distinct $V_{[d]_G, j}$ spaces. Then we obtain the orbital decomposition (4.4) of $\mathbb{F}(\mathbf{x})$ with respect to the group $A = G$.

Lemma 5.1. *Let $f \in \mathbb{F}(\mathbf{x})$. Then f is $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -summable if and only if f_0 and each $f_{[d]_G, j}$ are $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -summable for all $[d]_G \in T_G$ and $j \in \mathbb{N}^+$.*

Proof. The sufficiency is due to the linearity of difference operators Δ_{x_i} . For the necessity, suppose $f = \sum_{i=1}^n \Delta_{x_i}(g^{(i)})$ with $g^{(i)} \in \mathbb{F}(\mathbf{x})$. By the orbital decomposition of rational functions (5.1), we can write $f, g^{(i)}$ in the form

$$f = f_0 + \sum_j \sum_{[d]_G} f_{[d]_G, j} \quad \text{and} \quad g^{(i)} = g_0^{(i)} + \sum_j \sum_{[d]_G} g_{[d]_G, j}^{(i)} \quad \text{for } 1 \leq i \leq n.$$

By the linearity of Δ_{x_i} , we see

$$f = \sum_{i=1}^n \Delta_{x_i} \left(g_0^{(i)} \right) + \sum_j \sum_{[d]_G} \left(\sum_{i=1}^n \Delta_{x_i} \left(g_{[d]_G, j}^{(i)} \right) \right).$$

By Lemma 4.8, it is another expression of f with respect to $V_{[d]_G, j}$. Such a decomposition is unique, so $f_0 = \sum_{i=1}^n \Delta_{x_i} (g_0^{(i)})$ and $f_{[d]_G, j} = \sum_{i=1}^n \Delta_{x_i} (g_{[d]_G, j}^{(i)})$, which are $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -summable. \blacksquare

Using Lemma 5.1, we can reduce the summability problem of a rational function to its orbital components. Note that polynomials in x_1 are always (σ_{x_1}) -summable. Thus Problem 2.4 can be reduced to that for rational functions in $V_{[d]_G, j}$, which are of the form

$$f = \sum_{\tau} \frac{a_{\tau}}{\tau(d)^j}, \quad (5.2)$$

where $\tau \in G$, $a_{\tau} \in \mathbb{F}(\hat{\mathbf{x}}_1)[x_1]$, $d \in \mathbb{F}[\mathbf{x}]$ with $\deg_{x_1}(a_{\tau}) < \deg_{x_1}(d)$ and d is irreducible in x_1 over $\mathbb{F}(\hat{\mathbf{x}}_1)$.

Let σ be an automorphism on $\mathbb{F}(\mathbf{x})$ and $a, b \in \mathbb{F}(\mathbf{x})$. Then for any integer $k \in \mathbb{Z}$, we have the reduction formula

$$\frac{a}{\sigma^k(b)} = \sigma(h) - h + \frac{\sigma^{-k}(a)}{b}, \quad (5.3)$$

where $h = 0$ if $k = 0$, $h = \sum_{i=0}^{k-1} \frac{\sigma^{i-k}(a)}{\sigma^i(b)}$ if $k > 0$ and $h = -\sum_{i=0}^{-k-1} \frac{\sigma^i(a)}{\sigma^{i+k}(b)}$ if $k < 0$. For any $\tau = \sigma_{x_1}^{k_1} \cdots \sigma_{x_n}^{k_n} \in G$, applying the reduction formula (5.3) with $\sigma = \sigma_{x_i}$ for $i = 1, \dots, n$, we get

$$\frac{a}{\sigma_{x_1}^{k_1} \cdots \sigma_{x_n}^{k_n}(b)} = \sum_{i=1}^n (\sigma_{x_i}(h_i) - h_i) + \frac{\sigma_{x_1}^{-k_1} \cdots \sigma_{x_n}^{-k_n}(a)}{b}, \quad (5.4)$$

where

$$h_i = \begin{cases} 0, & \text{if } k_i = 0, \\ \sum_{\ell=0}^{k_i-1} \frac{\sigma_{x_i}^{\ell-k_i} \sigma_{x_{i-1}}^{-k_{i-1}} \cdots \sigma_{x_1}^{-k_1}(a)}{\sigma_{x_i}^{\ell} \sigma_{x_{i+1}}^{k_{i+1}} \cdots \sigma_{x_n}^{k_n}(b)}, & \text{if } k_i > 0, \\ -\sum_{\ell=0}^{-k_i-1} \frac{\sigma_{x_i}^{\ell} \sigma_{x_{i-1}}^{-k_{i-1}} \cdots \sigma_{x_1}^{-k_1}(a)}{\sigma_{x_i}^{\ell+k_i} \sigma_{x_{i+1}}^{k_{i+1}} \cdots \sigma_{x_n}^{k_n}(b)}, & \text{if } k_i < 0. \end{cases}$$

for $i = 1, \dots, n$. The equation (5.4) is called the $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -reduction formula. Rewriting every fraction of f in (5.2) by the reduction formula (5.4), we get the following lemma.

Lemma 5.2. *Let $f \in V_{[d]_G, j}$ be in the form (5.2). Then we can decompose it into the form*

$$f = \sum_{i=1}^n \Delta_{x_i}(g_i) + r \text{ with } r = \frac{a}{d^j}, \quad (5.5)$$

where $g_i \in \mathbb{F}(\mathbf{x})$, $a = \sum_{\tau} \tau^{-1}(a_{\tau})$ with $\deg_{x_1}(a) < \deg_{x_1}(d)$. In particular, f is $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -summable if and only if r is $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -summable.

Example 5.3. Consider the rational function $f_1 \in \mathbb{Q}(x, y, z)$ given in Example 4.9. Then $f_1 \in V_{[d_1]_{G,1}}$ and it can be written as

$$f_1 = \frac{x - z^2}{d_1} + \frac{x - y - 2z}{\sigma_y(d_1)} + \frac{y + z^2}{\sigma_x \sigma_y^3 \sigma_z^{-1}(d_1)},$$

where $d_1 = x^2 + 2xy + z^2$. By applying the $(\sigma_x, \sigma_y, \sigma_z)$ -reduction formula, we have

$$f_1 = \Delta_x(u_1) + \Delta_y(v_1) + \Delta_z(w_1) + r_1 \text{ with } r_1 = \frac{2x - 1}{d_1},$$

where

$$u_1 = \frac{y + z^2}{\sigma_y^3 \sigma_z^{-1}(d_1)}, \quad v_1 = \frac{x - y + 1 - 2z}{d_1} + \sum_{\ell=0}^2 \frac{y + \ell - 3 + z^2}{\sigma_y^\ell \sigma_z^{-1}(d_1)}, \quad w_1 = -\frac{y - 3 + z^2}{\sigma_z^{-1}(d_1)}.$$

Then f_1 is $(\sigma_x, \sigma_y, \sigma_z)$ -summable if and only if r_1 is $(\sigma_x, \sigma_y, \sigma_z)$ -summable.

The results in Lemmas 5.1 and 5.2 are summarized as follows. The following lemma reduces the rational summability problem from general rational functions to simple fractions.

Corollary 5.4. Let $f \in \mathbb{F}(\mathbf{x})$. Then we can decompose f into the form

$$f = \sum_{i=1}^n \Delta_{x_i}(g_i) + r \text{ with } r = \sum_{i=1}^I \sum_{j=1}^{J_i} \frac{a_{i,j}}{d_i^j}, \quad (5.6)$$

where $g_i \in \mathbb{F}(\mathbf{x})$, $a_{i,j} \in \mathbb{F}(\hat{\mathbf{x}}_1)[x_1]$, $d_i \in \mathbb{F}[\mathbf{x}]$ with $\deg_{x_1}(a_{i,j}) < \deg_{x_1}(d_i)$ and the d_i 's are monic irreducible polynomials in distinct G -orbits. Furthermore, f is $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -summable if and only if each $a_{i,j}/d_i^j$ is $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -summable for all i, j with $1 \leq i \leq I$ and $1 \leq j \leq J_i$.

5.2 Summability criteria

By Corollary 5.4, we reduce the rational summability problem to that for simple fractions

$$f = \frac{a}{d^j}, \quad (5.7)$$

where $j \in \mathbb{N}^+$, $a \in \mathbb{F}(\hat{\mathbf{x}}_1)[x_1]$ and $d \in \mathbb{F}[\mathbf{x}]$ is irreducible with $\deg_{x_1}(a) < \deg_{x_1}(d)$. In this section, we shall present a criterion on the summability for such simple fractions.

For the univariate summability problem, we recall the following well-known result in [2, 4, 10, 61, 64, 69]. Since the univariate case is the base of our induction method, we give a proof for the sake of completeness.

Lemma 5.5. Let $f \in \mathbb{F}(\mathbf{x})$ be of the form (5.7). Then f is (σ_{x_1}) -summable in $\mathbb{F}(\mathbf{x})$ if and only if $a = 0$.

Proof. The sufficiency is trivial since $f = \Delta_{x_1}(0)$. To show the necessity, suppose f is (σ_{x_1}) -summable but $a \neq 0$. Since $f = a/d^j \in V_{[d]_{G,j}}$, by the proof of Lemma 5.1 we can further assume $f = \Delta_{x_1}(g)$ for some $g \in V_{[d]_{G,j}}$. Write g in the form $g = \sum_{i=\ell_0}^{\ell_1} a_i/\sigma_{x_1}^i(d)^j$ with $a_{\ell_0} a_{\ell_1} \neq 0$. Then

$$f = \Delta_{x_1}(g) = \sum_{i=\ell_0}^{\ell_1+1} \frac{\tilde{a}_i}{\sigma_{x_1}^i(d)^j},$$

where $\tilde{a}_i = \sigma_{x_1}(a_{i-1}) - a_i$ for $\ell_0 + 1 \leq i \leq \ell_1$, $\tilde{a}_{\ell_0} = -a_{\ell_0}$ and $\tilde{a}_{\ell_1+1} = \sigma_{x_1}(a_{\ell_1})$. Note that \tilde{a}_{ℓ_0} and \tilde{a}_{ℓ_1+1} are nonzero. For any integer $i \in \mathbb{Z}$, $\sigma_{x_1}^i(d)$ is still an irreducible polynomial. However, there is only one irreducible factor in the denominator of $f = a/d^j$. So we must have $\sigma_{x_1}^i(d) = d$ for some nonzero integer i . It implies that d is free of x_1 . This is a contradiction because d has positive degree in x_1 . \blacksquare

For the multivariate summability problem with $n > 1$, let $G = \langle \sigma_{x_1}, \dots, \sigma_{x_n} \rangle$ and $H = \langle \sigma_{x_1}, \dots, \sigma_{x_{n-1}} \rangle$. The isotropy groups of the polynomial d in G and H are denoted by G_d and H_d , respectively, i.e.,

$$G_d = \{\tau \in G \mid \tau(d) = d\} \quad \text{and} \quad H_d = \{\tau \in H \mid \tau(d) = d\}.$$

By Lemma 4.4, we know either $\text{rank}(G_d/H_d) = 0$ or $\text{rank}(G_d/H_d) = 1$.

When $\text{rank}(G_d/H_d) = 0$, the summability problem in n variables can be reduced to that in $n - 1$ variables.

Lemma 5.6. *Let $f = a/d^j \in \mathbb{F}(\mathbf{x})$ be of the form (5.7). If $n > 1$ and $\text{rank}(G_d/H_d) = 0$, then f is $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -summable in $\mathbb{F}(\mathbf{x})$ if and only if f is $(\sigma_{x_1}, \dots, \sigma_{x_{n-1}})$ -summable in $\mathbb{F}(\mathbf{x})$.*

Proof. The sufficiency is obvious by definition. For the necessity, suppose f is $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -summable but not $(\sigma_{x_1}, \dots, \sigma_{x_{n-1}})$ -summable. By the orbital decomposition of f in (5.1) and Lemma 5.1, we get

$$f = \Delta_{x_1}(g_1) + \dots + \Delta_{x_n}(g_n) \tag{5.8}$$

with g_1, \dots, g_n in the same subspace $V_{[d]_{G,j}}$ as f . As an analogue to (5.5) in $n - 1$ variables x_1, \dots, x_{n-1} , we can decompose g_n as

$$g_n = \sum_{i=1}^{n-1} \Delta_{x_i}(u_i) + \sum_{\ell=0}^{\rho} \frac{\lambda_{\ell}}{\sigma_{x_n}^{\ell}(\mu)^j}, \tag{5.9}$$

where $u_i \in \mathbb{F}(\mathbf{x})$, $\rho \in \mathbb{N}$, $\lambda_{\ell} \in \mathbb{F}(\hat{\mathbf{x}}_1)[x_1]$, $\mu \in \mathbb{F}[\mathbf{x}]$ with $\deg_{x_1}(\lambda_{\ell}) < \deg_{x_1}(d)$ and μ is in the same G -orbit as d .

Furthermore, we can assume $\lambda_0 \lambda_{\rho} \neq 0$ and each nonzero $\lambda_{\ell}/\sigma_{x_n}^{\ell}(\mu)^j$ is not $(\sigma_{x_1}, \dots, \sigma_{x_{n-1}})$ -summable. Substituting g_n in (5.8) into (5.8), we get

$$f + \sum_{\ell=0}^{\rho+1} \frac{\tilde{\lambda}_{\ell}}{\sigma_{x_n}^{\ell}(\mu)^j} = \sum_{i=1}^{n-1} \Delta_{x_i}(h_i), \tag{5.10}$$

where $\tilde{\lambda}_0 = \lambda_0$, $\tilde{\lambda}_{\rho+1} = -\sigma_{x_n}(\lambda_{\rho})$, $\tilde{\lambda}_{\ell} = \lambda_{\ell} - \sigma_{x_n}(\lambda_{\ell-1})$ for all $1 \leq \ell \leq \rho$ and $h_i = g_i + \Delta_{x_n}(u_i)$ for all $1 \leq i \leq n - 1$.

Since $\text{rank}(G_d/H_d) = 0$ and $G_d = G_{\mu}$, it follows that all $\sigma_{x_n}^{\ell}(\mu)$ with $\ell \in \mathbb{Z}$ are in distinct H -orbits. In particular, $[\mu]_H, [\sigma_{x_n}(\mu)]_H, \dots, [\sigma_{x_n}^{\rho+1}(\mu)]_H$ are distinct H -orbits. On the other hand, the left hand side of (5.10) is $(\sigma_{x_1}, \dots, \sigma_{x_{n-1}})$ -summable, but $\tilde{\lambda}_0/\mu^j$ is not $(\sigma_{x_1}, \dots, \sigma_{x_{n-1}})$ -summable according to the assumption. By Lemma 5.1 (in $n - 1$ variables), the only choice is that $\mu \sim_H d$. Similarly, $\sigma_{x_n}^{\rho+1}(\mu) \sim_H d$ and hence $\mu \sim_H \sigma_{x_n}^{\rho+1}(\mu)$. This leads to a contradiction since ρ is a non-negative integer. \blacksquare

Lemma 5.7. *Let $f \in \mathbb{F}(\mathbf{x})$ and K be a subgroup of $G = \langle \sigma_{x_1}, \dots, \sigma_{x_n} \rangle$ with rank r ($1 \leq r \leq n$). If $\{\sigma_i\}_{i=1}^r$ and $\{\tau_i\}_{i=1}^r$ are two bases of K , then f is $(\sigma_1, \dots, \sigma_r)$ -summable if and only if f is (τ_1, \dots, τ_r) -summable.*

To prove the basis change property of the summability problem in Lemma 5.7, we first show the following lemma. It can be seen as a variant of the reduction formula (5.4). Since it is useful in computation, we give a detailed proof by induction.

Lemma 5.8. *Let $\sigma_1, \dots, \sigma_r$ be elements in G and $K = \langle \sigma_1, \dots, \sigma_r \rangle$ be the subgroup of G generated by $\sigma_1, \dots, \sigma_r$. Then for every $\tau \in K$,*

$$\tau - \mathbf{1} = (\sigma_1 - \mathbf{1})\tilde{\sigma}_1 + \dots + (\sigma_r - \mathbf{1})\tilde{\sigma}_r,$$

for some $\tilde{\sigma}_i \in \mathbb{F}[K]$.

Proof. We prove this lemma by induction on the number of σ_i . If $r = 1$, then $\tau = \sigma_1^{k_1}$ for some $k_1 \in \mathbb{Z}$. We have $\sigma_1^{k_1} - \mathbf{1} = (\sigma_1 - \mathbf{1})\mu$, where $\mu = 0$ if $k_1 = 0$, $\mu = \sum_{i=0}^{k_1-1} \sigma_1^i$ if $k_1 > 0$ and $\mu = -\sum_{i=0}^{-k_1-1} \sigma_1^{i+k_1}$ if $k_1 < 0$. If $r \geq 2$, assume that the conclusion holds for $r - 1$. Write $\tau = \sigma_1^{k_1} \dots \sigma_r^{k_r}$ for some $k_1, \dots, k_r \in \mathbb{Z}$. Then

$$\tau - \mathbf{1} = \left(\sigma_1^{k_1} - \mathbf{1} \right) \sigma_2^{k_2} \dots \sigma_r^{k_r} + \left(\sigma_2^{k_2} \dots \sigma_r^{k_r} - \mathbf{1} \right).$$

If $\sigma_2^{k_2} \dots \sigma_r^{k_r} = \mathbf{1}$, then we are done. Otherwise, by the inductive hypothesis, we get $\tau - \mathbf{1} = (\sigma_1 - \mathbf{1})\tilde{\sigma}_1 + \dots + (\sigma_r - \mathbf{1})\tilde{\sigma}_r$ for some $\tilde{\sigma}_1, \dots, \tilde{\sigma}_r \in \mathbb{F}[K]$. In fact, the above argument gives the following explicit expression

$$\tilde{\sigma}_i = \begin{cases} 0 & \text{if } k_i = 0, \\ \sum_{\ell=0}^{k_i-1} \sigma_i^\ell \sigma_{i+1}^{k_{i+1}} \dots \sigma_r^{k_r} & \text{if } k_i > 0, \\ -\sum_{\ell=0}^{-k_i-1} \sigma_i^{\ell+k_i} \sigma_{i+1}^{k_{i+1}} \dots \sigma_r^{k_r} & \text{if } k_i < 0, \end{cases}$$

for $i = 1, \dots, r$. ■

Proof of Lemma 5.7. Suppose f is (τ_1, \dots, τ_r) -summable. This means

$$f = \Delta_{\tau_1}(h_1) + \dots + \Delta_{\tau_r}(h_r) \tag{5.11}$$

for some $h_1, \dots, h_r \in \mathbb{F}(\mathbf{x})$. For each $i = 1, \dots, r$, since $\tau_i \in \langle \sigma_1, \dots, \sigma_r \rangle$, it follows from Lemma 5.8 that $\tau_i - \mathbf{1} = (\sigma_1 - \mathbf{1})\tilde{\sigma}_{i,1} + \dots + (\sigma_r - \mathbf{1})\tilde{\sigma}_{i,r}$ for some $\tilde{\sigma}_{i,j} \in \mathbb{F}[K]$ with K being the subgroup generated by $\sigma_1, \dots, \sigma_r$. Applying this operator to h_i yields that

$$\Delta_{\tau_i}(h_i) = \Delta_{\sigma_1}(h_{i,1}) + \dots + \Delta_{\sigma_r}(h_{i,r}), \tag{5.12}$$

where $h_{i,j} = \tilde{\sigma}_{i,j}(h_i)$ for $j = 1, \dots, r$. Combining Equations (5.11) and (5.12), we have

$$f = \sum_{i=1}^r \Delta_{\tau_i}(h_i) = \sum_{i=1}^r \sum_{j=1}^r \Delta_{\sigma_j}(h_{i,j}) = \sum_{j=1}^r \Delta_{\sigma_j} \left(\sum_{i=1}^r h_{i,j} \right),$$

where the last equality follows from the linearity of Δ_{σ_j} . Thus f is $(\sigma_1, \dots, \sigma_r)$ -summable. Similarly, the other direction is also true. ■

Theorem 5.9. *Let $f = a/d^j \in \mathbb{F}(\mathbf{x})$ be of the form (5.7). Let $\{\tau_i\}_{i=1}^r$ ($1 \leq r < n$) be a basis of G_d (take $\tau_1 = \mathbf{1}$, if $G_d = \{\mathbf{1}\}$). Then f is $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -summable if and only if*

$$a = \Delta_{\tau_1}(b_1) + \dots + \Delta_{\tau_r}(b_r)$$

for some $b_i \in \mathbb{F}(\hat{\mathbf{x}}_1)[x_1]$ with $\deg_{x_1}(b_i) < \deg_{x_1}(d)$ for all $1 \leq i \leq r$.

Proof. The sufficiency follows from the fact that $f = \sum_{i=1}^r \Delta_{\tau_i}(b_i/d^j)$ and Lemma 5.8. For the necessity, we proceed by induction on n . If $n = 1$, then G_d is a trivial group and the univariate case follows from Lemma 5.5. If $n > 1$, suppose the inductive hypothesis is true for $n - 1$ as follows.

If $\{\theta_i\}_{i=1}^s$ is a basis of H_d , then f is $(\sigma_{x_1}, \dots, \sigma_{x_{n-1}})$ -summable if and only if $a = \sum_{i=1}^s \Delta_{\theta_i}(b_i)$ for some $b_i \in \mathbb{F}(\hat{\mathbf{x}}_1)[x_1]$ with $\deg_{x_1}(b_i) < \deg_{x_1}(d)$ for all $1 \leq i \leq s$.

Now we proceed by a case distinction according to the rank of G_d/H_d which is either 0 or 1 by Lemma 4.4. If $\text{rank}(G_d/H_d) = 0$, then $H_d = G_d$. The conclusion follows from Lemma 5.6 and the inductive hypothesis. If $\text{rank}(G_d/H_d) = 1$, by Lemma 5.7, we may assume that $\{\tau_i\}_{i=1}^r$ is a basis of G_d such that $H_d = \langle \tau_1, \dots, \tau_{r-1} \rangle$ and $G_d/H_d = \langle \bar{\tau}_r \rangle$. Here $\bar{\tau}_r$ represents the element $\tau_r H_d$ with $\tau_r \in G_d$. Then we can choose $\tau_r = \sigma_{x_1}^{-k_1} \dots \sigma_{x_{n-1}}^{-k_{n-1}} \sigma_{x_n}^{k_n}$ such that k_n is a positive integer. Otherwise, replace τ_r by τ_r^{-1} . Since $\bar{\tau}_r$ is a generator of G_d/H_d , we have that k_n is the smallest positive integer such that $\sigma_{x_n}^{k_n}(d) \sim_H d$.

By the decomposition (4.4), we can assume $f = \Delta_{x_1}(g_1) + \dots + \Delta_{x_n}(g_n)$ with $g_i \in V_{[d]_{G,j}}$. In here, using Lemma 5.2, g_n can be decomposed as

$$g_n = \sum_{i=1}^{n-1} \Delta_{x_i}(u_i) + \sum_{\ell=0}^{k_n-1} \frac{\lambda_\ell}{\sigma_{x_n}^\ell(d)^j},$$

where $u_i \in \mathbb{F}(\mathbf{x})$ and $\lambda_\ell \in \mathbb{F}(\hat{\mathbf{x}}_1)[x_1]$ with $\deg_{x_1}(\lambda_\ell) < \deg_{x_1}(d)$. Then we have

$$f - \Delta_{x_n} \left(\sum_{\ell=0}^{k_n-1} \frac{\lambda_\ell}{\sigma_{x_n}^\ell(d)^j} \right) = \sum_{i=1}^{n-1} \Delta_{x_i}(h_i), \quad (5.13)$$

where $h_i = g_i + \Delta_{x_n}(u_i)$. Note that $\sigma_{x_n}^{k_n}(d) = \sigma_{x_1}^{k_1} \dots \sigma_{x_{n-1}}^{k_{n-1}}(d)$ and apply the reduction formula (5.4) to simplify (5.13). We get

$$\tilde{f} := \sum_{\ell=0}^{k_n-1} \frac{\tilde{\lambda}_\ell}{\sigma_{x_n}^\ell(d)^j} = \sum_{i=1}^{n-1} \Delta_{x_i}(\tilde{h}_i), \quad (5.14)$$

where $\tilde{h}_i \in \mathbb{F}(\mathbf{x})$, $\tilde{\lambda}_0 = a + \lambda_0 - \sigma_{x_1}^{-k_1} \dots \sigma_{x_{n-1}}^{-k_{n-1}} \sigma_{x_n}(\lambda_{k_n-1})$ and $\tilde{\lambda}_\ell = \lambda_\ell - \sigma_{x_n}(\lambda_{\ell-1})$ for $1 \leq \ell \leq k_n - 1$.

Note that $[d]_H, [\sigma_{x_n}(d)]_H, \dots, [\sigma_{x_n}^{k_n-1}(d)]_H$ are distinct H -orbits due to the minimality of k_n . From the equation (5.14), \tilde{f} is $(\sigma_{x_1}, \dots, \sigma_{x_{n-1}})$ -summable. So by Lemma 5.1, each $\frac{\tilde{\lambda}_\ell}{\sigma_{x_n}^\ell(d)^j}$ is $(\sigma_{x_1}, \dots, \sigma_{x_{n-1}})$ -summable for $0 \leq \ell \leq k_n - 1$. Let W denote the vector subspace of $\mathbb{F}(\mathbf{x})$ over \mathbb{F} consisting of all elements in the form of $\sum_{i=1}^{r-1} \Delta_{\tau_i}(b_i)$ with $b_i \in \mathbb{F}(\hat{\mathbf{x}}_1)[x_1]$ and $\deg_{x_1}(b_i) < \deg_{x_1}(d)$. (If $r = 1$, take $W = \{0\}$.) If two rational functions $g, h \in \mathbb{F}(\mathbf{x})$ satisfy the property that $g - h \in W$, we say g, h are congruent modulo W , denoted by $g \equiv h \pmod{W}$. Since $H_d = H_{\sigma_{x_n}^\ell(d)}$, we apply the inductive hypothesis to conclude that

$$\begin{cases} 0 \equiv a + \lambda_0 - \sigma_{x_1}^{-k_1} \dots \sigma_{x_{n-1}}^{-k_{n-1}} \sigma_{x_n}(\lambda_{k_n-1}) & \pmod{W} \\ 0 \equiv \lambda_1 - \sigma_{x_n}(\lambda_0) & \pmod{W} \\ \vdots & \\ 0 \equiv \lambda_{k_n-1} - \sigma_{x_n}(\lambda_{k_n-2}) & \pmod{W}. \end{cases}$$

Since W is G -invariant, it follows from the equations that

$$a \equiv \sigma_{x_1}^{-k_1} \cdots \sigma_{x_{n-1}}^{-k_{n-1}} \sigma_{x_n}^{k_n}(\lambda_0) - \lambda_0 \equiv \Delta_{\tau_r}(\lambda_0) \pmod{W}.$$

This completes the proof. \blacksquare

Remark 5.10. For the bivariate case with $n = 2$, Theorem 5.9 coincides with the known criterion in [47, Theorem 3.3] and [27, Theorem 3.7]. In this case, $\text{rank}(G_d) \leq 1$ and $H_d = \{\mathbf{1}\}$. If $\text{rank}(G_d) = 0$, then a/d^j is $(\sigma_{x_1}, \sigma_{x_2})$ -summable in $\mathbb{F}(\mathbf{x})$ if and only if $a = 0$. If $\text{rank}(G_d) = 1$ and G_d is generated by $\tau = \sigma_{x_1}^{\ell_1} \sigma_{x_2}^{-\ell_2} \in G$ for some $\ell_2 \neq 0$, then a/d^j is $(\sigma_{x_1}, \sigma_{x_2})$ -summable if and only if $a = \sigma_{x_1}^{\ell_1} \sigma_{x_2}^{-\ell_2}(b) - b$ for some $b \in \mathbb{F}(\hat{\mathbf{x}}_1)[x_1]$ with $\deg_{x_1}(b) < \deg_{x_1}(d)$.

Example 5.11. Let $f = 1/(x_1^s + \cdots + x_n^s) \in \mathbb{Q}(x_1, \dots, x_n)$ with $s, n \in \mathbb{N} \setminus \{0\}$. Let G_d be the isotropy group of $d = x_1^s + \cdots + x_n^s$ in $G = \langle \sigma_{x_1}, \dots, \sigma_{x_n} \rangle$. Then we can decide for all cases the $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -summability of f in $\mathbb{Q}(x_1, \dots, x_n)$.

- (1) If $s = 1$ and $n > 1$, then d is irreducible. The rank of G_d is $n - 1$ and one basis is given by $\tau_1, \dots, \tau_{n-1}$ with $\tau_i = \sigma_{x_i} \sigma_{x_{i+1}}^{-1}$ for $i = 1, \dots, n - 1$. Since $1 = \tau_1(x_1) - x_1$, it follows that f is $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -summable. In fact, we have

$$\frac{1}{x_1 + \cdots + x_n} = \Delta_{x_1} \left(\frac{x_1}{x_1 + \cdots + x_n} \right) + \Delta_{x_2} \left(\frac{-x_1 - 1}{x_1 + \cdots + x_n} \right).$$

This means f is $(\sigma_{x_1}, \sigma_{x_2})$ -summable, so is $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -summable.

- (2) If $s \geq 1$ and $n = 1$, then $f = 1/x_1^s$. Since the isotropy group of x_1 in $\langle \sigma_{x_1} \rangle$ is $\{\mathbf{1}\}$, by Theorem 5.9, we get that f is not (σ_{x_1}) -summable.
- (3) If $s > 1$ and $n = 2$, then $f = 1/(x_1^s + x_2^s) = \sum_{j=1}^s a_j/(x_1 - \beta_j x_2)$, where β_j 's are distinct roots of $z^s = -1$ and $a_j = 1/(s(\beta_j x_2)^{s-1})$. There exists $j \in \{1, \dots, s\}$ such that $\beta_j \notin \mathbb{Z}$. Then for $d_j = x_1 - \beta_j x_2$, we have $G_{d_j} = \{\mathbf{1}\}$. So a_j/d_j is not $(\sigma_{x_1}, \sigma_{x_2})$ -summable in $\mathbb{C}(x_1, x_2)$ and by Lemma 5.1, neither is f . Hence f is not $(\sigma_{x_1}, \sigma_{x_2})$ -summable in $\mathbb{Q}(x_1, x_2)$. This result has appeared in [27, Example 3.8].
- (4) If $s > 1$ and $n > 2$, then d is irreducible. Since $G_d = \{\mathbf{1}\}$, by Theorem 5.9, it follows that f is not $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -summable.

Now we transfer the (τ_1, \dots, τ_r) -summability problem to the $(\sigma_{x_1}, \dots, \sigma_{x_r})$ -summability problem.

Proposition 5.12. Let $\{\tau_i\}_{i=1}^r$ ($1 \leq r \leq n$) be a family of linearly independent elements in $G = \langle \sigma_{x_1}, \dots, \sigma_{x_n} \rangle$. Then there exists an \mathbb{F} -automorphism ϕ of $\mathbb{F}(\mathbf{x})$ such that ϕ is a difference isomorphism between the difference fields $(\mathbb{F}(\mathbf{x}), \tau_i)$ and $(\mathbb{F}(\mathbf{x}), \sigma_{x_i})$ for all $i = 1, \dots, r$. Therefore, for any $f \in \mathbb{F}(\mathbf{x})$, f is (τ_1, \dots, τ_r) -summable in $\mathbb{F}(\mathbf{x})$ if and only if $\phi(f)$ is $(\sigma_{x_1}, \dots, \sigma_{x_r})$ -summable in $\mathbb{F}(\mathbf{x})$.

Proof. Assume $\tau_i = \sigma_{x_1}^{\alpha_{i,1}} \cdots \sigma_{x_m}^{\alpha_{i,m}}$ with $\alpha_{i,j} = 0$ if $j > n$ and write $\alpha_i = (\alpha_{i,1}, \dots, \alpha_{i,m}) \in \mathbb{Z}^m$ viewed as a vector in \mathbb{Q}^m for $i = 1, \dots, r$. Then $\alpha_1, \dots, \alpha_r$ are linearly independent over \mathbb{Q} . So we can find the other vectors $\alpha_{r+1}, \dots, \alpha_m$ such that $\{\alpha_1, \dots, \alpha_m\}$ forms a basis of \mathbb{Q}^m . Let $\alpha_i = (\alpha_{i,1}, \dots, \alpha_{i,m})$ for $i = r + 1, \dots, m$ and $A = (\alpha_{i,j}) \in \mathbb{Q}^{m \times m}$. Then A is an invertible matrix. Thus we define an \mathbb{F} -automorphism $\phi : \mathbb{F}(\mathbf{x}) \rightarrow \mathbb{F}(\mathbf{x})$ by

$$(\phi(x_1), \dots, \phi(x_m)) := (x_1, \dots, x_m)A.$$

Let $u_j := \phi(x_j) = \sum_{i=1}^m \alpha_{i,j} x_i$ for all $1 \leq j \leq m$. Then ϕ satisfies the relation $\phi \circ \tau_i = \sigma_{x_i} \circ \phi$ for all $i = 1, \dots, r$, which means the following diagrams

$$\begin{array}{ccc}
\mathbb{F}(\mathbf{x}) & \xrightarrow{\phi} & \mathbb{F}(\mathbf{x}) \\
\tau_1 \downarrow & & \downarrow \sigma_1 \\
\mathbb{F}(\mathbf{x}) & \xrightarrow{\phi} & \mathbb{F}(\mathbf{x})
\end{array}
\quad \dots \quad
\begin{array}{ccc}
\mathbb{F}(\mathbf{x}) & \xrightarrow{\phi} & \mathbb{F}(\mathbf{x}) \\
\tau_r \downarrow & & \downarrow \sigma_r \\
\mathbb{F}(\mathbf{x}) & \xrightarrow{\phi} & \mathbb{F}(\mathbf{x})
\end{array}$$

are commutative. This is true since for any $f \in \mathbb{F}(x_1, \dots, x_m)$, we have

$$\begin{aligned}
\phi(\tau_i(f(x_1, \dots, x_m))) &= \phi(f(x_1 + \alpha_{i,1}, \dots, x_m + \alpha_{i,m})) \\
&= f(u_1 + \alpha_{i,1}, \dots, u_m + \alpha_{i,m})
\end{aligned}$$

and

$$\begin{aligned}
\sigma_{x_i}(\phi(f(x_1, \dots, x_m))) &= \sigma_{x_i}(f(u_1, \dots, u_m)) \\
&= f(u_1 + \alpha_{i,1}, \dots, u_m + \alpha_{i,m}).
\end{aligned}$$

It follows that

$$f = \sum_{i=1}^r \Delta_{\tau_i}(g_i) \quad \text{if and only if} \quad \phi(f) = \sum_{i=1}^r \Delta_{x_i}(\phi(g_i))$$

whenever $f, g_1, \dots, g_r \in \mathbb{F}(\mathbf{x})$. This proves our assertion. \blacksquare

Combining Theorem 5.9 and Proposition 5.12, the summability problem 2.4 in n variables can be reduced to that in fewer variables. So we can design the following recursive algorithm for testing $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -summability of multivariate rational functions. Furthermore, the (τ_1, \dots, τ_n) -summability problem can also be solved via the transformation in Proposition 5.12.

Recall that a map $\phi : \mathbb{F}(\mathbf{x}) \rightarrow \mathbb{F}(\mathbf{x})$ is called a \mathbb{Q} -affine map if $\phi(f(\mathbf{x})) = f(\mathbf{x} \cdot A + \mathbf{b})$, where A is an invertible matrix in $\text{GL}_m(\mathbb{Q})$ and \mathbf{b} is a vector in \mathbb{Q}^m . Note that the identity map, all shift operations and all difference isomorphisms constructed in Proposition 5.12 are \mathbb{Q} -affine maps. The composition of two \mathbb{Q} -affine maps is still a \mathbb{Q} -affine map. If f is $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -summable in $\mathbb{F}(\mathbf{x})$, Algorithm 5.13 will output *unnormalised certificates* for f in the form

$$g = \sum_{\ell=1}^{\rho} \prod_{k=1}^{K_{\ell}} \psi_{\ell,k}(u_{\ell,k}),$$

where $u_{\ell,k} \in \mathbb{F}(\mathbf{x})$ and the $\psi_{\ell,k}$'s are \mathbb{Q} -affine maps.

Algorithm 5.13 (Constructive Testing of the Rational Summability).

IsSummable($f, [x_1, \dots, x_n]$).

INPUT: a multivariate rational function $f \in \mathbb{F}(\mathbf{x})$ and a list $[x_1, \dots, x_n]$ of variable names;

OUTPUT: unnormalised certificates g_1, \dots, g_n for f if f is $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -summable in $\mathbb{F}(\mathbf{x})$; false otherwise.

- 1 using shift equivalence testing and partial fraction decomposition, decompose f into $f = f_0 + \sum_{j \in \mathbb{N}^+} \sum_{[d]_G} f_{[d]_G, j}$ as in Equation (5.1).
- 2 apply the reduction to f_0 and each nonzero component $f_{[d]_G, j}$ such that

$$f = \Delta_{x_1}(g_1) + \dots + \Delta_{x_n}(g_n) + r \quad \text{with} \quad r = \sum_{i=1}^I \sum_{j=1}^{J_i} \frac{a_{i,j}}{d_i^j},$$

where $a_{i,j}/d_i^j$ is the remainder of $f_{[d_i]_{G,j}}$ described in Lemma 5.2.

3 if $r = 0$, then **return** g_1, \dots, g_n .

4 **for** $i = 1, \dots, I$ **do**

5 by Remark 4.2, one can compute a basis $\tau_{i,1}, \dots, \tau_{i,r_i}$ for the isotropy group G_{d_i} of d_i .

6 **for** $j = 1, \dots, J_i$ **do**

7 if $n = 1$ or $G_{d_i} = \{\mathbf{1}\}$ then

8 **return** false if $a_{i,j} \neq 0$.

9 else

10 find an \mathbb{F} -automorphism ϕ_i of $\mathbb{F}(\mathbf{x})$ given in Proposition 5.12 such that $\phi_i \circ \tau_{i,\ell} = \sigma_{x_\ell} \circ \phi_i$
for $\ell = 1, \dots, r_i$.

11 set $\tilde{a}_{i,j} = \phi_i(a_{i,j})$.

12 execute **IsSummable**($\tilde{a}_{i,j}, [x_1, \dots, x_{r_i}]$).

13 if $\tilde{a}_{i,j}$ is $(\sigma_{x_1}, \dots, \sigma_{x_{r_i}})$ -summable in $\mathbb{F}(\mathbf{x})$, let

$$\tilde{a}_{i,j} = \Delta_{x_1} \left(\tilde{b}_{i,j}^{(1)} \right) + \dots + \Delta_{x_{r_i}} \left(\tilde{b}_{i,j}^{(r_i)} \right);$$

return false otherwise.

14 applying ϕ_i^{-1} to the previous equation yields that

$$a_{i,j} = \Delta_{\tau_{i,1}} \left(b_{i,j}^{(1)} \right) + \dots + \Delta_{\tau_{i,r_i}} \left(b_{i,j}^{(r_i)} \right),$$

where $(b_{i,j}^{(1)}, \dots, b_{i,j}^{(r_i)}) = (\phi_i^{-1}(\tilde{b}_{i,j}^{(1)}), \dots, \phi_i^{-1}(\tilde{b}_{i,j}^{(r_i)}))$.

15 using Lemma 5.8, compute $h_{i,j}^{(1)}, \dots, h_{i,j}^{(n)} \in \mathbb{F}(\mathbf{x})$ such that

$$\frac{a_{i,j}}{d_i^j} = \sum_{\ell=1}^{r_i} \Delta_{\tau_{i,\ell}} \left(\frac{b_{i,j}^{(\ell)}}{d_i^j} \right) = \sum_{\ell=1}^n \Delta_{x_\ell} \left(h_{i,j}^{(\ell)} \right).$$

16 update $g_\ell = g_\ell + h_{i,j}^{(\ell)}$ for $\ell = 1, \dots, n$.

17 **return** g_1, \dots, g_n .

We now analyse the complexity of Algorithm 5.13 for $\mathbb{F} = \mathbb{Q}$. The following theorem shows that the rational summability problem can be solved in polynomial time.

Theorem 5.14. *Let δ and M be two positive integers and $f(\mathbf{x})$ be a multivariate rational function in $\mathbb{Q}(\mathbf{x})_\delta$ with $\|f\| = M$. If m and M are fixed, i.e., $m, M \in O(1)$, then the total runtime of Algorithm 5.13 is $O(\delta^{O(1)})$ ops in \mathbb{Q} .*

Proof. In each recursion, the input rational function $\tilde{a}_{i,j}$ in Step 12 is in $\mathbb{Q}(\mathbf{x})_{O(\delta^{O(1)})}$ with maximum $O((M\delta)^{\delta^{O(1)}})$ by Theorem 3.8 and the timing of the first three steps is dominated by that of the irreducible factorization of the denominator of the input, which is $O(\delta^{O(1)})$ ops in \mathbb{Q} according to Fact 2.8. Since there are at most m recursive calls and $m \in O(1)$, the total runtime is $O(\delta^{O(1)})$ ops in \mathbb{Q} . ■

Example 5.15. *Let $G = \langle \sigma_x, \sigma_y, \sigma_z \rangle$ and $f = f_1 + f_2 + f_3 \in \mathbb{Q}(x, y, z)$ be the same as in Example 4.10.*

(1) After the $(\sigma_x, \sigma_y, \sigma_z)$ -reduction for f_1 , see Example 5.3, we get

$$f_1 = \Delta_x(u_1) + \Delta_y(v_1) + \Delta_z(w_1) + r_1 \text{ with } r_1 = \frac{2x-1}{d_1}, \quad (5.15)$$

where $u_1, v_1, w_1 \in \mathbb{Q}(x, y, z)$ and $d_1 = x^2 + 2xy + z^2$. By Example 4.6 (1), the isotropy group $G_{d_1} = \{\mathbf{1}\}$ is trivial. By Theorem 5.9, we see that r_1 is not $(\sigma_x, \sigma_y, \sigma_z)$ -summable because its numerator $a_1 = 2x - 1$ is not zero. Hence f_1 is not $(\sigma_x, \sigma_y, \sigma_z)$ -summable.

(2) For $f_2 = a_2/d_2$ with $a_2 = x + z$ and $d_2 = (x - 3y)^2(y + z) + 1$, we know from Example 4.6 (2) that a basis of G_{d_2} is $\{\sigma_x^3 \sigma_y \sigma_z^{-1}\}$. For any $\{\mu, \nu\} \subseteq \{x, y, z\}$, since the isotropy group of d_2 in $\langle \sigma_\mu, \sigma_\nu \rangle$ is trivial, we get that f_2 is not (σ_μ, σ_ν) -summable in $\mathbb{Q}(x, y, z)$. By Theorem 5.9, we see that f_2 is $(\sigma_x, \sigma_y, \sigma_z)$ -summable in $\mathbb{Q}(x, y, z)$ if and only if a_2 is (τ) -summable in $\mathbb{Q}(x, y, z)$ with $\tau = \sigma_x^3 \sigma_y \sigma_z^{-1}$. Choose one \mathbb{Q} -automorphism ϕ_2 of $\mathbb{Q}(x, y, z)$ given in Proposition 5.12 as follows

$$\phi_2(h(x, y, z)) = h(3x, x + y, -x + z),$$

for any $h \in \mathbb{Q}(x, y, z)$. Then $\phi_2 \circ \tau = \sigma_x \circ \phi_2$. Hence a_2 is (τ) -summable in $\mathbb{Q}(x, y, z)$ if and only if $\phi_2(a_2)$ is (σ_x) -summable in $\mathbb{Q}(x, y, z)$. Since

$$\phi_2(a_2) = 2x + z = \Delta_x((x - 1)(x + z)) \quad (5.16)$$

is (σ_x) -summable, it follows that f_2 is $(\sigma_x, \sigma_y, \sigma_z)$ -summable. In fact, applying ϕ_2^{-1} to Equation (5.16) yields that

$$a_2 = x + z = \Delta_\tau(b) \text{ with } b = \frac{1}{9}(x - 3)(2x + 3z).$$

By Lemma 5.8, we have

$$f_2 = \Delta_\tau\left(\frac{b}{d_2}\right) = \Delta_x(u_2) + \Delta_y(v_2) + \Delta_z(w_2), \quad (5.17)$$

where $u_2 = \sum_{\ell=0}^2 \sigma_x^\ell \sigma_y \sigma_z^{-1} \left(\frac{b}{d_2}\right)$, $v_2 = \sigma_z^{-1} \left(\frac{b}{d_2}\right)$ and $w_2 = -\sigma_z^{-1} \left(\frac{b}{d_2}\right)$.

(3) For $f_3 = a_3/d_3^2$ with $a_3 = y + z/(y^2 + z - 1) - 1/(y^2 + z)$ and $d_3 = x + 2y + z$, we know from Example 4.6 (2) that a basis of G_{d_3} is $\{\tau_1, \tau_2\}$, where $\tau_1 = \sigma_x^2 \sigma_y^{-1}$ and $\tau_2 = \sigma_x \sigma_z^{-1}$. To decide the $(\sigma_x, \sigma_y, \sigma_z)$ -summability of f_3 , we construct a \mathbb{Q} -automorphism ϕ_3 of $\mathbb{Q}(x, y, z)$ such that $\phi_3 \circ \tau_1 = \sigma_x \circ \phi_3$ and $\phi_3 \circ \tau_2 = \sigma_y \circ \phi_3$ as follows

$$\phi_3(h(x, y, z)) = h(2x + y, -x, -y + z),$$

for any $h \in \mathbb{Q}(x, y, z)$. Then it remains to decide the (σ_x, σ_y) -summability of

$$\phi_3(a_3) = -x + \frac{z - y}{\underbrace{x^2 - y + z - 1}_{\sigma_y(\tilde{d})}} - \frac{1}{\underbrace{x^2 - y + z}_{\tilde{d}}}$$

in $\mathbb{Q}(x, y, z)$. So we use the (σ_x, σ_y) -reduction to reduce $\phi_3(a_3)$ and obtain

$$\phi_3(a_3) = \Delta_x(\tilde{b}_1) + \Delta_y(\tilde{b}_2) + \frac{z - y}{x^2 - y + z}, \quad (5.18)$$

where $\tilde{b}_1 = -\frac{1}{2}x(x-1)$ and $\tilde{b}_2 = \frac{z-y+1}{x^2-y+z}$. Since the isotropy group of \tilde{d} in $\langle \sigma_x, \sigma_y \rangle$ is trivial, $\phi_3(a_3)$ is not (σ_x, σ_y) -summable. Hence f_3 is not $(\sigma_x, \sigma_y, \sigma_z)$ -summable. Even so, in this case, using the above calculation, we can further decompose f_3 into a summable part and a remainder. We now show how to do this. Starting from the decomposition (5.18) of $\phi_3(a_3)$ with respect to the (σ_x, σ_y) -summability problem, we apply ϕ_3^{-1} to both sides of this decomposition to obtain that

$$a_3 = \Delta_{\tau_1}(b_1) + \Delta_{\tau_2}(b_2) + \frac{z}{y^2+z},$$

where $b_1 = \phi_3^{-1}(\tilde{b}_2) = -\frac{1}{2}y(y+1)$ and $b_2 = \phi_3^{-1}(\tilde{b}_2) = \frac{z+1}{y^2+z}$. By Lemma 5.8 with $\tau = \tau_1, \tau_2$, we have

$$\begin{aligned} f_3 &= \frac{a_3}{d_3^2} = \Delta_{\tau_1}\left(\frac{b_1}{d_3^2}\right) + \Delta_{\tau_2}\left(\frac{b_2}{d_3^2}\right) + \underbrace{\frac{z}{(y^2+z)d_3^2}}_{r_3} \\ &= \Delta_x(u_3) + \Delta_y(v_3) + \Delta_z(w_3) + r_3, \end{aligned} \quad (5.19)$$

where $u_3 = \sum_{\ell=0}^1 \sigma_x^\ell \sigma_y^{-1}\left(\frac{b_1}{d_3^2}\right) + \sigma_z^{-1}\left(\frac{b_2}{d_3^2}\right)$, $v_3 = -\sigma_y^{-1}\left(\frac{b_1}{d_3^2}\right)$ and $w_3 = -\sigma_z^{-1}\left(\frac{b_2}{d_3^2}\right)$.

(4) For $f = f_1 + f_2 + f_3$, from Example 4.10 we know that f_1, f_2, f_3 are in distinct $V_{[d]G,j}$ spaces. Since f_1 is not $(\sigma_x, \sigma_y, \sigma_z)$ -summable, it follows from Lemma 5.1 that f is not $(\sigma_x, \sigma_y, \sigma_z)$ -summable. Moreover, combining Equations (5.15), (5.17) and (5.19), we decompose f into

$$f = \Delta_x(u) + \Delta_y(v) + \Delta_z(w) + r \text{ with } r = \frac{2x-1}{d_1} + \frac{z}{(y^2+z)d_3^2},$$

where $u = \sum_{i=1}^2 u_i$, $v = \sum_{i=1}^2 v_i$ and $w = \sum_{i=1}^2 w_i$ are rational functions in $\mathbb{Q}(x, y, z)$.

As we discussed in the above example, given a rational function $f \in \mathbb{F}(\mathbf{x})$, we can compute rational functions $g_1, \dots, g_n, r \in \mathbb{F}(\mathbf{x})$ such that

$$f = \Delta_{x_1}(g_1) + \dots + \Delta_{x_n}(g_n) + r \quad (5.20)$$

satisfying the property that f is $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -summable if and only if $r = 0$. This process can be achieved by induction on n . However, this remainder r is not unique, which depends on the choice of the difference isomorphisms ϕ_i 's. It remains an open problem to make the remainder r minimal in terms of degrees and arithmetic sizes. Moreover, better choices of the isomorphism might also lead to more efficient reductions.

6 The existence problem of telescopers

Similar to the summability problem, there are mainly two steps of solving the existence problem 2.2 of telescopers. First we use the orbital decomposition and Abramov's reduction to simplify the existence problem in Section 6.1. Then in Section 6.2, we use the exponent separation introduced in [22] to further reduce the existence problem to simple fractions and use the summability criteria in Section 5.2 to derive the existence criteria.

6.1 Orbital reduction for existence of telescopers

Let f be a rational function in $\mathbb{K}(t, \mathbf{x})$, where $\mathbf{x} = \{x_1, \dots, x_m\}$. Let n be an integer such that $1 \leq n \leq m$. We consider the existence problem of telescopers of type $(\sigma_t; \sigma_{x_1}, \dots, \sigma_{x_n})$ for the rational function f in $\mathbb{K}(t, \mathbf{x})$. Let $G_t = \langle \sigma_t, \sigma_{x_1}, \dots, \sigma_{x_n} \rangle$ be the free abelian group generated by the shift operators $\sigma_t, \sigma_{x_1}, \dots, \sigma_{x_n}$. Taking $\mathbb{E} = \mathbb{K}(t, \hat{\mathbf{x}}_1)$ and $A = G_t$ in Equality (4.2), we get

$$V_{[d]_{G_t}, j} = \text{Span}_{\mathbb{E}} \left\{ \frac{a}{\tau(d)^j} \mid a \in \mathbb{E}[x_1], \tau \in G_t, \deg_{x_1}(a) < \deg_{x_1}(d) \right\},$$

where $j \in \mathbb{N}^+$ and $d \in \mathbb{E}[\mathbf{x}]$ is irreducible with $\deg_{x_1}(d) > 0$. Then f can be decomposed as

$$f = f_0 + \sum_j \sum_{[d]_{G_t}} f_{[d]_{G_t}, j}, \quad (6.1)$$

where $f_0 \in V_0 = \mathbb{E}[x_1]$ and $f_{[d]_{G_t}, j}$ are in distinct $V_{[d]_{G_t}, j}$ spaces. It induces the following orbital decomposition of $\mathbb{K}(t, \mathbf{x})$ with respect to the group G_t

$$\mathbb{K}(t, \mathbf{x}) = V_0 \bigoplus \left(\bigoplus_{j \in \mathbb{N}^+} \bigoplus_{[d]_{G_t} \in T_{G_t}} V_{[d]_{G_t}, j} \right)$$

as a vector space over $\mathbb{K}(t, \hat{\mathbf{x}}_1)$. This orbital decomposition is G_t -invariant. Moreover, for any L in $\mathbb{K}(t)\langle S_t \rangle$, if $f \in V_{[d]_{G_t}, j}$, then $L(f) \in V_{[d]_{G_t}, j}$. Note that such an operator L commutes with the difference operator Δ_{x_i} for $i = 1, \dots, n$. So by Remark 2.5 and the similar argument as in the proof of Lemma 5.1, we arrive at the following lemma.

Lemma 6.1. *Let $f \in \mathbb{K}(t, \mathbf{x})$. Then f has a telescoper of type $(\sigma_t; \sigma_{x_1}, \dots, \sigma_{x_n})$ if and only if f_0 and each $f_{[d]_{G_t}, j}$ have a telescoper of the same type for all $[d]_{G_t} \in T_{G_t}$ and $j \in \mathbb{N}^+$.*

Since $f_0 \in V_0 = \mathbb{K}(t, \hat{\mathbf{x}}_1)[x_1]$ is always (σ_{x_1}) -summable, it follows that $L = 1$ is a telescoper for f_0 . For $f \in V_{[d]_{G_t}, j}$, it can be written as

$$f = \sum_{\tau} \frac{a_{\tau}}{\tau(d)^j}, \quad (6.2)$$

where $\tau \in G_t$, $a_{\tau} \in \mathbb{K}(t, \hat{\mathbf{x}}_1)[x_1]$, $d \in \mathbb{K}[t, \mathbf{x}]$ with $\deg_{x_1}(a_{\tau}) < \deg_{x_1}(d)$ and d is irreducible in x_1 over $\mathbb{K}(t, \hat{\mathbf{x}}_1)$. Each $\tau \in G_t$ is in the form of $\tau = \sigma_t^{k_0} \sigma_{x_1}^{k_1} \dots \sigma_{x_n}^{k_n}$ for some $k_0, k_1, \dots, k_n \in \mathbb{Z}$. Using the $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -reduction formula (5.4), we get the following decomposition.

Lemma 6.2. *Let $f \in V_{[d]_{G_t}, j}$ be in the form (6.2). Then we can decompose it into the form*

$$f = \sum_{i=1}^n \Delta_{x_i}(g_i) + r \text{ with } r = \sum_{\ell=0}^{\rho} \frac{a_{\ell}}{\sigma_t^{\ell}(\mu)^j},$$

where $\rho \in \mathbb{N}$, $g_i \in \mathbb{K}(t, \mathbf{x})$, $a_{\ell} \in \mathbb{K}(t, \hat{\mathbf{x}}_1)[x_1]$, $\mu \in \mathbb{K}[t, \mathbf{x}]$, $\deg_{x_1}(a_{\ell}) < \deg_{x_1}(d)$, μ is in the same G_t -orbit as d , and $\sigma_t^{\ell}(\mu)$, $\sigma_t^{\ell'}(\mu)$ are not G -equivalent for $0 \leq \ell \neq \ell' \leq \rho$. Therefore f has a telescoper of type $(\sigma_t; \sigma_{x_1}, \dots, \sigma_{x_n})$ if and only if r has a telescoper of the same type.

Example 6.3. *Let $\mathbb{K} = \mathbb{Q}$, $G_t = \langle \sigma_t, \sigma_x, \sigma_y, \sigma_z \rangle$ and $G = \langle \sigma_x, \sigma_y, \sigma_z \rangle$.*

(1) Consider the rational function f in $\mathbb{Q}(t, x, y, z)$ of the form

$$f = \frac{2x-1}{d} + \frac{y}{\sigma_t(d)} + \frac{1}{\sigma_t^3 \sigma_x \sigma_y \sigma_z(d)}$$

where $d = x^2 + 2xy + z^2 + t$. Then $f \in V_{[d]_{G_t}, 1}$ and applying the $(\sigma_x, \sigma_y, \sigma_z)$ -reduction formula to f yields

$$f = \Delta_x(u_0) + \Delta_y(v_0) + \Delta_z(w_0) + \frac{2x-1}{d} + \frac{y}{\sigma_t(d)} + \frac{1}{\sigma_t^3(d)}, \quad (6.3)$$

where

$$u_0 = \frac{1}{\sigma_t^3 \sigma_y \sigma_z(d)}, \quad v_0 = \frac{1}{\sigma_t^3 \sigma_z(d)} \quad \text{and} \quad w_0 = \frac{1}{\sigma_t^3(d)}.$$

Since there is no nonzero integer s such that $\sigma_t^s(d)$ and d are G -equivalent, the equation (6.3) gives a required decomposition for f in Lemma 6.2.

(2) Consider the rational function f in $\mathbb{Q}(t, x, y, z)$ of the form

$$f = \frac{1}{t(t+y+2z)d} + \frac{y+z-1}{(t+3z)\sigma_t(d)} - \frac{y+z}{(t+3z)\sigma_t \sigma_x^3 \sigma_y^2(d)},$$

where $d = 3y + (x+z)^2 + t$. Then $f \in V_{[d]_{G_t}, 1}$ and applying the $(\sigma_x, \sigma_y, \sigma_z)$ -reduction formula to f yields that

$$f = \Delta_x(u_0) + \Delta_y(v_0) + \Delta_z(w_0) + \frac{1}{t(t+y+2z)d} + \frac{1}{(t+3z)\sigma_t(d)}, \quad (6.4)$$

where

$$u_0 = -\sum_{\ell=0}^2 \frac{y+z}{(t+3z)\sigma_t \sigma_x^\ell \sigma_y^2(d)}, \quad v_0 = -\sum_{\ell=0}^1 \frac{y+\ell-2+z}{(t+3z)\sigma_t \sigma_y^\ell(d)} \quad \text{and} \quad w_0 = 0.$$

Since the isotropy group of d in G_t is $G_{t,d} = \langle \sigma_t^3 \sigma_y^{-1}, \sigma_x \sigma_z^{-1} \rangle$, the minimal positive integer s such that $\sigma_t^s(d)$ and d are G -equivalent is $s = 3$. So d and $\sigma_t(d)$ are not G -equivalent. Thus the equation (6.4) gives a required decomposition for f in Lemma 6.2.

6.2 Criteria on the existence of telescopers

Combining Lemmas 6.1 and 6.2, we reduce the existence problem (2.2) to that for rational functions in the form

$$f = \sum_{i=0}^I \frac{a_i}{\sigma_t^i(d)^j}, \quad (6.5)$$

where $j \in \mathbb{N}^+$, $a_i \in \mathbb{K}(t, \hat{\mathbf{x}}_1)[x_1]$, $d \in \mathbb{K}[t, \mathbf{x}]$, $\deg_{x_1}(a_i) < \deg_{x_1}(d)$ and d is irreducible such that $\sigma_t^i(d)$ and $\sigma_t^{i'}(d)$ are not G -equivalent for $0 \leq i \neq i' \leq I$.

Let $G_t = \langle \sigma_t, \sigma_{x_1}, \dots, \sigma_{x_n} \rangle$ and $G = \langle \sigma_{x_1}, \dots, \sigma_{x_n} \rangle$ be a subgroup of G_t . Let G_d and $G_{t,d}$ be the isotropy groups of the polynomial d in G and G_t , respectively. By Lemma 4.4, the quotient group $G_{t,d}/G_d$ is free and of rank 0 or 1.

In the case of $\text{rank}(G_{t,d}/G_d) = 0$, the existence problem of telescopers is equivalent to the summability problem.

Lemma 6.4. *Let $f \in \mathbb{K}(t, \mathbf{x})$ be in the form (6.5). If $\text{rank}(G_{t,d}/G_d) = 0$, then f has a telescoper of type $(\sigma_t; \sigma_{x_1}, \dots, \sigma_{x_n})$ if and only if each $a_i/\sigma_t^i(d)^j$ is $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -summable in $\mathbb{K}(t, \mathbf{x})$ for $0 \leq i \leq I$.*

Proof. Suppose that each $a_i/\sigma_t^i(d)^j$ is $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -summable for $0 \leq i \leq I$. By the linearity of the difference operators Δ_{x_i} , we see that $L = \mathbf{1}$ is a telescoper for f . Conversely, assume that $L = \sum_{\ell=0}^{\rho} e_{\ell} S_t^{\ell}$ with $e_{\ell} \in \mathbb{K}(t)$ is a telescoper of type $(\sigma_t; \sigma_{x_1}, \dots, \sigma_{x_n})$ for f . Without loss of generality, we may suppose that $e_0 \neq 0$. Then we have

$$L(f) = \sum_{\ell=0}^{\rho} \sum_{i=0}^I e_{\ell} \sigma_t^{\ell} \left(\frac{a_i}{\sigma_t^i(d)^j} \right) = \sum_{\ell=0}^{I+\rho} \left(\frac{\sum_{i=0}^{\ell} e_i \sigma_t^i(a_{\ell-i})}{\sigma_t^{\ell}(d)^j} \right)$$

is $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -summable where $e_{\ell} = 0$ if $\ell > \rho$ and $a_i = 0$ if $i > I$. Since $\text{rank}(G_{t,d}/G_d) = 0$, all $\sigma_t^{\ell}(d)$ with $\ell \in \mathbb{Z}$ are in distinct G -orbits. By Lemma 5.1, for any ℓ with $0 \leq \ell \leq \rho$, there exist $g_{\ell,1}, \dots, g_{\ell,n} \in \mathbb{K}(t, \mathbf{x})$ such that

$$\frac{\sum_{i=0}^{\ell} e_i \sigma_t^i(a_{\ell-i})}{\sigma_t^{\ell}(d)^j} = \Delta_{x_1}(g_{\ell,1}) + \dots + \Delta_{x_n}(g_{\ell,n}). \quad (6.6)$$

To show that each $a_i/\sigma_t^i(d)^j$ is $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -summable for $0 \leq i \leq I$, we proceed by induction. For $i = 0$, substituting $\ell = 0$ into (6.6), we get $a_0/d^j = \Delta_{x_1}(g_{0,1}/e_0) + \dots + \Delta_{x_n}(g_{0,n}/e_0)$. Suppose that $a_i/\sigma_t^i(d)^j$ is $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -summable for $i = 0, \dots, s-1$ with $s \leq I$. Taking $\ell = s$ in Equation (6.6) yields that

$$\frac{a_s}{\sigma_t^s(d)^j} = \Delta_{x_1} \left(\frac{g_{s,1}}{e_0} \right) + \dots + \Delta_{x_n} \left(\frac{g_{s,n}}{e_0} \right) - \frac{1}{e_0} \sum_{i=1}^s e_i \sigma_t^i \left(\frac{a_{s-i}}{\sigma_t^{s-i}(d)^j} \right).$$

By the inductive hypothesis, we have $a_{s-i}/\sigma_t^{s-i}(d)^j$ is $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -summable for $1 \leq i \leq s$. Note that $e_i \in \mathbb{K}(t)$ is free of \mathbf{x} . Due to the commutativity between σ_t and σ_{x_i} for $i = 1, \dots, n$, we get $\frac{1}{e_0} \sum_{i=1}^s e_i \sigma_t^i \left(\frac{a_{s-i}}{\sigma_t^{s-i}(d)^j} \right)$ is $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -summable. Hence $a_s/\sigma_t^s(d)^j$ is also $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -summable. \blacksquare

Example 6.5. We continue Example 6.3 (1) and write $f \in \mathbb{Q}(t, x, y, z)$ as

$$f = \Delta_x(u_0) + \Delta_y(v_0) + \Delta_z(w_0) + r \text{ with } r = \frac{2x-1}{d} + \frac{y}{\sigma_t(d)} + \frac{1}{\sigma_t^3(d)},$$

where $u_0, v_0, w_0 \in \mathbb{Q}(t, x, y, z)$ and $d = x^2 + 2xy + z^2 + t$. Note that the isotropy groups $G_{t,d}$ and G_d are trivial. The first term $(2x-1)/d$ of r is not $(\sigma_x, \sigma_y, \sigma_z)$ -summable in $\mathbb{Q}(t, x, y, z)$ by the similar reason as in Example 5.15 (1). Since $\text{rank}(G_{t,d}/G_d) = 0$, we know from Lemma 6.4 that r does not have any telescoper of type $(\sigma_t; \sigma_x, \sigma_y, \sigma_z)$ and neither does f .

Lemma 6.6. Let $f = \sum_{i=0}^I a_i/\sigma_t^i(d)^j \in \mathbb{K}(t, \mathbf{x})$ be in the form (6.5). If $\text{rank}(G_{t,d}/G_d) = 1$, then f has a telescoper of type $(\sigma_t; \sigma_{x_1}, \dots, \sigma_{x_n})$ if and only if each $a_i/\sigma_t^i(d)^j$ has a telescoper of the same type for $0 \leq i \leq I$.

Proof. Sufficiency follows from Remark 2.5. The proof of necessity is a natural generalization from the trivariate case [22, lemma 5.3] to the multivariate case. Suppose $L = \sum_{i=0}^{\ell} e_i S_t^i \in \mathbb{K}(t) \langle S_t \rangle$ is a telescoper for f . Since $\text{rank}(G_{t,d}/G_d) = 1$, there is a minimal positive integer k_0 such that $\sigma_t^{k_0}(d) = \sigma_{x_1}^{k_1} \dots \sigma_{x_n}^{k_n}(d)$ for some integers k_1, \dots, k_n . In the expression (6.5), we require that $\sigma_t^i(d)$ and $\sigma_t^{i'}(d)$ are not G -equivalent for any $0 \leq i \neq i' \leq I$. By the minimality of k_0 , we may assume $f = \sum_{i=0}^{k_0-1} a_i/\sigma_t^i(d)^j$. The k_0 -exponent separation of L (see [22, Section 4]) is defined as follows

$$L = L_0 + L_1 + \dots + L_{k_0-1},$$

where $L_i = \sum_{j=0}^{\ell} e_{jk_0+i} S_t^{jk_0+i}$ and $e_i = 0$ if $i > \ell$. Since $L(f)$ is $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -summable, by Lemma 5.1 each orbital component of $L(f)$ is summable. So we have

$$\begin{cases} L_0 \frac{a_0}{d^j} + L_{k_0-1} \frac{a_1}{\sigma_t(d)^j} + \dots + L_1 \frac{a_{k_0-1}}{\sigma_t^{k_0-1}(d)^j} & \equiv 0 \\ L_1 \frac{a_0}{d^j} + L_0 \frac{a_1}{\sigma_t(d)^j} + \dots + L_2 \frac{a_{k_0-1}}{\sigma_x^{k_0-1}(d)^j} & \equiv 0 \\ \dots & \\ L_{k_0-1} \frac{a_0}{d^j} + L_{k_0-2} \frac{a_1}{\sigma_t(d)^j} + \dots + L_0 \frac{a_{k_0-1}}{\sigma_t^{k_0-1}(d)^j} & \equiv 0, \end{cases} \quad (6.7)$$

where $f \equiv 0$ means that f is $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -summable in $\mathbb{K}(t, \mathbf{x})$. Taking

$$\mathcal{V} = \left[\frac{a_0}{d^j}, \frac{a_1}{\sigma_t(d)^j}, \dots, \frac{a_{k_0-1}}{\sigma_t^{k_0-1}(d)^j} \right]^T,$$

then Equation (6.7) can be written as

$$\mathcal{L}_{k_0} \cdot \mathcal{V} \equiv 0,$$

where

$$\mathcal{L}_{k_0} = \begin{bmatrix} L_0 & L_{k_0-1} & L_{k_0-2} & \dots & L_1 \\ L_1 & L_0 & L_{k_0-1} & \dots & L_2 \\ L_2 & L_1 & L_0 & \dots & L_3 \\ \vdots & \vdots & \vdots & & \vdots \\ L_{k_0-1} & L_{k_0-2} & L_{k_0-3} & \dots & L_0 \end{bmatrix}.$$

According to [22, Proposition 4.3], there exist nonzero operators $T_0, \dots, T_{k_0-1} \in \mathbb{K}(t)\langle S_t \rangle$ and a matrix \mathcal{M} over $\mathbb{K}(t)\langle S_t \rangle$ such that

$$\mathcal{M} \cdot \mathcal{L}_{k_0} = \text{diag}(T_0, \dots, T_{k_0-1}).$$

For each $0 \leq i \leq k_0 - 1$, we know that T_i is a telescoper of type $(\sigma_t; \sigma_{x_1}, \dots, \sigma_{x_n})$ for $a_i/\sigma_t^i(d)^j$, because the operators in $T_i \in \mathbb{K}(t)\langle S_t \rangle$ commute with the difference operators $\Delta_{x_1}, \dots, \Delta_{x_n}$. ■

Now we consider the existence problem of telescopers for simple fractions in the form

$$f = \frac{a}{d^j} \quad (6.8)$$

where $j \in \mathbb{N}^+$, $a \in \mathbb{K}(t, \hat{\mathbf{x}}_1)[x_1]$, $d \in \mathbb{K}[t, \mathbf{x}]$, $\deg_{x_1}(a) < \deg_{x_1}(d)$ and d is irreducible such that $\text{rank}(G_{t,d}/G_d) = 1$.

Theorem 6.7. *Let $f \in \mathbb{K}(t, \mathbf{x})$ be as in (6.8). Let $\{\tau_0, \tau_1, \dots, \tau_r\}$ ($1 \leq r < n$) be a basis of $G_{t,d}$ such that $G_{t,d}/G_d = \langle \tau_0 \rangle$ and $\{\tau_1, \dots, \tau_r\}$ is a basis of G_d (take $\tau_1 = \mathbf{1}$, if $G_d = \{\mathbf{1}\}$). Let $\tau_0 = \sigma_t^{k_0} \sigma_{x_1}^{-k_1} \dots \sigma_{x_n}^{-k_n}$ for some $k_i \in \mathbb{Z}$ and set $T_0 = S_t^{k_0} S_{x_1}^{-k_1} \dots S_{x_n}^{-k_n}$. Then f has a telescoper of type $(\sigma_t; \sigma_{x_1}, \dots, \sigma_{x_n})$ if and only if there exists a nonzero operator $L \in \mathbb{K}(t)\langle T_0 \rangle$ such that*

$$L(a) = \Delta_{\tau_1}(b_1) + \dots + \Delta_{\tau_r}(b_r)$$

for some $b_i \in \mathbb{K}(t, \hat{\mathbf{x}}_1)[x_1]$ with $\deg_{x_1}(b_i) < \deg_{x_1}(d)$ for $1 \leq i \leq r$.

Proof. Firstly, suppose that $L_0 = \sum_{\ell=0}^{\rho} e_{\ell} T_0^{\ell} \in \mathbb{K}(t)\langle T_0 \rangle$ is a nonzero operator such that $L_0(a) = \sum_{i=1}^r \Delta_{\tau_i}(b_i)$ for some $b_i \in \mathbb{K}(t, \hat{\mathbf{x}}_1)[x_1]$ with $\deg_{x_1}(b_i) < \deg_{x_1}(d)$. Set $L = \sum_{\ell=0}^{\rho} e_{\ell} S_t^{\ell k_0}$. Then

$$\begin{aligned}
L(f) &= \sum_{\ell=0}^{\rho} \frac{e_{\ell} \sigma_t^{\ell k_0}(a)}{\sigma_t^{\ell k_0}(d)^j} = \sum_{\ell=0}^{\rho} \frac{e_{\ell} \sigma_t^{\ell k_0}(a)}{\sigma_{x_1}^{\ell k_1} \cdots \sigma_{x_n}^{\ell k_n}(d)^j} \\
&= \sum_{i=1}^n \Delta_{x_i}(g_i) + \frac{\sum_{\ell=0}^{\rho} e_{\ell} \sigma_t^{\ell k_0} \sigma_{x_1}^{-\ell k_1} \cdots \sigma_{x_n}^{-\ell k_n}(a)}{d^j} \quad \text{for some } g_i \in \mathbb{K}(t, \mathbf{x}) \\
&= \sum_{i=1}^n \Delta_{x_i}(g_i) + \frac{L_0(a)}{d^j} \\
&= \sum_{i=1}^n \Delta_{x_i}(g_i) + \frac{1}{d^j} \sum_{i=1}^r (\tau_i(b_i) - b_i) \\
&= \sum_{i=1}^n \Delta_{x_i}(g_i) + \sum_{i=1}^r \left(\tau_i \left(\frac{b_i}{d^j} \right) - \frac{b_i}{d^j} \right) \\
&= \sum_{i=1}^n \Delta_{x_i}(g_i + h_i) \quad \text{for some } h_i \in \mathbb{K}(t, \mathbf{x}).
\end{aligned} \tag{6.9}$$

The last equality follows from Lemma 5.8.

Conversely, let L be a telescoper of type $(\sigma_t; \sigma_{x_1}, \dots, \sigma_{x_n})$ for f . By the k_0 -exponent separation (see [22, Section 4]) of L and Lemma 5.1, without loss of generality, we may assume $L = \sum_{\ell=0}^{\rho} e_{\ell} S_t^{\ell k_0} \in \mathbb{K}(t)\langle S_t \rangle$ is a telescoper for f . Then

$$L\left(\frac{a}{d^j}\right) = \sum_{\ell=0}^{\rho} \frac{e_{\ell} \sigma_t^{\ell k_0}(a)}{\sigma_{x_1}^{\ell k_1} \cdots \sigma_{x_n}^{\ell k_n}(d)^j} = \sum_{i=1}^n \Delta_{x_i}(h_i) + \frac{1}{d^j} h$$

for some $h_1, \dots, h_n, h \in \mathbb{K}(t, \mathbf{x})$ with

$$h = \sum_{\ell=0}^{\rho} e_{\ell} \sigma_t^{\ell k_0} \sigma_{x_1}^{-\ell k_1} \cdots \sigma_{x_n}^{-\ell k_n}(a) = \sum_{\ell=0}^{\rho} e_{\ell} \tau_0^{\ell}(a). \tag{6.11}$$

Since $L(a/d^j)$ is $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -summable and $\{\tau_1, \dots, \tau_r\}$ is a basis of G_d , by Theorem 5.9 with $\mathbb{F} = \mathbb{K}(t)$ we get

$$h = \Delta_{\tau_1}(b_1) + \cdots + \Delta_{\tau_r}(b_r) \tag{6.12}$$

for some $b_i \in \mathbb{K}(t, \hat{\mathbf{x}}_1)[x_1]$ with $\deg_{x_1}(b_i) < \deg_{x_1}(d)$ for $1 \leq i \leq r$. Combining Equations (6.11) and (6.12) yields that a has a telescoper $L_0 = \sum_{\ell=0}^{\rho} e_{\ell} T_0^{\ell}$ of type $(\tau_0; \tau_1, \dots, \tau_r)$. \blacksquare

Proposition 6.8. *Let $\tau \in G_t \setminus G$ and $f = a/b$ with $a, b \in \mathbb{K}[t, \mathbf{x}]$ and $\gcd(a, b) = 1$. Then there exist $e_0, \dots, e_r \in \mathbb{K}(t)$, not all zero, such that $\sum_{i=0}^r e_i \tau^i(f) = 0$ if and only if $b = b_1 b_2$ with $b_1 \in \mathbb{K}[t]$ and $b_2 \in \mathbb{K}[t, \mathbf{x}]$ satisfying that $\tau(b_2) = b_2$.*

Proof. First we suppose $b = b_1 b_2$ with b_1, b_2 satisfying the above conditions. Then for any $i \in \mathbb{N}$,

$$\tau^i(f) = \frac{\tau^i(a)}{\tau^i(b_1 b_2)} = \frac{\tau^i(a)}{\tau^i(b_1) b_2} = \frac{\tau^i(a/b_1)}{b_2}. \tag{6.13}$$

Note that $b_1 \in \mathbb{K}[t]$ and the total degrees of the polynomials $\tau^i(a)$ in \mathbf{x} are the same as that of a . Thus all shifts of a/b_1 lie in a finite dimensional linear space over $\mathbb{K}(t)$. So there exist $e_0, e_1, \dots, e_r \in \mathbb{K}(t)$, not all zero, such that $\sum_{i=0}^r e_i \tau^i(a/b_1) = 0$. This implies $\sum_{i=0}^r e_i \tau^i(f) = 0$.

Conversely, suppose $\sum_{i=0}^r e_i \tau^i(f) = 0$. Let b_1 and b_2 be the content and primitive part of b as a polynomial in \mathbf{x} over $\mathbb{K}(t)$. If $b_2 \in \mathbb{K}$, then we are done. Now we assume that $b_2 \notin \mathbb{K}$. Then all of its irreducible factors have positive total degrees in \mathbf{x} . Assume that there exists an irreducible polynomial p such that $\tau(p) \neq p$. By Lemma 4.3, the quotient group $G_t/G_{t,p}$ is free, so is torsion free. So for any integer $i \neq 0$, $\tau^i(p) \neq p$. Among all of such irreducible factors of b_2 , we can find one factor p such that $\tau^i(p) \nmid b_2$ for any integer $i < 0$. Let s be the largest integer such that $\tau^s(p) \mid b_2$. Then the irreducible polynomial $\tau^{r+s}(p)$ divides $\tau^r(b_2)$, but $\tau^{r+s}(p) \nmid \tau^i(b_2)$ for any $0 \leq i \leq r-1$. Otherwise $\tau^{r+s-i}(p) \mid b_2$, which contradicts the choice of s . Therefore we have $\sum_{i=0}^r e_i \tau^i(f) \neq 0$, since p depends on \mathbf{x} and the coefficients e_i are in $\mathbb{K}(t)$. This leads to a contradiction. So every irreducible factor p of b_2 satisfies the property that $\tau(p) = p$. This implies that $\tau(b_2) = b_2$. ■

Lemma 6.9. *Let $\tau \in G_t \setminus G$ and $f = a/(b_1 b_2)$ with $b_1 \in \mathbb{K}[t]$, $a, b_2 \in \mathbb{K}[t, \mathbf{x}]$ and $\tau(b_2) = b_2$. Then we can compute $e_0, \dots, e_r \in \mathbb{K}[t]$, not all zero, such that $\sum_{i=0}^r e_i \tau^i(f) = 0$.*

Proof. By Proposition 6.12 below, we can construct a difference isomorphism between $(\mathbb{K}(t, \mathbf{x}), \tau)$ and $(\mathbb{K}(t, \mathbf{x}), \sigma_t)$ such that $\varphi \circ \tau = \sigma_t \circ \varphi$ and $\varphi(\mathbb{K}[t]) \subseteq \mathbb{K}[t]$. Then $\sigma_t(\varphi(b_2)) = \varphi(\tau(b_2)) = \varphi(b_2)$ and for all $e_i(t) \in \mathbb{K}[t]$,

$$\sum_{i=0}^r e_i(t) \tau^i(f) = 0 \quad \text{if and only if} \quad \sum_{i=0}^r e_i(\varphi(t)) \sigma_t^i(\varphi(f)) = 0.$$

So we only need to consider the case $\tau = \sigma_t$. Now suppose that $f = a/(b_1 b_2)$ with $b_1 \in \mathbb{K}[t]$, $a, b_2 \in \mathbb{K}[t, \mathbf{x}]$ and $\sigma_t(b_2) = b_2$. It suffices to find a nonzero operator $L \in \mathbb{K}[t]\langle S_t \rangle$ such that $L(f) = 0$. We write $a = \sum_{i=0}^s a_i t^i$ with $a_i \in \mathbb{K}[\mathbf{x}]$. For each $0 \leq i \leq s$, let $L_i = t^i \sigma_t(b_1) S_t - (t+1)^i b_1$. Then L_i is an operator in $\mathbb{K}[t]\langle S_t \rangle$ such that $L_i(t^i/b_1) = 0$. Since $\sigma_t(b_2) = b_2$ and $\sigma_t(a_i) = a_i$, we have $L_i(a_i t^i/(b_1 b_2)) = (a_i/b_2) L_i(t^i/b_1) = 0$. Let $L \in \mathbb{K}[t]\langle S_t \rangle$ be the LCLM of L_i for all $i = 0, \dots, s$ and write $L = R_i L_i$ with $R_i \in \mathbb{K}[t]\langle S_t \rangle$. Then L is a nonzero operator and

$$L(f) = L \left(\sum_{i=0}^s \frac{a_i t^i}{b_1 b_2} \right) = \sum_{i=0}^s L \left(\frac{a_i t^i}{b_1 b_2} \right) = \sum_{i=0}^s R_i L_i \left(\frac{a_i t^i}{b_1 b_2} \right) = 0.$$

This completes the proof. ■

Remark 6.10. *For the bivariate case with $m = n = 1$, our existence criterion coincides with the known result in [6, Theorem 1] and [26, Theorem 4.11]. Let $f = a/d^j \in \mathbb{K}(t, \mathbf{x})$, where $j \in \mathbb{N}^+$, $a \in \mathbb{K}(t, \hat{\mathbf{x}}_1)[x_1]$, $d \in \mathbb{K}[t, \mathbf{x}]$, $\deg_{x_1}(a) < \deg_{x_1}(d)$ and d is irreducible. Let $G_t = \langle \sigma_t; \sigma_{x_1} \rangle$ and $G = \langle \sigma_{x_1} \rangle$. Since the degree of d in x_1 is positive, we have $G_d = \{\mathbf{1}\}$. If $\text{rank}(G_{t,d}/G_d) = 0$, then by Lemma 6.4 and Theorem 5.9, f has a telescoper of type (σ_t, σ_{x_1}) if and only if $a = 0$. If $\text{rank}(G_{t,d}/G_d) = 1$, then there exists $\tau = \sigma_t^s \sigma_{x_1}^k \in G_{t,d}$ with $s > 0$ such that $G_{t,d}/G_d = \langle \bar{\tau} \rangle$. By Theorem 6.7 and Proposition 6.8, f has a telescoper of type $(\sigma_t; \sigma_{x_1})$ if and only if $a = c/b$ with $c \in \mathbb{K}[t, \mathbf{x}]$, $b \in \mathbb{K}[t, \hat{\mathbf{x}}_1]$, $\text{gcd}(b, c) = 1$, where b can be written as $b = b_1 b_2$ with $b_1 \in \mathbb{K}[t]$ and $b_2 \in \mathbb{K}[t, \hat{\mathbf{x}}_1]$ such that $\tau(b_2) = b_2$. Since $m = n = 1$, we have $b \in \mathbb{K}[t]$ and hence f always has a telescoper of type $(\sigma_t; \sigma_{x_1})$.*

Example 6.11. *Let $f = 1/(t^s + x_1^s + \dots + x_n^s) \in \mathbb{Q}(t, x_1, \dots, x_n)$ with $s, n \in \mathbb{N} \setminus \{0\}$. Then $d = t^s + x_1^s + \dots + x_n^s$ is irreducible over \mathbb{Q} if $n > 1$. Let $G_{t,d}$ and G_d be the isotropy groups of d in $G_t = \langle \sigma_t, \sigma_{x_1}, \dots, \sigma_{x_n} \rangle$ and $G = \langle \sigma_{x_1}, \dots, \sigma_{x_n} \rangle$, respectively. Then we can decide the existence of telescopers of type $(\sigma_t; \sigma_{x_1}, \dots, \sigma_{x_n})$ for all cases of f .*

- (1) If $s = 1$, then d is irreducible. Since $G_{t,d} = \langle \tau \rangle$ with $\tau = \sigma_t \sigma_{x_1}^{-1}$ and $G_d = \{\mathbf{1}\}$, we have $G_{t,d}/G_d = \langle \bar{\tau} \rangle$ and $\text{rank}(G_{t,d}/G_d) = 1$. Observing that $\tau - 1$ is an annihilator of the numerator of f , by Theorem 6.7 we get f has a telescoper. In fact, $L(f) = \Delta_{x_1}(f) + \Delta_{x_2}(0) + \cdots + \Delta_{x_n}(0)$, where $L = S_t - 1$.
- (2) If $s > 1$ and $n = 1$, then $f = 1/(t^s + x_1^s) = \sum_{j=1}^s a_j/(t - \beta_j x_1)$, where β_j 's are distinct roots of $z^s = -1$ and $a_j = 1/(s(\beta_j x_1)^{s-1})$. There exists $j \in \{1, \dots, s\}$ such that $\beta^j \notin \mathbb{Z}$. Then for $d_j = t - \beta_j x_1$, we have $G_{t,d_j} = G_{d_j} = \{\mathbf{1}\}$. So a_j/d_j is not σ_{x_1} -summable in $\mathbb{C}(t, x_1)$ and neither is f . By Lemma 6.4, we get that f does not have any telescoper of type (σ_t, σ_{x_1}) in $\mathbb{C}(t)\langle S_t \rangle$. Hence f does not have any telescoper of the same type in $\mathbb{Q}(t)\langle S_t \rangle$.
- (3) If $s > 1$ and $n > 1$, then d is irreducible. Since $G_d = \{\mathbf{1}\}$, f is not $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -summable by Theorem 5.9. Since $G_{t,d} = \{\mathbf{1}\}$ and $\text{rank}(G_{t,d}/G_d) = 0$, by Lemma 6.4, we conclude that f does not have any telescoper.

Proposition 6.12. Let $\{\tau_0, \tau_1, \dots, \tau_r\}$ ($1 \leq r \leq n$) be a family of \mathbb{Z} -linearly independent elements in G_t such that $\tau_0 \in G_t \setminus G$ and $\{\tau_1, \dots, \tau_r\} \subseteq G$. Then there exists a \mathbb{K} -automorphism φ of $\mathbb{K}(t, \mathbf{x})$ such that φ is a difference isomorphism between the difference fields $(\mathbb{K}(t, \mathbf{x}), \tau_0)$ and $(\mathbb{K}(t, \mathbf{x}), \sigma_t)$, and simultaneously a difference isomorphism between $(\mathbb{K}(t, \mathbf{x}), \tau_i)$ and $(\mathbb{K}(t, \mathbf{x}), \sigma_{x_i})$ for all $i = 1, \dots, r$. Furthermore, $\varphi(\mathbb{K}(t)) \subseteq \mathbb{K}(t)$ and hence for any $f \in \mathbb{K}(t, \mathbf{x})$, f has a telescoper of type $(\tau_0; \tau_1, \dots, \tau_r)$ if and only if $\varphi(f)$ has a telescoper of type $(\sigma_t; \sigma_{x_1}, \dots, \sigma_{x_r})$.

Proof. Let $\tau_i = \sigma_t^{\alpha_{i,0}} \sigma_{x_1}^{\alpha_{i,1}} \cdots \sigma_{x_m}^{\alpha_{i,m}}$, where $\alpha_{i,j} = 0$ if $j > n$. Define $\alpha_i = (\alpha_{i,0}, \alpha_{i,1}, \dots, \alpha_{i,m}) \in \mathbb{Z}^{m+1}$ for $i = 0, 1, \dots, r$. Since $\alpha_0, \alpha_1, \dots, \alpha_r$ are linearly independent over \mathbb{Q} , we can find vectors $\alpha_{r+1}, \dots, \alpha_m \in \mathbb{Q}^{m+1}$ such that $\{\alpha_0, \alpha_1, \dots, \alpha_m\}$ is a basis of \mathbb{Q}^{m+1} over \mathbb{Q} . Write $\alpha_i = (\alpha_{i,0}, \alpha_{i,1}, \dots, \alpha_{i,m})$ for $i = r+1, \dots, m$. Since $\tau_0 \in G_t \setminus G$ and $\{\tau_1, \dots, \tau_r\} \subseteq G$, we have $\alpha_{0,0} \neq 0$ and $\alpha_{i,0} = 0$ for $i = 1, \dots, r$. So we can further assume that $\alpha_{i,0} = 0$ for $i = r+1, \dots, m$. Let $A = (\alpha_{i,j}) \in \mathbb{Q}^{(m+1) \times (m+1)}$ which is invertible. Let φ be a \mathbb{K} -automorphism of $\mathbb{K}(t, \mathbf{x})$ defined by

$$(\varphi(t), \varphi(x_1), \dots, \varphi(x_m)) := (t, x_1, \dots, x_m)A.$$

Then $\varphi(t) = \alpha_{0,0} \cdot t$ and $\varphi(x_j) = \alpha_{0,j} \cdot t + \sum_{i=1}^m \alpha_{i,j} \cdot x_i$ for $j = 1, \dots, m$. It can be checked that $\varphi \circ \tau_0 = \sigma_t \circ \varphi$ and $\varphi \circ \tau_i = \sigma_{x_i} \circ \varphi$ for $1 \leq i \leq r$. This means the following diagrams are commutative.

$$\begin{array}{ccc} \mathbb{K}(t, \mathbf{x}) & \xrightarrow{\varphi} & \mathbb{K}(t, \mathbf{x}) \\ \tau_0 \downarrow & & \downarrow \sigma_t \\ \mathbb{K}(t, \mathbf{x}) & \xrightarrow{\varphi} & \mathbb{K}(t, \mathbf{x}) \end{array} \quad \cdots \quad \begin{array}{ccc} \mathbb{K}(t, \mathbf{x}) & \xrightarrow{\varphi} & \mathbb{K}(t, \mathbf{x}) \\ \tau_i \downarrow & & \downarrow \sigma_{x_i} \\ \mathbb{K}(t, \mathbf{x}) & \xrightarrow{\varphi} & \mathbb{K}(t, \mathbf{x}) \end{array}$$

Note that $\varphi(\mathbb{K}(t)) \subseteq \mathbb{K}(t)$. It follows that

$$\sum_{\ell=0}^{\rho} e_{\ell}(t) \tau_0^{\ell}(f) = \sum_{i=1}^r \Delta_{\tau_i}(g_i) \quad \text{if and only if} \quad \sum_{\ell=0}^{\rho} e_{\ell}(\alpha_{0,0} t) \sigma_t^{\ell}(\varphi(f)) = \sum_{i=1}^r \Delta_{x_i}(\varphi(g_i)),$$

whenever $e_{\ell}(t) \in \mathbb{K}(t)$ and $f, g_i \in \mathbb{K}(t, \mathbf{x})$. This completes our proof. \blacksquare

The way of constructing a difference isomorphism in Proposition 6.12 is almost the same as that in Proposition 5.12. The only difference is that in Proposition 6.12, we require $\varphi(\mathbb{K}(t)) \subseteq \mathbb{K}(t)$.

Let $f = a/d^j$ be in the form (6.8) with $\text{rank}(G_{t,d}/G_d) = 1$. By Theorem 6.7, there are two cases according to whether G_d is trivial or not. If $G_d = \{\mathbf{1}\}$, then a/d^j has a telescoper of type

$(\sigma_t; \sigma_{x_1}, \dots, \sigma_{x_n})$ if and only if there exists a nonzero operator $L \in \mathbb{K}(t)\langle T_0 \rangle$ such that $L(a) = 0$. This problem is solved by Proposition 6.8. If G_d is nontrivial, we can apply the transformation in Proposition 6.12 to reduce the existence problem of telescopers to that of fewer variables. Moreover, the general existence of telescopers of type $(\tau_0; \tau_1, \dots, \tau_n)$ for rational functions has also been solved.

Similar to the treatment in Algorithm 5.13, if f has a telescoper of type $(\sigma_t; \sigma_{x_1}, \dots, \sigma_{x_n})$, Algorithm 6.13 will output *unnormalised certificates* for f in the form

$$g = \sum_{\ell=1}^{\rho} \prod_{k=1}^{K_{\ell}} \psi_{\ell,k}(v_{\ell,k}),$$

where $v_{\ell,k} \in \mathbb{F}(t, \mathbf{x})$ and the $\psi_{\ell,k}$'s are \mathbb{Q} -affine maps.

Algorithm 6.13 (Constructive Testing of the Existence of Telescopers).

IsTelescopable $(f, [x_1, \dots, x_n], t)$.

INPUT: a multivariate rational function $f \in \mathbb{K}(t, \mathbf{x})$, a set $\{x_1, \dots, x_n\}$ of variable names and a variable name t for telescoping;

OUTPUT: a telescoper L and its unnormalised certificates g_1, \dots, g_n if f has a telescoper of type $(\sigma_t; \sigma_{x_1}, \dots, \sigma_{x_n})$; false otherwise.

- 1 using shift equivalence testing and irreducible partial fraction decomposition, decompose f into $f = f_0 + \sum_{j \in \mathbb{N}^+} \sum_{[d]_{G_t}} f_{[d]_{G_t}, j}$ as in Equation (6.1).
- 2 apply the reduction to f_0 and each nonzero component $f_{[d]_{G_t}, j}$ such that

$$f = \Delta_{x_1}(g_1) + \dots + \Delta_{x_n}(g_n) + r \text{ with } r = \sum_{i=1}^I \sum_{j=1}^{J_i} \sum_{\ell=0}^{s_{i,j}} \frac{a_{i,j,\ell}}{\sigma_t^{\ell}(d_i)^j},$$

where $\sum_{\ell=0}^{s_{i,j}} \frac{a_{i,j,\ell}}{\sigma_t^{\ell}(d_i)^j}$ is the remainder of $f_{[d_i]_{G_t}, j}$ described in Lemma 6.2.

- 3 if $r = 0$, then **return** $L = \mathbf{1}$ and g_1, \dots, g_n .
- 4 **for** $i = 1, \dots, I$ **do**
- 5 using Remark 4.5, one can find elements $\tau_{i,0}, \tau_{i,1}, \dots, \tau_{i,r_i} \in G_{t,d_i}$ such that $G_{t,d_i}/G_{d_i} = \langle \bar{\tau}_{i,0} \rangle$ and $\{\tau_{i,1}, \dots, \tau_{i,r_i}\}$ forms a basis for G_{d_i} .
- 6 **for** $j = 1, \dots, J_i$, $0 = 1, \dots, s_{i,j}$ **do**
- 7 if $\text{rank}(G_{t,d_i}/G_{d_i}) = 0$ then
- 8 execute Algorithm 5.13 with **IsSummable** $(r_{i,j,\ell}, [x_1, \dots, x_n])$, where $r_{i,j,\ell} := \frac{a_{i,j,\ell}}{\sigma_t^{\ell}(d_i)^j}$.
- 9 if $r_{i,j,\ell}$ is $(\sigma_{x_1}, \dots, \sigma_{x_n})$ -summable in $\mathbb{F}(\mathbf{x})$, let

$$r_{i,j,\ell} = \Delta_{x_1} \left(h_{i,j,\ell}^{(1)} \right) + \dots + \Delta_{x_n} \left(h_{i,j,\ell}^{(n)} \right)$$

and set $L_{i,j,\ell} = \mathbf{1}$; **return** false otherwise.

- 10 if $\text{rank}(G_{t,d_i}/G_{d_i}) = 1$ then
- 11 choose $\tau_{i,0} = \sigma_t^{k_{i,0}} \sigma_{x_1}^{-k_{i,1}} \dots \sigma_{x_n}^{-k_{i,n}}$ with $k_{i,0} > 0$.
- 12 set $T_{i,0} = S_t^{k_{i,0}} S_{x_1}^{-k_{i,1}} \dots S_{x_n}^{k_{i,n}}$.
- 13 if $n = 1$ or $G_{d_i} = \{\mathbf{1}\}$ then
- 14 using Proposition 6.8, check whether there exists a nonzero operator $\bar{L}_{i,j,\ell}(t, T_{i,0}) \in \mathbb{K}(t)\langle T_{i,0} \rangle$ such that $\bar{L}_{i,j,\ell}(t, T_{i,0})(a_{i,j,\ell}) = 0$. If so, use Lemma 6.9 to find such an operator $\bar{L}_{i,j,\ell}(t, T_{i,0})$ and set $L_{i,j,\ell}(t, S_t) = \bar{L}_{i,j,\ell}(t, S_t^{k_{i,0}})$. By Equation (6.9) we obtain

$$L_{i,j,\ell} \left(\frac{a_{i,j,\ell}}{\sigma_t^{\ell}(d_i)^j} \right) = \sum_{\lambda=1}^n \Delta_{x_{\lambda}} \left(h_{i,j,\ell}^{(\lambda)} \right) + \underbrace{\frac{\bar{L}_{i,j,\ell}(a_{i,j,\ell})}{\sigma_t^{\ell}(d_i)^j}}_{=0};$$

15 **return** false otherwise.
 else
 16 find a \mathbb{K} -automorphism φ_i of $\mathbb{K}(t, \mathbf{x})$ given in Proposition 6.12 such that $\varphi_i \circ \tau_{i,0} = \sigma_t \circ \varphi_i$ and $\varphi_i \circ \tau_{i,\lambda} = \sigma_{x_\lambda} \circ \varphi_i$ for $\lambda = 1, \dots, r_i$.
 17 set $\tilde{a}_{i,j,\ell} = \varphi_i(a_{i,j,\ell})$.
 18 execute **IsTelescopable**($\tilde{a}_{i,j,\ell}, [x_1, \dots, x_{r_i}], t$).
 19 if $\tilde{a}_{i,j,\ell}$ has a telescoper of type $(\sigma_t; \sigma_{x_1}, \dots, \sigma_{x_{r_i}})$, let

$$\tilde{L}_{i,j,\ell}(t, S_t)(\tilde{a}_{i,j,\ell}) = \sum_{\lambda=1}^{r_i} \Delta_{x_\lambda} \left(\tilde{b}_{i,j,\ell}^{(\lambda)} \right);$$

return false otherwise.
 20 apply φ_i^{-1} to both sides of the previous equation to get

$$\bar{L}_{i,j,\ell}(t, T_{i,0})(a_{i,j,\ell}) = \sum_{\lambda=1}^{r_i} \Delta_{\tau_{i,\lambda}} \left(b_{i,j,\ell}^{(\lambda)} \right),$$

where $\bar{L}_{i,j,\ell}(t, T_{i,0}) = \tilde{L}_{i,j,\ell}(t/k_{i,0}, T_{i,0})$ and $b_{i,j,\ell}^{(\lambda)} = \varphi_i^{-1}(\tilde{b}_{i,j,\ell}^{(\lambda)})$ for all $\lambda = 1, \dots, r_i$.
 21 set $L_{i,j,\ell}(t, S_t) = \bar{L}_{i,j,\ell}(t, S_t^{k_{i,0}})$ and by Equations (6.9) and (6.10) we obtain

$$\begin{aligned} L_{i,j,\ell} \left(\frac{a_{i,j,\ell}}{\sigma_t^\ell(d_i)^j} \right) &= \sum_{\lambda=1}^n \Delta_{x_\lambda} \left(u_{i,j,\ell}^{(\lambda)} \right) + \frac{\bar{L}_{i,j,\ell}(a_{i,j,\ell})}{\sigma_t^\ell(d_i)^j} \\ &= \sum_{\lambda=1}^n \Delta_{x_\lambda} \left(h_{i,j,\ell}^{(\lambda)} \right) \end{aligned}$$

for some $u_{i,j,\ell}^{(\lambda)}, h_{i,j,\ell}^{(\lambda)} \in \mathbb{K}(t, \mathbf{x})$.
 22 let $L \in \mathbb{K}(t)\langle S_t \rangle$ be the LCLM of $L_{i,j,\ell}$ for all i, j, ℓ and write

$$L = R_{i,j,\ell} L_{i,j,\ell}$$

for some $R_{i,j,\ell} \in \mathbb{K}(t)\langle S_t \rangle$.

23 update $g_\lambda = L(g_\lambda) + \sum_{i=1}^I \sum_{j=1}^{J_i} \sum_{\ell=0}^{s_{i,j}} R_{i,j,\ell}(h_{i,j,\ell}^{(\lambda)})$ for all $\lambda = 1, \dots, n$.

24 **return** L and g_1, \dots, g_n .

Example 6.14. Let $\mathbb{K} = \mathbb{Q}$ and $f \in \mathbb{Q}(t, x, y, z)$. We will decide constructively the existence of telescopers of type $(\sigma_t; \sigma_x, \sigma_y, \sigma_z)$ for various cases of f . Let $G_t = \langle \sigma_t, \sigma_x, \sigma_y, \sigma_z \rangle$ and $G = \langle \sigma_x, \sigma_y, \sigma_z \rangle$.

(1) For $f = \frac{1}{(t+1)(t+2z)d}$ with $d = (t-3y+x)^2(t+y)(t+z) + 1$, a basis of the isotropy group $G_{t,d}$ is $\{\tau_0\}$, where $\tau_0 = \sigma_t \sigma_x^{-4} \sigma_y^{-1} \sigma_z^{-1}$. Then $G_{t,d}/G_d = \langle \bar{\tau}_0 \rangle$. Since $\text{rank}(G_{t,d}/G_d) = 1$ and $G_d = \{\mathbf{1}\}$ is a trivial group, we know from Theorem 6.7 that f has a telescoper of type $(\sigma_t; \sigma_x, \sigma_y, \sigma_z)$ if and only if there exists a nonzero operator $L_0 \in \mathbb{Q}(t)\langle T_0 \rangle$ with $T_0 = S_t S_x^{-4} S_y^{-1} S_z^{-1}$ such that

$$L_0(a) = 0, \text{ where } a = fd = \frac{1}{(t+1)(t+2z)}.$$

Note that the prime part of the denominator $b = (t+1)(t+2z)$ of a with respect to the variables $\{x, y, z\}$ is $b_2 = t+2z$ and $\tau_0(b_2) \neq b_2$. By Proposition 6.8, there does not exist any operator $L_0 \in \mathbb{Q}(t)\langle T_0 \rangle$ such that $L_0(a) = 0$. So f does not have any telescoper of type $(\sigma_t; \sigma_x, \sigma_y, \sigma_z)$.

(2) For $f = \frac{1}{(t+1)d}$ with d being the same as in Example 6.14 (1), it is easy to check that for $a = \frac{1}{t+1}$,

$$L_0(a) = 0 \text{ with } L_0 = T_0 - \frac{t+1}{t+2},$$

where $T_0 = S_t S_x^{-4} S_y^{-1} S_z^{-1}$. So by Theorem 6.7, f has a telescoper L of type $(\sigma_t; \sigma_x, \sigma_y, \sigma_z)$. In fact, we can take $L = S_t - \frac{t+1}{t+2}$. Then

$$\begin{aligned} L(f) &= \frac{\sigma_t(a)}{\sigma_t(d)} - \frac{t+1}{t+2} \cdot \frac{a}{d} = \frac{\sigma_t(a)}{\sigma_x^4 \sigma_y \sigma_z(d)} - \frac{t+1}{t+2} \cdot \frac{a}{d} \\ &= \Delta_x(u) + \Delta_y(v) + \Delta_z(w) + \underbrace{\frac{\tau_0(a) - ((t+1)/(t+2))a}{d}}_{=L_0(a)/d=0} \\ &= \Delta_x(u) + \Delta_y(v) + \Delta_z(w), \end{aligned}$$

where $u = \sum_{\ell=0}^3 \frac{\sigma_t(a)}{\sigma_x^\ell \sigma_y \sigma_z(d)}$, $v = \frac{\sigma_t(a)}{\sigma_z(d)}$, and $w = \frac{\sigma_t(a)}{d}$. Additionally, this is a non-trivial example in two respects. Firstly, since $G_d = \{\mathbf{1}\}$, this rational function f is not $(\sigma_x, \sigma_y, \sigma_z)$ -summable in $\mathbb{Q}(t, x, y, z)$. Secondly, for any $\{\mu, \nu\} \subseteq \{x, y, z\}$, since the isotropy group of d in $\langle \sigma_t, \sigma_\mu, \sigma_\nu \rangle$ is trivial and f is not (σ_μ, σ_ν) -summable, by Lemma 6.4, f does not have any telescoper in $\mathbb{Q}(t)\langle S_t \rangle$ of type $(\sigma_t; \sigma_\mu, \sigma_\nu)$.

(3) We continue Example 6.3 (2) and write f in the form

$$f = \Delta_x(u_0) + \Delta_y(v_0) + \Delta_z(w_0) + r_1 + r_2,$$

where $u_0, v_0, w_0 \in \mathbb{Q}(t, x, y, z)$ and $r_1 = \frac{1}{t(t+y+2z)d}$, $r_2 = \frac{1}{(t+3z)\sigma_t(d)}$ with $d = 3y + (x+z)^2 + t$.

(a) For $r_1 = a_1/d$ with $a_1 = 1/(t(t+y+2z))$, we find that a basis of $G_{t,d}$ is $\{\tau_0, \tau_1\}$, where $\tau_0 = \sigma_t^3 \sigma_y^{-1}$ and $\tau_1 = \sigma_x \sigma_z^{-1}$. Then by Theorem 6.7, r_1 has a telescoper of type $(\sigma_t; \sigma_x, \sigma_y, \sigma_z)$ if and only if a_1 has a telescoper of type $(\tau_0; \tau_1)$. Choose one \mathbb{Q} -automorphism ϕ_1 of $\mathbb{Q}(t, x, y, z)$ given in Proposition 6.12 as follows

$$\phi_1(h(t, x, y, z)) = h(3t, x, -t+y, -x+z),$$

for any $h \in \mathbb{Q}(x, y, z)$. Then $\phi_1 \circ \tau_0 = \sigma_t \circ \phi_1$ and $\phi_1 \circ \tau_1 = \sigma_x \circ \phi_1$. So a_1 has a telescoper of type $(\tau_0; \tau_1)$ if and only if $\phi_1(a_1)$ has a telescoper of type $(\sigma_t; \sigma_x)$. A direct calculation yields that

$$\phi_1(a_1) = \frac{1}{\underbrace{3t(2t+y-2x+2z)}_{\tilde{d}}}.$$

Again consider the isotropy group of \tilde{d} in $\langle \sigma_t, \sigma_x \rangle$, which is generated by $\tilde{\tau}_0 = \sigma_t \sigma_x^2$. Since $(\tilde{\tau}_0 - \frac{t}{t+1})(\frac{1}{3t}) = 0$, by similar arguments used in Example 6.14 (2), we see that $\phi_1(a_1)$ has a telescoper $\tilde{L}_1 \in \mathbb{Q}(t)\langle S_t \rangle$ of type $(\sigma_t; \sigma_x)$ and in particular we find

$$\tilde{L}_1(t, S_t)(\phi_1(a_1)) = \Delta_x(\tilde{b}_1) \tag{6.14}$$

with $\tilde{L}_1 = S_t - \frac{t}{t+1}$ and $\tilde{b}_1 = -\frac{1}{3(t+1)(2t+y-2x+2z)}$. So by Theorem 6.7, r_1 has a telescoper $L \in \mathbb{Q}(t)\langle S_t \rangle$ of type $(\sigma_t; \sigma_x, \sigma_y, \sigma_z)$. In fact, we can find an explicit expression for L . Applying ϕ_1^{-1} to Equation (6.14) yields that

$$\bar{L}_1(t, T_0)(a_1) = \Delta_{\tau_1}(b_1),$$

where $T_0 = S_t^3 S_y^{-1}$, $\bar{L}_1(t, T_0) = \tilde{L}_1(\frac{t}{3}, T_0) = T_0 - \frac{t}{t+3}$, $b_1 = \phi_1^{-1}(\tilde{b}_1) = -\frac{1}{(t+3)(t+y+2z+2)}$.
Let $L_1(t, S_t) = \bar{L}_1(t, S_t^3) = S_t^3 - \frac{t}{t+3}$. Then we have

$$\begin{aligned} L_1(r_1) &= \frac{\sigma_t^3(a_1)}{\sigma_t^3(d)} - \frac{t}{t+3} \cdot \frac{a_1}{d} = \frac{\sigma_t^3(a_1)}{\sigma_y(d)} - \frac{t}{t+3} \cdot \frac{a_1}{d} \\ &= \Delta_x(0) + \Delta_y(v_1) + \Delta_z(0) + \frac{\bar{L}_1(a_1)}{d} \text{ with } v_1 = \frac{\sigma_t^3 \sigma_y^{-1}(a_1)}{d} \end{aligned}$$

and using Lemma 5.8 with $\tau = \tau_1$, we get

$$\frac{\bar{L}_1(a_1)}{d} = \Delta_{\tau_1} \left(\frac{b_1}{d} \right) = \Delta_x(u_1) + \Delta_y(0) + \Delta_z(w_1)$$

with $u_1 = \sigma_z^{-1} \left(\frac{b_1}{d} \right)$ and $w_1 = -\sigma_z^{-1} \left(\frac{b_1}{d} \right)$. Hence L_1 is a telescoper of type $(\sigma_t; \sigma_x, \sigma_y, \sigma_z)$ for r_1 and

$$L_1(r_1) = \Delta_x(u_1) + \Delta_y(v_1) + \Delta_z(w_1).$$

(b) Similarly, for $r_2 = a_2/\sigma_t(d)$ with $a_2 = 1/(t+3z)$, we apply the algorithm IsTelescopable to r_2 . The result is true and we obtain

$$L_2(r_2) = \Delta_x(u_2) + \Delta_y(v_2) + \Delta_z(w_2),$$

where $L_2 = S_t^3 - 1$, $u_2 = \sigma_z^{-1} \left(\frac{b_2}{\sigma_t(d)} \right)$, $v_2 = \frac{\sigma_t^3 \sigma_y^{-1}(a_2)}{\sigma_t(d)}$ and $w_2 = -\sigma_z^{-1} \left(\frac{b_2}{\sigma_t(d)} \right)$ with $b_2 = -\frac{1}{t+3z+3}$.

(c) For $r = r_1 + r_2$, using the LCLM algorithm to compute the least common multiple L of L_1 and L_2 in $\mathbb{Q}(t)\langle S_t \rangle$, we obtain

$$L = R_1 L_1 = R_2 L_2 = S_t^6 - \frac{2(t+3)}{t+6} S_t^3 + \frac{t}{t+6}$$

with $R_1 = S_t^3 - \frac{t+3}{t+6}$ and $R_2 = S_t^3 - \frac{t}{t+6}$. Then

$$L(r) = \Delta_x(u) + \Delta_y(v) + \Delta_z(w),$$

where $u = \sum_{i=1}^2 R_i(u_i)$, $v = \sum_{i=1}^2 R_i(v_i)$ and $w = \sum_{i=1}^2 R_i(w_i)$ are rational functions in $\mathbb{Q}(t, x, y, z)$. Updating $u = u + L(u_0)$, $v = v + L(v_0)$ and $w = w + L(w_0)$, we get

$$L(f) = \Delta_x(u) + \Delta_y(v) + \Delta_z(w).$$

So L is a telescoper of type $(\sigma_t; \sigma_x, \sigma_y, \sigma_z)$ for f .

6.3 Examples and applications

Creative telescoping is a powerful tool for proving combinatorial identities algorithmically [65]. The following example shows an application of telescopers for multivariate rational functions.

Example 6.15. We show that

$$F(t) = \sum_{x=0}^t \sum_{y=0}^t \sum_{z=0}^t f(t, x, y, z) = 0,$$

where

$$f(t, x, y, z) = \frac{(2y - t)(2x - t)(2z - t)}{(y + t + 1)(-2t + y - 1)(x + t + 1)(-2t + x - 1)(z + t + 1)(-2t + z - 1)}.$$

Applying Algorithm 6.13 to f , we find that f has a telescoper $L = S_t - 1$ of type $(\sigma_t; \sigma_x, \sigma_y, \sigma_z)$ with certificates (u, v, w) , where

$$u = \frac{(-2y + t + 1)(-2z + t + 1)(8t^2 - 2tx - x^2 + 19t - 2x + 11)}{(x + t + 1)(2t - x + 2)(2t - x + 3)(y + t + 2)(2t - y + 3)(z + t + 2)(2t - z + 3)},$$

$$v = \frac{(-2x + t)(-2z + t + 1)(8t^2 - 2ty - y^2 + 19t - 2y + 11)}{(x + t + 1)(2t - x + 1)(y + t + 1)(2t - y + 2)(2t - y + 3)(z + t + 2)(2t - z + 3)}$$

and

$$w = \frac{(-2x + t)(-2y + t)(8t^2 - 2tz - z^2 + 19t - 2z + 11)}{(x + t + 1)(2t - x + 1)(y + t + 1)(2t - y + 1)(z + t + 1)(2t - z + 2)(2t - z + 3)}.$$

Thus we have

$$\begin{aligned} & \sum_{x=0}^t \sum_{y=0}^t \sum_{z=0}^t (f(t+1, x, y, z) - f(t, x, y, z)) \\ &= \sum_{x=0}^t \sum_{y=0}^t \sum_{z=0}^t (\Delta_x(u) + \Delta_y(v) + \Delta_z(w)) \\ &= \sum_{y=0}^t \sum_{z=0}^t (u(t, t+1, y, z) - u(t, 0, y, z)) + \sum_{x=0}^t \sum_{z=0}^t (v(t, x, t+1, z) - v(t, x, 0, z)) \\ & \quad + \sum_{x=0}^t \sum_{y=0}^t (w(t, x, y, t+1) - w(t, x, y, 0)). \end{aligned}$$

Then applying L to $F(t)$ yields

$$\begin{aligned} & F(t+1) - F(t) \\ &= \sum_{y=0}^t \sum_{z=0}^t \underbrace{(u(t, t+1, y, z) - u(t, 0, y, z) + f(t+1, t+1, y, z))}_{g_1} \\ & \quad + \sum_{x=0}^t \sum_{z=0}^t \underbrace{(v(t, x, t+1, z) - v(t, x, 0, z) + f(t+1, x, t+1, z))}_{g_2} \\ & \quad + \sum_{x=0}^t \sum_{y=0}^t \underbrace{(w(t, x, y, t+1) - w(t, x, y, 0) + f(t, x, y, t+1))}_{g_3} \\ & \quad + \sum_{x=0}^t f(t+1, x, t+1, t+1) + \sum_{y=0}^t f(t+1, t+1, y, t+1) + \sum_{z=0}^t f(t+1, t+1, t+1, z) \\ & \quad + f(t+1, t+1, t+1, t+1). \end{aligned}$$

So we reduce the triple sum to double sums and single sums. One can check that $g_1 = 0$. By Algorithm 5.13, one can find that g_2 is σ_x -summable and g_3 is (σ_x, σ_y) -summable. Similarly, we

further reduce the double sums to the single sums. Applying the Algorithm 5.13 (specialized to the univariate case) again, we simplify the single sums and finally obtain that $F(t+1) - F(t) = 0$. Since the initial value of $F(t)$ is $F(0) = f(0, 0, 0, 0) = 0$, we conclude that $F(t) = 0$. This completes the proof.

Under some assumptions, there are several packages to compute the creative telescoping in more than two variables. The Mathematica package “HolonomicFunctions” developed by Koutschan contains two functions “CreativeTelescoping” and “FindCreativeTelescoping” to construct telescopers for holonomic functions in different ways [56]. Another Mathematica function “FindRecurrence”, the core of the Mathematica package “MultiSum” by Wegschaider, is designed to find telescopers for proper hypergeometric functions [79]. For rational functions in three variables, an effective algorithm has been presented in [21] to compute their minimal telescopers.

Experiments show that Algorithm 6.13 is more efficient to test the existence of telescopers and construct one telescoper. For example, for the rational function

$$f(t, x, y) = \frac{4t + 2}{(45t + 5x + 10y + 47)(45t + 5x + 10y + 2)(63t - 5x + 2y + 58)(63t - 5x + 2y - 5)},$$

Algorithm 6.13 tells us that it has a telescoper of type $(\sigma_t; \sigma_x, \sigma_y)$ and outputs a telescoper and its corresponding certificates within two seconds in Maple, while the algorithm in [21] takes about three minutes and the other three functions of the two Mathematica packages use much more timings as shown in [21]. Given a rational function, one could use our algorithm to pre-check the existence of its telescopers and find a telescoper if such a telescoper exists. After that one may apply the other efficient methods to find a telescoper with lower degree in S_t .

7 Implementations and timings

We have implemented Algorithms 3.6, 5.13 and 6.13 in the computer algebra system Maple 2020. In this section, we compare the efficiency of the algorithms for solving the SET problem and illustrate the usage of our package “RationalWZ” by several examples. Our maple code and more examples are available for download at

<http://www.mmrc.iss.ac.cn/~schen/RationalWZ-2022.html>

We have implemented the G algorithm, the KS algorithm, the DOS algorithm, and the algorithm applying \mathbf{a} -degree cover to Algorithm 3.6 (ADC) in Maple 2020 with $\mathbb{F} = \mathbb{Q}$.

Fixing one admissible cover, there are two methods to calculate it and then to implement Algorithm 3.6. A direct method is expanding $p(\mathbf{x} + \mathbf{a}) - q(\mathbf{x})$ with $2n$ variables to get the set of its coefficients in \mathbf{x} and then the admissible cover, while another is obtaining the members of the admissible cover successively by computing partial derivatives dynamically. For efficiency, the DOS algorithm and the ADC algorithm are realized by partial derivatives and expansion respectively.

The test suite was generated as follows.

Let $n, d, \mu, d' \in \mathbb{N}$ and $d' < d$. Let $\mathbf{x} = \{x_1, x_2, \dots, x_n\}$. We first generated randomly a μ -term polynomial $p(\mathbf{x})$ of degree d , as well as a polynomial $dis(\mathbf{x})$ of degree d' . Then we generated randomly a vector $\mathbf{s} = (s_1, s_2, \dots, s_n) \in \mathbb{Z}^n$ and let $q(\mathbf{x}) = p(\mathbf{x} + \mathbf{s}) + dis(\mathbf{x})$. Therefore, the calculation is most likely to terminate after computing $\mathbb{V}_{\mathbb{F}}(\cup_{i=0}^{d-1-d'} S_i^H)$. By setting $0 \leq s_i \leq 99$, we can avoid the case where the memory is not enough to complete the computation.

Note that, in all the tests, the algorithms take the expanded forms of examples given above as input. All timings are measured in seconds on a macOS Monterey (Version 12.0.1) MacBook Pro with 32GB Memory and Apple M1 Pro Chip.

For a selection of random polynomials and vectors for different choices of n, μ, d, d' as above, we first tabulate the timings of the G algorithm, the KS algorithm, the DOS algorithm and the ADC algorithm. Note that $d' = -\infty$ means $dis = 0$, implying that p is shift equivalent to q .

n	μ	d	d'	G	KS	DOS	ADC
3	10	15	13	5.476	2.090	0.014	0.008
3	10	15	10	0.243	1.124	0.023	0.020
3	10	15	5	21.719	1.809	0.050	0.032
3	10	15	0	573.178	2.576	0.068	0.039
3	10	15	$-\infty$	18.491	0.714	0.043	0.036
3	100	15	13	0.205	10.025	0.044	0.028
3	100	15	10	0.482	9.997	0.046	0.046
3	100	15	5	22.114	11.317	0.061	0.062
3	100	15	0	2152.378	19.470	0.083	0.069
3	100	15	$-\infty$	1200.473	13.640	0.085	0.068

The experimental results illustrate that the DOS algorithm and the ADC algorithm outperform the other two algorithms. Furthermore, we conducted experiments in more complicated cases.

n	μ	d	d'	DOS	ADC
5	100	40	35	199.177	59.889
5	100	40	30	24.684	90.159
5	100	40	20	379.835	95.761
5	100	40	10	681.189	665.885
5	100	40	0	182.671	67.261
5	100	40	$-\infty$	709.223	77.880
5	10000	20	18	2.724	122.744
5	10000	20	15	3.088	163.258
5	10000	20	10	5.290	139.685
5	10000	20	5	10.755	125.359
5	10000	20	0	23.949	151.010
5	10000	20	$-\infty$	24.562	136.187

The experimental results indicate that the ADC algorithm outperforms the other for most of non-dense testing examples, while the DOS algorithm has a clear advantage for dense ones. It may be because the timing of expansion grows fast with the number of terms in the input polynomial. In conclusion, we present an algorithm, the ADC algorithm, which is complementary to the DOS algorithm for solving the SET problem.

From the runtime comparison, we implemented the ADC algorithm in the package RationalWZ. In our setting, the base field \mathbb{F} can be \mathbb{Q} or the field of rational functions $\mathbb{Q}(u_1, \dots, u_s)$. The following instructions show how to load the modules.

```
> read "RationalWZ.mm";
> with(ShiftEquivalenceTesting):
> with(OrbitalDecomposition):
> with(RationalReduction):
> with(RationalSummation):
> with(RationalTelescoping):
```

Example 7.1. Compute the dispersion set (over \mathbb{Z}) of two polynomials.

(1) For $p, q \in \mathbb{Q}[x, y]$ in Example 3.4, we type

> ShiftEquivalent($x^2 + 2xy + y^2 + 2x + 6y$, $x^2 + 2xy + y^2 + 4x + 8y + 11$, $[x, y]$)

$[-1, 2]$

which implies $Z_{p,q} = \{(-1, 2)\}$. So $p(x-1, y+2) = q(x, y)$.

(2) For $p, q \in \mathbb{Q}[x, y, z]$ in Example 3.21 (1), we type

> ShiftEquivalent($x^4 + x^3y + xy^2 + z^2$, $x^4 + x^3(y+1) + x(y+1)^2 + (z+2)^2 + xy$, $[x, y, z]$)

$[\]$

which implies $Z_{p,q} = \emptyset$. So p, q are not shift equivalent.

Example 7.2. Decide the $(\sigma_x, \sigma_y, \sigma_z)$ -summability of a rational function $f \in \mathbb{Q}(x, y, z)$. Let f_3, r_3 be the same as in Example 5.15 (3).

(1) Applying the function “IsSummable” to $f = f_3$, we see that f is not $(\sigma_x, \sigma_y, \sigma_z)$ -summable.

> IsSummable($(y + z/(y^2 + z - 1) - 1/(y^2 + z))/(x + 2y + z)^2$, $[x, y, z]$)

false

(2) Applying the function “IsSummable” to $f = f_3 - r_3$, we see that f is $(\sigma_x, \sigma_y, \sigma_z)$ -summable and its certificates are as follows:

> IsSummable($(y + z/(y^2 + z - 1) - 1/(y^2 + z))/(x + 2y + z)^2 - z/((y^2 + z)*(x + 2y + z)^2)$, $[x, y, z]$)

true, $\left[-\frac{y(y-1)}{2(x-2+2y+z)^2} - \frac{y(y-1)}{2(x-1+2y+z)^2} + \frac{z}{(y^2+z-1)(x-1+2y+z)^2}, \frac{y(y-1)}{2(x-2+2y+z)^2}, -\frac{z}{(y^2+z-1)(x-1+2y+z)^2} \right]$

Example 7.3. Decide the existence of telescopers of type $(\sigma_t; \sigma_x, \sigma_y, \sigma_z)$ for a rational function $f \in \mathbb{Q}(t, x, y, z)$.

(1) Applying the function “IsTelescopable” to f in Example 6.14 (1), we see that f does not have a telescoper in $\mathbb{Q}(t)\langle S_t \rangle$ of type $(\sigma_t; \sigma_x, \sigma_y, \sigma_z)$.

> IsTelescopable($1/((t+1)*(t+2z))*((t-3y+x)^2*(t+y)*(t+z)+1)$, $[x, y, z], t, 'St'$)

false

(2) Applying the function “IsTelescopable” to $f = r_1$ in Example 6.14 (3), we see that f has a telescoper $L = -\frac{t}{t+3} + St^3$ of type $(\sigma_t; \sigma_x, \sigma_y, \sigma_z)$ and its certificates are as follows:

> IsTelescopable($1/(t*(t+y+2z))*(3y+(x+z)^2+t)$, $[x, y, z], t, 'St'$)

true, $-\frac{t}{t+3} + St^3, \left[-\frac{1}{6(\frac{t}{2} + \frac{y}{2} + z)(\frac{t}{3} + 1)(x^2 + 2x(z-1) + (z-1)^2 + t + 3y)}, \frac{1}{(t+3)(t+2+y+2z)(x^2 + 2xz + z^2 + t + 3y)}, \frac{1}{6(\frac{t}{3} + 1)(\frac{t}{2} + \frac{y}{2} + z)(x^2 + 2x(z-1) + (z-1)^2 + t + 3y)} \right]$

8 Conclusion and future work

In this paper, we constructively solve the summability problem and the existence problem of telescopers for multivariate rational functions, and present a new efficient algorithm for solving the shift equivalence testing (SET) problem of multivariate polynomials.

Our algorithm can compute a telescoper for a given multivariate rational function if the existence of telescopers is guaranteed, but the computed telescoper may not be of minimal order. So a natural question is how to compute the minimal telescoper (which is unique if it is monic) for multivariate rational function if it exists. Similar to the trivariate case [21], we may first need an additive decomposition to decompose a rational function f as a sum of a summable function and a remainder r , as shown in Example 5.15, such that f is summable if and only if the remainder r is zero. Then we need to deal with the problem that the sum of two remainders in the additive decomposition may not be a remainder. The similar problem appears and has been solved in the case for trivariate rational functions [21, Section 4] and the case for bivariate hypergeometric terms [23, Section 5].

For the efficiency, we may need to consider how to choose a “minimal” remainder, which may depend on the choices of difference isomorphisms. Choosing a “good” admissible cover may help us to discover a more efficient SET algorithm. In theory, we use an irreducible partial fraction decomposition in summation algorithms, but in practice, an incomplete partial fraction decomposition would be enough, like in the univariate case [9, 64].

In the future research, we hope to explore more summation algorithms for other classes of functions, like multivariate hypergeometric terms [79]. This would be an extension of Gosper’s algorithm [37] which only works for the summation of univariate hypergeometric terms and has many applications in proving combinatorial identities [65]. Some algorithms have been developed for special bivariate hypergeometric terms [29] and for multiple binomial sums [15].

In the differential case, telescopers always exist for D-finite functions [82]. One interesting problem is how to find a telescoper [15, 25], especially the minimal one. Another problem is the integrability problem proposed by Picard [66–68], which is a continuous analogue of the summability problem. Given a rational function $f \in \mathbb{F}(x_1, \dots, x_m)$, the integrability problem is deciding whether there exist rational functions $g_1, \dots, g_m \in \mathbb{F}(x_1, \dots, x_m)$ such that

$$f = \partial_{x_1}(g_1) + \dots + \partial_{x_m}(g_m),$$

where ∂_{x_i} is the usual partial derivative with respect to x_i . When $m = 1$, it can be solved by Ostrogradsky-Hermite reduction [46, 62]. When $m = 2$, it was solved by Picard [67, p. 475–479]. In more than two variables, there is no complete algorithm for deciding the integrability of rational functions. Under a regularity assumption, some related results are listed in [15].

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