D-finite Generating Functions in Enumerative Combinatorics

Shaoshi Chen

KLMM, AMSS
Chinese Academy of Sciences

CGT2018, August 23–26, 2018, Anhui, China

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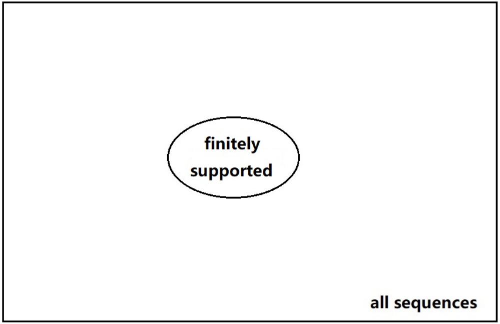
joint work with Christoph Koutschan (Austrian Academy of Sciences)
and Jason P. Bell (University of Waterloo)

Combinatorial sequences

Counting combinatorial objects leads to sequences $s : \mathbb{N} \to \mathbb{K}$.



Example: $s(n) = \binom{10}{n}$ is finitely supported.

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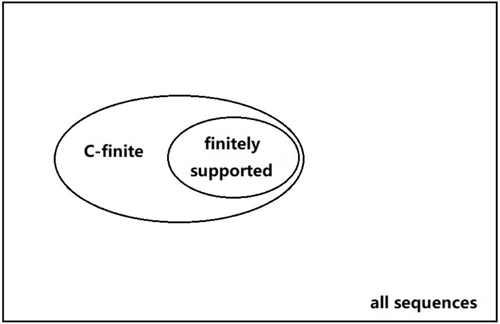

Example: The Fibonacci sequences $s(n) := F_n$ satisfying

$$s(n+2) - s(n+1) - s(n) = 0 \quad \text{with } s(0) = 0 \text{ and } s(1) = 1$$

is C-finite.

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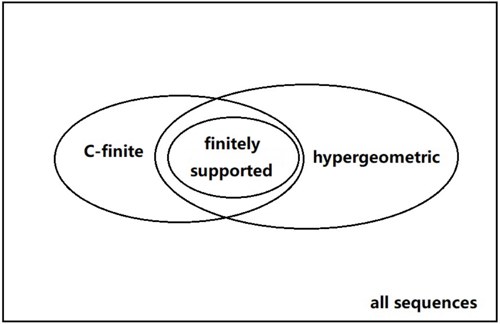

Example: The factorial sequences $s(n) := n!$ satisfying

$$s(n+1) - (n+1)s(n) = 0 \quad \text{with } s(0) = 1$$

is hypergeometric.

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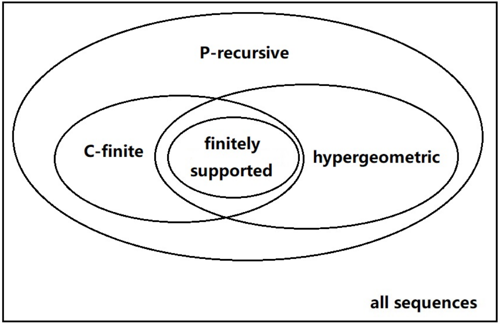

Example: The harmonic sequences $s(n) := \sum_{k=1}^{n} \frac{1}{k}$ satisfying

$$(n+2)s(n+2) - (2n+3)s(n+1) + (n+1)s(n) = 0$$

is P-recursive.

Generating functions

The generating function of a sequence $s : \mathbb{N} \to \mathbb{K}$ is

$$f(x) = \sum_{n=0}^{+\infty} s(n) x^n \in \mathbb{K}[[x]].$$

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Proposition.

$s(n)$ is finitely supported $\Leftrightarrow$ $f(x) = \sum_{n=0}^{+\infty} s(n)x^n$ is polynomial

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Proposition.

$$s(n) \text{ is C-finite} \quad \Leftrightarrow \quad f(x) = \sum_{n=0}^{+\infty} s(n)x^n \text{ is rational}$$

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Example. Bell numbers b_n count the number of partitions of a set

$1, 1, 2, 5, 15, 52, 203, 877, 4140, \ldots$ (sequence A000110 in OEIS)

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$$\sum_{n=0}^{+\infty} \frac{b_n x^n}{n!} = \exp(\exp(x) - 1).$$

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$$\exp(\exp(x) - 1) \text{ is not D-finite!}$$

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b_n is not P-recursive!

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Let $\mathbb{K}$ be a field of characteristic zero (e.g. $\mathbb{Q}, \mathbb{R}, \mathbb{C}$).

Definition. A function $f(x_1, \ldots, x_d)$ is D-finite over $\mathbb{K}(x_1, \ldots, x_d)$ if for each $i \in \{1, \ldots, d\}$, f satisfies a LPDE:

$$p_{i,r_i} \frac{\partial^{r_i} f}{\partial x_i^{r_i}} + p_{i,r_i-1} \frac{\partial^{r_i-1} f}{\partial x_i^{r_i-1}} + \cdots + p_{i,0} f = 0,$$

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R. P. Stanley. Differentiably Finite Power Series. *European Journal of Combinatorics*, 1: 175–188, 1980.

L. Lipshitz. D-Finite Power Series. *Journal of Algebra*, 122: 353–373, 1989.

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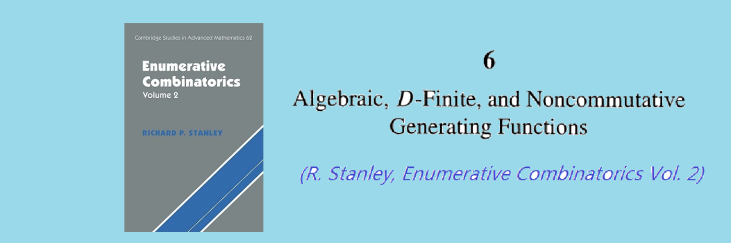

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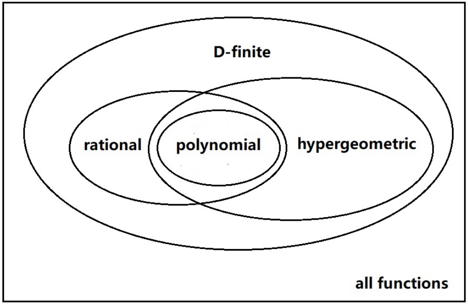

Algebraic properties

Let $\mathbf{n} = n_1, \ldots, n_d$, $\mathbf{x} = x_1, \ldots, x_d$, and $\mathbf{x}^{\mathbf{n}} = x_1^{n_1} \cdots x_d^{n_d}$.

Definition. Let $f = \sum a(\mathbf{n})\mathbf{x}^{\mathbf{n}}$ and $g = \sum b(\mathbf{n})\mathbf{x}^{\mathbf{n}}$ be in $\mathbb{K}[[\mathbf{x}]]$. The Hadamard product of f and g is

$$f \odot g = \sum a(\mathbf{n})b(\mathbf{n})\mathbf{x}^{\mathbf{n}}.$$

The diagonal of f is defined as $\mathrm{diag}(f) = \sum a(n, \ldots, n)x^n \in \mathbb{K}[[x]]$.

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Theorem (Lipshitz1989). Let $\mathscr{D} := \{f \in \mathbb{K}[[\mathbf{x}]] \mid f \text{ is D-finite}\}$. Then

(i) if $f, g \in \mathscr{D}$, then $f + g$, $f \cdot g$, and $f \odot g$ are in $\mathscr{D}$;

(ii) if $f \in \mathscr{D}$, $\operatorname{diag}(f)$ is D-finite in $\mathbb{K}[[x]]$;

(iii) if $f \in \mathscr{D}$, and $\alpha_1, \ldots, \alpha_d \in \mathbb{K}[[\mathbf{y}]]$ are algebraic over $\mathbb{K}(\mathbf{y})$ and the substitution makes sense, then $f(\alpha_1, \ldots, \alpha_d)$ is D-finite.

Analytic properties

Theorem. If $f(x) = \sum_{n=0}^{+\infty} a(n)x^n$ is D-finite, then $f(x)$ has only finitely many singularities and

$$a(n) \quad \sim \quad \zeta^{-n} \cdot \exp(P(n^{1/r})) \cdot n^{\theta} \cdot (\log n)^{\ell},$$

where $\theta \in \overline{\mathbb{Q}}$, $r, \ell \in \mathbb{N}$ and ζ is a singularity of $f(x)$.

Example. Both $1/\sin(x)$ and $\sum_{n=0}^{+\infty} \log(\log(n))x^n$ are not D-finite.

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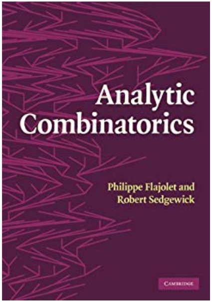

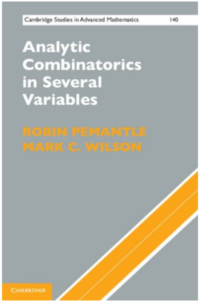

Classification problems

Problem. For a class of combinatorial sequences, decide whether their generating functions are rational, algebraic, or D-finite?

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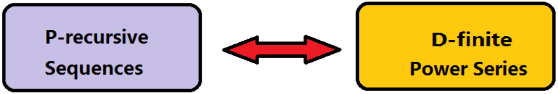

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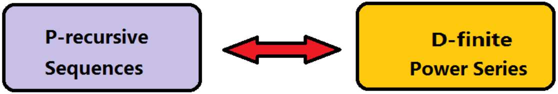

Remark. This correspondence is not true for multivariate sequences.

$$\sum_{n_1, n_2 \geq 0} \frac{1}{n_1^2 + n_2^2 + 1} \cdot y_1^{n_1} y_2^{n_2} \text{ is not D-finite!}$$

Problem. How to decide the D-finiteness for multivariate P-recursive sequences?

Mixed hypergeometric terms

Let $\mathbb{K}$ be a field of char. zero and algebraically closed.

$$\mathbf{x} = (x_1, \ldots, x_p), \qquad \mathbf{n} = (n_1, \ldots, n_q)$$

$$D_i : \underbrace{\partial/\partial x_i}_{\text{derivations}}, \qquad S_j : \underbrace{n_j \to n_j + 1}_{\text{shifts}}$$

Definition. $h(\mathbf{x}, \mathbf{n})$ is mixed hypergeometric over $\mathbb{K}(\mathbf{x}, \mathbf{n})$ if

all $\dfrac{D_i(h)}{h}$ and $\dfrac{S_j(h)}{h}$ are rational functions in $\mathbb{K}(\mathbf{x}, \mathbf{n})$.

Remark. Mixed hypergeometric terms are solutions of systems of first-order homogeneous differential and difference equations.

Examples

- Rational functions:

$$x_1 + x_2 + n_1, \quad \frac{1}{(x_1 + x_2)}, \quad \frac{x_1 + n_1 + 1}{x_1 + x_2 + n_1^2 + 3}, \quad \ldots$$

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- Hyperexponential functions:

$$\exp(x_1 + x_2^2), \quad (x_1^2 + x_2 + 1)^{\sqrt{5}}, \quad \exp\left(\int \frac{1}{x_1 + x_2}\right), \quad \ldots$$

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- Symbolic powers:

$$x_1^{n_1}, \quad (x_1 + x_2)^{n_1} \cdot (x_2 + x_3^2)^{n_2}, \quad \ldots$$

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- Hypergeometric terms:

$$2^{n_1}, \quad n_1!, \quad (n_1 + 2n_2 + \sqrt{3})!, \quad \ldots$$

Structure theorem

Theorem. Any mixed hypergeometric term $h(\mathbf{x},\mathbf{n})$ is of the form

$$f(\mathbf{x},\mathbf{n})\cdot\prod_{j=1}^{q}\beta_j(\mathbf{x})^{n_j}\cdot\exp(g_0(\mathbf{x}))\cdot\prod_{\ell=1}^{L}g_\ell(\mathbf{x})^{c_\ell}\cdot\prod_{\lambda}(\mathbf{v}_\lambda\cdot\mathbf{n}+p_\lambda)!^{e_\lambda}$$

where f is a rational function in $\mathbb{K}(\mathbf{x},\mathbf{n})$.

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Proper terms. A mixed hypergeometric term $h(\mathbf{x},\mathbf{n})$ is proper if it is of the form

$$P(\mathbf{x},\mathbf{n})\cdot\prod_{j=1}^{q}\beta_j(\mathbf{x})^{n_j}\cdot\exp(g_0(\mathbf{x}))\cdot\prod_{\ell=1}^{L}g_\ell(\mathbf{x})^{c_\ell}\cdot\prod_{\lambda}(\mathbf{v}_\lambda\cdot\mathbf{n}+p_\lambda)!^{e_\lambda}$$

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Remark. Properness can be verified algorithmically :-)

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Example. $1/(n_1^2 + n_2^2 + 1)$ is not proper.

Wilf–Zeilberger conjecture: D-finite $\Leftrightarrow$ Proper

In the fundamental paper by Wilf and Zeilberger:

Invent. math. 108: 575–633 (1992)

Inventiones mathematicae

An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities

Herbert S. Wilf * and Doron Zeilberger **

Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104, USA
Department of Mathematics, Temple University, Philadelphia, PA 19122, USA

In Page 585, they said:

Our examples are all proper-hypergeometric. We conjecture that a hypergeometric term is proper-hypergeometric if and only if it is holonomic.

Wilf–Zeilberger conjecture: D-finite $\Leftrightarrow$ Proper

Previous work:

- Continuous case

  Bernstein1971, Kashiwara1978, Takayama1992, Christopher1999, ...

  Ingredients: Bernstein-Kashiwara equivalence, Structure of multivariate hyperexponential functions

- Discrete case

  Payne1997, Hou(2001, 2004), Abramov–Petkovšek(2001, 2003), ...

  Ingredients: Lipshitz's theorem, struture of multivariate hypergeometric terms (Ore-Sato theorem)

Proof of the WZ conjecture: the general case

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Proof of the Wilf–Zeilberger conjecture for mixed hypergeometric terms ☆

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[b] School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China
[c] Johann Radon Institute for Computational and Applied Mathematics (RICAM), Austrian Academy of Sciences, Altenberger Straße 69, 4040 Linz, Austria

Ingredients:

- Structure theorem for mixed hypergeometric terms;
- Algebraic properties of D-finite functions;
- Elimination theory in algebraic D-modules.

Rationality theorems on D-finite power series

Problem. When a D-finite power series is rational?

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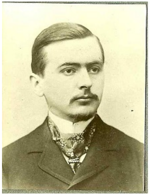


Pierre Fatou (1878-1929)

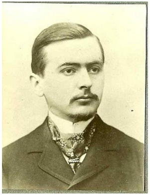
Pierre Fatou, *Séries trigonométriques et séries de Taylor*, Acta Math. **30** (1906), no. 1, 335–400.

$$f(x) = \sum_{n\geq 0} a_n x^n, \quad \text{where } a_n \in \Delta \text{ with } |\Delta| < +\infty.$$

Fatou's Theorem (1906).
A power series with coefficients from a finite set is either rational or transcendental over $\mathbb{C}(x)$.

Rationality theorems on D-finite power series

Problem. When a D-finite power series is rational?



Pierre Fatou (1878-1929)

Pierre Fatou, *Séries trigonométriques et séries de Taylor*, Acta Math. **30** (1906), no. 1, 335–400.

$$f(x) = \sum_{n \geq 0} a_n x^n, \quad \text{where } a_n \in \Delta \text{ with } |\Delta| < +\infty.$$

Corollary.
An algebraic power series with coefficients from a finite set is rational.

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From 1917 to 1922, there are four papers with the same title:

Über Potenzreihen mit endlich vielen verschiedenen Koeffizienten.

Power Series with Finitely Distinct Coefficients

1. G. Polya in 1917, Math. Ann.
2. R. Jentzsch in 1918, Math. Ann.
3. F. Carlson in 1919, Math. Ann.
4. G. Szego in 1922, Math Ann.

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Szegö's Theorem (1922)

A power series with coefficients from a finite set is either rational or has the unit circle as its natural boundary.

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Power series with coefficients from a finite set ☆

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$$f(x_1,\ldots,x_d) = \sum a_{n_1,\ldots,n_d} x_1^{n_1} \cdots x_d^{n_d}, \quad \text{where } a_{n_1,\ldots,n_d} \in \Delta \text{ with } |\Delta| < +\infty.$$

Theorem. A multivariate D-finite power series with coefficients form a finite set is rational.

Nonnegative integer points on algebraic varieties

Let V be an algebraic variety over an algebraically closed field K of characteristic zero. We define the listing generating function

$$F_V(x_1, \ldots, x_d) := \sum_{(n_1,\ldots,n_d)\in V\cap\mathbb{N}^d} x_1^{n_1}\cdots x_d^{n_d}$$

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We may ask the following questions:

When F_V is zero?

Remark. This is Hilbert Tenth Problem when K is $\mathbb{Q}$. In 1970, Matiyasevich proved that this problem is undecidable.

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We may ask the following questions:

When F_V is a polynomial?

Remark. In 1929, Siegel proved that a smooth algebraic curve C of genus $g\geq 1$ has only finitely many integer points over a number field K.

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We may ask the following questions:

When F_V is a rational function?

Remark. If V is defined by linear polynomials over $\mathbb{Q}$, then F_V is rational.

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We may ask the following questions:

When F_V is a D-finite function?

Corollary.

$$F_V \text{ is } D\text{-finite} \quad \Leftrightarrow \quad F_V \text{ is rational.}$$

Nonnegative integer points on algebraic varieties

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$$F_V(x_1, \ldots, x_d) := \sum_{(n_1,\ldots,n_d) \in V \cap \mathbb{N}^d} x_1^{n_1} \cdots x_d^{n_d}$$

We may ask the following questions:

When F_V is a D-finite function?

Theorem.

The problem of testing whether F_V is rational is undecidable!

Nonnegative integer points on algebraic varieties

Let V be an algebraic variety over an algebraically closed field K of characteristic zero. We define the listing generating function

$$F_V(x_1, \ldots, x_d) := \sum_{(n_1,\ldots,n_d) \in V \cap \mathbb{N}^d} x_1^{n_1} \cdots x_d^{n_d}$$

We may ask the following questions:

When F_V is a differentially algebraic function?

Definition. $F \in K[[x_1, \ldots, x_d]]$ is differentially algebraic if the transcendence degree of the filed generated by the derivatives $D_{x_1}^{i_1} \cdots D_{x_d}^{i_d}(F)$ with $i_j \in \mathbb{N}$ over $K(x_1, \ldots, x_d)$ is finite.

Open problems

Conjecture. Let V be an algebraic variety over $\mathbb{C}$. Then the power series

$$\sum_{(n_1,\ldots,n_d)\in V\cap\mathbb{N}^d} x_1^{n_1}\cdots x_d^{n_d}$$

is differentially algebraic if and only if it is rational.

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Example. Let $p = x^2 - y$. Then the power series

$$F_p(x,y) := \sum_{m\geq 0} x^m y^{m^2}$$

is not differentially algebraic, otherwise, $F_p(x,2) = \sum 2^{m^2} x^m$ is differentially algebraic. By Mahler's lemma, we get a contradiction

$$2^{m^2} \ll (m!)^c \quad \text{for any positive constant } c.$$

Open problems

Conjecture. Let V be an algebraic variety over $\mathbb{C}$. Then the power series

$$\sum_{(n_1,\ldots,n_d)\in V\cap\mathbb{N}^d} x_1^{n_1}\cdots x_d^{n_d}$$

is differentially algebraic if and only if it is rational.

Conjecture (Chowla-Chowla-Lipshitz-Rubel). The power series

$$f := \sum_{n\in\mathbb{N}} x^{n^3} \in \mathbb{C}[[x]]$$

is not differentially algebraical, i.e., satisfies no ADE.

Remark. The power series $\sum x^{n^2}$ is differentially algebraic.

Igor Pak. Complexity problems in enumerative combinatorics. Proceedings of ICM2018, 3:3139–3166, 2018.

Summary

- Proof of the Wilf–Zeilberger conjecture

  S. Chen, C. Koutschan. Proof of the Wilf-Zeilberger Conjecture for Mixed Hypergeometric Terms. Journal of Symbolic Computation (Available online 15 June 2018).

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Thank you!