Reduction-based Algorithms for Creative Telescoping

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based on joint papers with A. Bostan, F. Chyzak, H. Huang, M. Kauers, C. Koutschan, Z. Li, M. van Hoeij, and G. Xin

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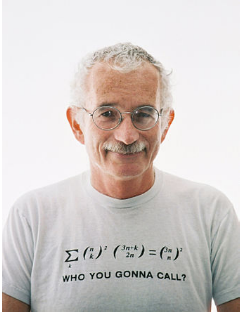

D. Zeilberger

Telescoping

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- Hypergeometric sums

$$\sum_{k=0}^{n} \frac{\binom{2k}{k}^2}{(k+1)4^{2k}} = \sum_{k=0}^{n} \Delta_k \left(\frac{4k\binom{2k}{k}^2}{4^{2k}}\right) = \frac{4(n+1)\binom{2n+2}{n+1}^2}{4^{2n+2}}$$

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$$L = (n+1)S_n - 4n - 2 \quad \text{and} \quad g = \frac{(2k-3n-3)k^2\binom{n}{k}^2}{(k-n-1)^2}$$

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- Since $f(n,k) = 0$ when $k < 0$ or $k > n$, we have

$$\sum_{k=-\infty}^{+\infty} \binom{n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}^2$$

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- Taking sums on both sides of $L(f) = \Delta_k(g)$:

$$\sum_{k=-\infty}^{+\infty} L(f) = L\left(\sum_{k=-\infty}^{+\infty} f\right) = g(n,+\infty) - g(n,-\infty) = 0$$

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- Verify the initial condition:

$$F(1) = 2 = \binom{2}{1}$$

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$$\sum_{n=0}^{\infty}\sum_{k_1}\sum_{k_2} \frac{u^n n!}{(a+1)_n}\binom{n+a}{n-k_1}\frac{(-x)^{k_1}}{k_1!}\binom{n+a}{n-k_2}\frac{(-y)^{k_2}}{k_2!}$$

$$= (1-u)^{-a-1}\exp\left\{-\frac{(x+y)^u}{1-u}\right\}\sum_{n}\frac{1}{n!(a+1)_n}\left(\frac{xyu}{(1-u)^2}\right)^n$$

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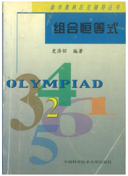

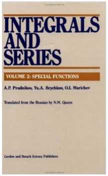

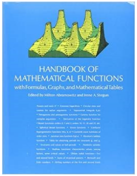

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A discrete function $f(n,k)$ is said to be hypergeometric if

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Example: $f(n,k) = (n+k)2^n(-1)^k \frac{(n+k)!(2n-k)!(2n-2k)!}{(n+2k)!^2}$

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Proc. Natl. Acad. Sci. USA
Vol. 75, No. 1, pp. 40–42, January 1978
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Decision procedure for indefinite hypergeometric summation

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Example. $k! = \Delta_k(\text{No solution!})$

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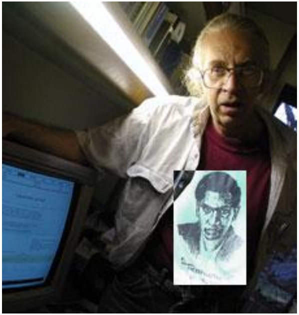

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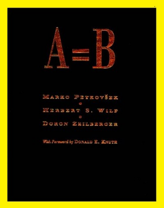

Petkovsek, Wilf & Zeilberger

Variants of Zeilberger's algorithm

Analogous algorithms have been formulated for

- q-hypergeometric terms (Wilf-Zeilberger)
- hyperexponential terms (Almkvist-Zeilberger)
- holonomic functions (Takayama, Chyzak)
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Common feature:

Telescopers and their certificates are computed at the same time!

Telescoper

Example.

$$H = \frac{k^{10}}{n+k}$$

The telescoper of minimal order L for H is

$$L = n^{10} S_n - (n+1)^{10}$$

Telescoper

Example.

$$H = \frac{k^{10}}{n+k}$$

The telescoper of minimal order L for H is

$$L = n^{10} S_n - (n+1)^{10}$$

Guess the certificate of L?

Certificate

$$\frac{1}{2520(n+k)}(2100k^8n^2 - 84n^3 - 68460k^6n^4 - 840n^4 - 3720n^5 + 140700k^4n^6 - 9480n^6 - 15024n^7 - 10500k^2n^8 - 14808n^8 - 8400n^9 - 79590n^2k^7 + 284235n^4k^5 - 143640n^6k^3 + 210nk^8 - 26250n^3k^6 + 133035n^5k^4 - 35700n^7k^2 + 252k^{11} + 18900k^9n - 213780k^7n^3 + 368340k^5n^5 - 110460k^3n^7 - 2100n^{10} + 1890k^9 - 1764k^7 + 1260k^5 - 378k^3 - 1260k^{10} - 294nk^2 + 700nk^4 - 588nk^6 + 63504k^{11}n^5 + 52920k^{11}n^4 + 30240k^{11}n^3 + 11340k^{11}n^2 - 2940n^2k^2 - 13080n^3k^2 - 33780n^4k^2 - 55116n^5k^2 - 57348n^6k^2 - 17360k^3n^2 - 48860k^3n^3 - 94920k^3n^4 - 135156k^3n^5 - 55440k^3n^8 - 13860k^3n^9 - 3780k^3n + 7000n^2k^4 + 31185n^3k^4 + 80850n^4k^4 + 90090n^7k^4 + 27720n^8k^4 + 57141k^5n^2 + 155610k^5n^3 + 347886k^5n^6 + 238392k^5n^7 + 110880k^5n^8 + 27720k^5n^9 + 12600k^5n - 5880n^2k^6 - 114114n^5k^6 - 123816n^6k^6 - 83160n^7k^6 - 27720n^8k^6 - 379830k^7n^4 - 469128k^7n^5 - 411840k^7n^6 - 257400k^7n^7 - 110880k^7n^8 - 27720k^7n^9 - 17640k^7n + 9405n^3k^8 + 24750n^4k^8 + 42075n^5k^8 + 47520n^6k^8 + 34650n^7k^8 + 13860n^8k^8 + 85085k^9n^2 + 398475k^9n^4 + 23100k^9n^9 + 480480k^9n^5 + 92400k^9n^8 + 235620k^9n^7 + 227150k^9n^3 + 404250k^9n^6 - 12628k^{10}n - 13860k^{10}n^9 - 152460k^{10}n^3 - 60060k^{10}n^8 - 267960k^{10}n^4 - 157080k^{10}n^7 - 271656k^{10}n^6 - 56980k^{10}n^2 - 323400k^{10}n^5 + 2520k^{11}n + 2520k^{11}n^9 + 11340k^{11}n^8 + 30240k^{11}n^7 + 52920k^{11}n^6)$$

Certificate

$$\frac{1}{2520(n+k)}(2100k^8n^2-84n^3-68460k^6n^4-840n^4-3720n^5+140700k^4n^6-9480n^6-15024n^7-10500k^2n^8-14808n^8-8400n^9-79590n^2k^7+284235n^4k^5-143640n^6k^3+210nk^8-26250n^3k^6+133035n^5k^4-35700n^7k^2+252k^{11}+18900k^9n-213780k^7n^3+368340k^5n^5-110460k^3n^7-2100n^{10}+1890k^9-1764k^7+1260k^5-378k^3-1260k^{10}-294nk^2+700nk^4-588nk^6+63504k^{11}n^5+52920k^{11}n^4+30240k^{11}n^3+11340k^{11}n^2-2940n^2k^2-13080n^3k^2-33780n^4k^2-55116n^5k^2-57348n^6k^2-17360k^3n^2-48860k^3n^3-94920k^3n^4-135156k^3n^5-55440k^3n^8-13860k^3n^9-3780k^3n+7000n^2k^4+31185n^3k^4+80850n^4k^4+90090n^7k^4+27720n^8k^4+57141k^5n^2+155610k^5n^3+347886k^5n^6+238392k^5n^7+110880k^5n^8+27720k^5n^9+12600k^5n-5880n^2k^6-114114n^5k^6-123816n^6k^6-83160n^7k^6-27720n^8k^6-379830k^7n^4-469128k^7n^5-411840k^7n^6-257400k^7n^7-110880k^7n^8-27720k^7n^9-17640k^7n+9405n^3k^8+24750n^4k^8+42075n^5k^8+47520n^6k^8+34650n^7k^8+13860n^8k^8+85085k^9n^2+398475k^9n^4+23100k^9n^9+480480k^9n^5+92400k^9n^8+235620k^9n^7+227150k^9n^3+404250k^9n^6-12628k^{10}n-13860k^{10}n^9-152460k^{10}n^3-60060k^{10}n^8-267960k^{10}n^4-157080k^{10}n^7-271656k^{10}n^6-56980k^{10}n^2-323400k^{10}n^5+2520k^{11}n+2520k^{11}n^9+11340k^{11}n^8+30240k^{11}n^7+52920k^{11}n^6)$$

Very often, certificates are not needed!

Telescoping without certificates

Problem. Can we compute the telescopers without also computing the certificates?

Reduction-based Algorithms: $L(x, \partial_x)(f(x,y)) = \partial_y(g(x,y))$

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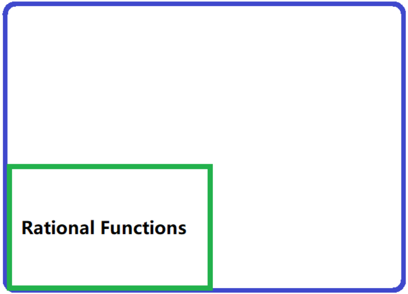

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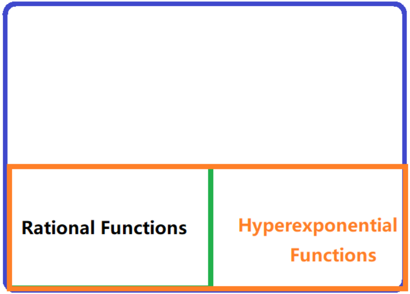

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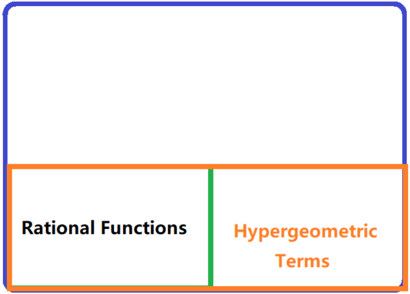

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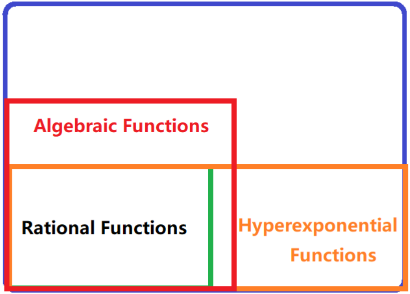

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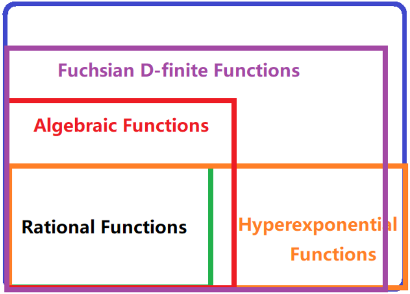

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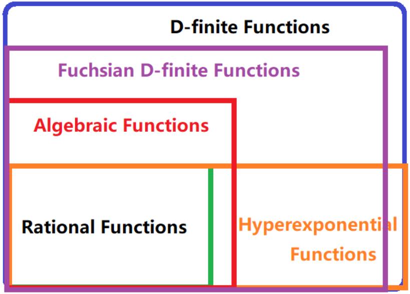

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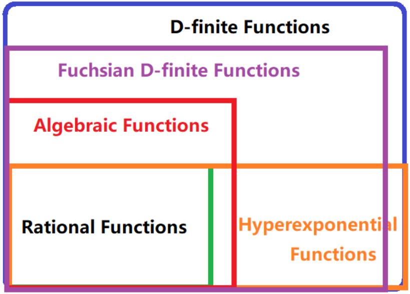

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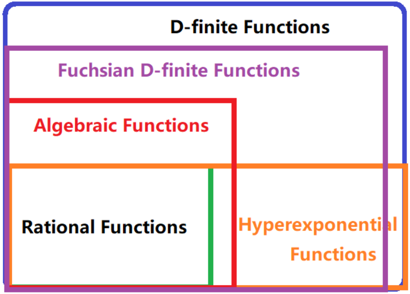

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Reduction: the rational differential case

Fact:

$$\frac{1}{(y+1)^2}+\frac{1}{y+1}=D_y\left(\frac{-1}{y+1}\right)+\frac{1}{y+1}$$

Reduction: the rational differential case

Fact:

$$\frac{1}{(y+1)^s} + \cdots + \frac{1}{y+1} = D_y\left(\frac{-(s-1)^{-1}}{(y+1)^{s-1}} + \cdots + \frac{-1}{y+1}\right) + \frac{1}{y+1}$$

Ostrogradsky-Hermite reduction: For any $f \in F(y)$,

$$f = D_y(g) + \frac{a}{b},$$

where $g \in F(y), \deg_y(a) < \deg_y(b)$ and b is squarefree.

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Remark. The above reduction decomposes a rational integral as

$$\int f(y)\,dy \;=\; \text{rational part} + \text{transcendental part}.$$

Reduction: the rational shift case

Fact:

$$\frac{1}{k+1} = \Delta_k\left(\frac{1}{k}\right) + \frac{1}{k}$$

Reduction: the rational shift case

Fact:

$$\frac{1}{k+2} = \Delta_k \left(\frac{1}{k+1} + \frac{1}{k} \right) + \frac{1}{k}$$

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Fact:

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Remark. The decomposition above is not unique, e.g.

$$\frac{2k+1}{k(k+1)} = \Delta_k\left(\frac{1}{k}\right)+\frac{2}{k} = \Delta_k\left(-\frac{1}{k}\right)+\frac{2}{k+1}$$

Reduction: the hyperexponential case

Let T be hyperexponential w.r.t. y with $f = D_y(T)/T \in \mathbb{F}(y)$.

$$f = \frac{D_y(r)}{r} + K \quad \leftrightsquigarrow \quad T = r \cdot H \quad \text{with } \frac{D_y(H)}{H} = K,$$

where $K = c/d$ satisfies $\gcd(c - iD_y(d), d) = 1$ for all $i \in \mathbb{Z}$

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$$T = D_y(\cdots) + \left(\frac{a}{b} + \frac{p}{d}\right) \cdot H,$$

where $a, b, p \in \mathbb{F}[y]$ with $\deg_y(a) < \deg_y(b)$, b square-free and p in a f.d. vector space V_K over $\mathbb{F}$.

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Reduction: the algebraic case

Let K be an algebraic extension over $\mathbb{F}(y)$ and $W := \{\omega_1, \ldots, \omega_n\}$ be an integral basis of K w.r.t. y. Then any $f \in K$ is of the form

$$f = \frac{1}{D}\sum_{i=1}^{n} a_i\omega_i \quad \text{with } a_i, D \in \mathbb{F}[y].$$

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Modified Trager's reduction:

$$f = D_y(g) + \sum_{i=1}^{n} \left(\frac{p_i}{d} + \frac{q_i}{y^{\tau_i} e} \right) \omega_i, \quad \text{where}$$

- $\tau_1, \dots, \tau_n \in \mathbb{N}, d, e \in \mathbb{F}[y]$ are squarefree, only depend on W;
- $p_i, q_i \in \mathbb{F}[y]$ with $\deg_y(p_i) < \deg_y(d)$ and q_i's are in a finitely dimensional $\mathbb{F}$-vector space.

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Proposition.

$$f = D_y(g) \quad \text{for some } g \in K \quad \Leftrightarrow \quad p_i = q_i = 0$$

Telescoping via reductions

Consider

$$f = \frac{1}{xy^3 + xy + 1}$$

Find nonzero $L \in K(x)\langle D_x \rangle$ s.t. $L(f) = D_y(g)$ for $g \in K(x,y)$.

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$$D_x(f) = D_y(g_1) + \frac{-4x^2 + 6xy + 9}{x(4x^2 + 27)(xy^3 + xy + 1)}$$

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$$(8x^2 - 3) \cdot \frac{1}{xy^3 + xy + 1}$$

$$+ (16x^3 + 27x) \cdot \frac{-4x^2 + 6xy + 9}{x(4x^2 + 27)(xy^3 + xy + 1)}$$

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$$= \frac{-12xy^4 + 18xy^2 + 4x + 3y}{(xy^3 + xy + 1)^2}$$

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&= \Delta_k\left(g_1\right) + \left(-\frac{1/n}{n+k} + \frac{1}{n}\right)H \\
S_n^2(T) &= \Delta_k\left(\cdots\right) + \left(-\frac{1/(n+1)}{n+k+1} + \frac{1}{n+1}\right)H \\
&= \Delta_k\left(g_2\right) + \left(-\frac{1/(n(n+1))}{n+k} + \frac{n-1}{n(n+1)}\right)H
\end{aligned}$$

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$$-1 \cdot \frac{1}{n+k}$$

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$$\begin{aligned} G &= (n+1) \cdot g_2 - (n-1) \cdot g_1 - 1 \cdot g_0 \\ &= \frac{k!}{(n+k)(n+k+1)} \end{aligned}$$

Timings (in seconds)

Let

$$T = \frac{f(n,k)}{g_1(n+k)g_2(2n+k)} \frac{\Gamma(2\alpha n+k)}{\Gamma(n+\alpha k)}$$

with

- $g_i(z) = p_i(z)p_i(z+\lambda)p_i(z+\mu)$, $\alpha, \lambda, \mu \in \mathbb{N}$,
- $\deg(p_1) = \deg(p_2) = m$ and $\deg(f) = n$.

(m,n,α,λ,μ)	Zeilberger	RCT+cert	RCT	order
$(2,0,1,5,10)$	354.46	58.01	4.93	4
$(2,0,2,5,10)$	576.31	363.25	53.15	6
$(2,0,3,5,10)$	2989.18	1076.50	197.75	7
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Thank you!