How to generate all possible WZ-pairs algorithmically?

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What is a Wilf–Zeilberger pair?

Definition. A pair $(F(x,y),G(x,y))$ is called a WZ-pair if

$$\partial_x(F) = \partial_y(G), \qquad \text{(WZ-equation)}$$

where $\partial_x \in \{D_x, \Delta_x, \Delta_{q,x}\}$ and $\partial_y \in \{D_y, \Delta_y, \Delta_{q,y}\}$.

Example.

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Example.

Continuous WZ-pair:

$$(F, G) := \left(\sqrt{x} \cdot \exp(xy^2), \qquad \frac{y \cdot \exp(xy^2)}{2\sqrt{x}}\right)$$

satisfies the continuous WZ-equation

$$D_x(F) = D_y(G).$$

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Example.

Discrete WZ-pair:

$$(F,G) := \left(\frac{\binom{n}{k}^2}{\binom{2n}{n}}, \qquad \frac{2k-3n-3}{4n+2} \frac{\binom{n}{k-1}^2}{\binom{2n}{n}} \right)$$

satisfies the discrete WZ-equation

$$\Delta_n(F) = \Delta_k(G).$$

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Mixed WZ-pair:

$$(F,G) := \left(x^k \cdot \sqrt{1-4x} \cdot \binom{2k}{k}, \qquad -\frac{x^{k-1}k}{\sqrt{1-4x}} \cdot \binom{2k}{k} \right)$$

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$$D_x(F) = \Delta_k(G).$$

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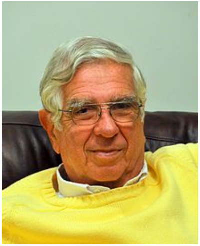
(



Herbert S. Wilf

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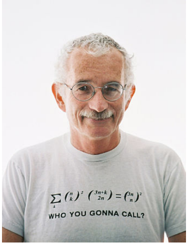

Doron Zeilberger

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Applications of WZ-pairs: proving identities

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Applying $\sum_{k=0}^{n}$ to the both sides of $\Delta_n(F) = \Delta_k(G)$ yields

$$\sum_{k=0}^{n} \Delta_n(F) = \Delta_n\left(\sum_{k=0}^{n} F\right) - \binom{2n+2}{n+1}^{-1}$$

$$\sum_{k=0}^{n} \Delta_k(G) = G(n,n+1) - G(n,0) = -\binom{2n+2}{n+1}^{-1}$$

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$$\Downarrow$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Applications of WZ-pairs: convergence accelerations

Theorem (Zeilberger, 1993) For any discrete WZ-pair $(F(n,k),G(n,k))$, we have

$$\sum_{n=0}^{\infty} G(n,0) = \sum_{n=1}^{\infty} \left(F(n,n-1)+G(n-1,n-1)\right).$$

Remark. The idea of using WZ-pairs for convergence accelerations goes back to Andrei Markov in 1890 for computing $\zeta(3)$.

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Example. Applying Theorem to the discrete WZ-pair

$$\left(F := \frac{(-1)^k k!^2 (n-k-1)!}{2(k+1)(n+k+1)!}, \quad G := \frac{(-1)^k k!^2 (n-k)!}{(n+1)^2 (n+k+1)!}\right)$$

yields the formula

$$\zeta(3) := \sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{\binom{2n}{n} n^3}.$$

Bringing order to chaos

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1 How to discover binomial-coefficient identities algorithmically?
2 How to generate all possible WZ-pairs algorithmically?

Rational WZ-pairs

Problem. Find all possible rational WZ-pairs, i.e., determine

$$\mathscr{P}_{(\partial_x,\partial_y)} := \{(f,g) \mid f,g \in \mathbb{C}(x,y) \text{ such that } \partial_x(f) = \partial_y(g)\}.$$

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Different types of WZ-pairs:

- Differential case: $\partial_x = D_x$ and $\partial_y = D_y$
- (q)-Shift cases: $\partial_x \in \{\Delta_x, \Delta_{q,x}\}$ and $\partial_y \in \{\Delta_y, \Delta_{q,y}\}$
- Mixed cases: $\partial_x \in \{\Delta_x, \Delta_{q,x}\}$ and $\partial_y = \partial/\partial y$

Structure of rational WZ-pairs: the differential case

Definition. A pair (f, g) is a log-derivative pair if $\exists\ h \in \mathbb{C}(x, y)$ s.t.

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Theorem (Christopher, 1999). Let $f, g \in \mathbb{C}(x, y)$ be such that $D_x(f) = D_y(g)$. Then $\exists\ a, b_1, \ldots, b_n \in \mathbb{C}(x, y)$ and nonzero $c_1, \ldots, c_n \in \mathbb{C}$ s.t.

$$f = D_y(a) + \sum_{i=1}^{n} c_i \frac{D_y(b_i)}{b_i} \quad \text{and} \quad g = D_x(a) + \sum_{i=1}^{n} c_i \frac{D_x(b_i)}{b_i}.$$

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Corollary. Any rational WZ-pair of type (D_x, D_y) is a linear combination of exact and log-derivative pairs.

Shift invariant rational functions

Notation. Let $\theta_x \in \{\sigma_x, \tau_{q,x}\}$ and $\theta_y \in \{\sigma_y, \tau_{q,y}\}$.

- Shift operators: σ_x, σ_y

$$\sigma_x(f(x,y)) = f(x+1,y) \quad \text{and} \quad \sigma_y(f(x,y)) = f(x,y+1).$$

- q-shift operators: $\tau_{q,x}, \tau_{q,y}$ with $q \in \mathbb{C} \setminus \{0\}$

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Definition. A rational function $f \in \mathbb{C}(x,y)$ is (θ_x, θ_y)-invariant if $\exists\ m,n \in \mathbb{Z}$, not all zero, such that $\theta_x^m \theta_y^n (f) = f$.

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Prop. Let $f \in \mathbb{C}(x,y)$ be (θ_x, θ_y)-invariant and $\bar{n} = n/\gcd(m,n)$ and $\bar{m} = m/\gcd(m,n)$. Then

1. if $\theta_x = \sigma_x$ and $\theta_y = \sigma_y$, then $f = g(\bar{n}x - \bar{m}y)$ for some $g \in \mathbb{C}(z)$;
2. if $\theta_x = \tau_{q,x}$, $\theta_y = \tau_{q,y}$, then $f = g(x^{\bar{n}}y^{-\bar{m}})$ for some $g \in \mathbb{C}(z)$;
3. if $\theta_x = \sigma_x$, $\theta_y = \tau_{q,y}$, then $f \in \mathbb{C}(x)$ if $m = 0$, $f \in \mathbb{C}(y)$ if $n = 0$, and $f \in \mathbb{C}$ if $mn \neq 0$.

Cyclic pairs

Notation. For $n \in \mathbb{Z}, h \in \mathbb{C}(x,y)$ we define

$$\frac{\theta_y^n - 1}{\theta_y - 1} \bullet h = \begin{cases} \sum_{j=0}^{n-1} \theta_y^j(h), & n \geq 0; \\ -\sum_{j=1}^{-n} \theta_y^{-j}(h), & n < 0. \end{cases}$$

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Definition. A pair (f,g) is a cyclic pair if $\exists\ h \in \mathbb{C}(x,y)$ s.t.

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Example. Let $p = 2x + 3y$. Then the pair (f,g) with

$$f = \frac{1}{p} + \frac{1}{\sigma_x(p)} + \frac{1}{\sigma_x^2(p)} \quad \text{and} \quad g = \frac{1}{p} + \frac{1}{\sigma_y(p)}$$

is a cyclic WZ-pair with respect to (Δ_x, Δ_y).

Structure of rational WZ-pairs: the (q)-shift case

Let $\theta_x \in \{\sigma_x, \tau_{q,x}\}$ and $\theta_y \in \{\sigma_y, \tau_{q,y}\}$.

Theorem. Let $f, g \in \mathbb{C}(x,y)$ be such that $\theta_x(f) - f = \theta_y(g) - g$. Then $\exists\ a, b_1, \ldots, b_n \in \mathbb{C}(x,y)$ s.t.

$$f = \theta_y(a) - a + \sum_{i=1}^{n} \frac{\theta_x^{s_i} - 1}{\theta_x - 1} \bullet b_i \quad \text{and} \quad g = \theta_x(a) - a + \sum_{i=1}^{n} \frac{\theta_y^{t_i} - 1}{\theta_y - 1} \bullet b_i,$$

where the b_i's are (θ_x, θ_y)-invariant, i.e., for each $i \in \{1, \ldots, n\}$ we have $\theta_x^{s_i}(b_i) = \theta_y^{t_i}(b_i)$ for some $s_i \in \mathbb{N}$ and $t_i \in \mathbb{Z}$ with s_i, t_i not all zero.

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Corollary. Any rational WZ-pair of type $(\theta_x - 1, \theta_y - 1)$ is a linear combination of exact and cyclic pairs.

Structure of rational WZ-pairs: the mixed case

Let $\theta_x \in \{\sigma_x, \tau_{q,x}\}$ and $\partial_y = D_y$.

Definition. A pair (f, g) is a constant pair if $\partial_x(f) = \partial_y(g) = 0$.

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Theorem. Let $f, g \in \mathbb{C}(x, y)$ be such that $\theta_x(f) - f = D_y(g)$. Then $\exists\ h \in \mathbb{C}(x, y)$, $u \in \mathbb{C}(y)$ and $v \in \mathbb{C}(x)$ s.t.

$$f = D_y(h) + u \quad \text{and} \quad g = \theta_x(h) - h + v.$$

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Stable transformations of WZ-pairs

Problem. Given a WZ pair $P := (F(x,y), G(x,y))$ satisfying

$$F(x+1,y) - F(x,y) = G(x,y+1) - G(x,y),$$

find a transformation ϕ such that $\phi(P)$ is also a WZ-pair?

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$$(F(x,y), G(x,y)) \quad \mapsto \quad (F(-x,y), -G(-x-1,k)).$$

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Thank you!