Rational-Transcendental Dichotomy Theorems on Power Series with Arithmetic Restrictions

Shaoshi Chen

KLMM, AMSS
Chinese Academy of Sciences

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Joint with J. P. Bell, E. Hossain, K. Nguyen and U. Zannier

Rational integrals

Theorem. Let $f \in \mathbb{C}(x)$ be a rational function over $\mathbb{C}$. Then

$$\int f(x)\, dx$$

is either rational or transcendental over $\mathbb{C}(x)$.

Theorem (van der Poorten, 1971). Let $f \in \mathbb{Q}(x)$ be a rational function over $\mathbb{Q}$. Then

$$\int_a^b f(x)\, dx \quad \text{with } a, b \in \mathbb{Q}$$

is either rational or transcendental over $\mathbb{Q}$.

Remark. Proof by using Baker's theorem on linear forms in the logarithms of algebraic numbers.

Polya-Cantor theorem

Problem. For $f(x) \in \mathbb{C}(x)$, when $\int f(x)\,dx$ is rational?

Theorem (Polya, 1921, Cantor, 1965). Let

$$f = \sum_{n=0}^{+\infty} a_n x^n \in \mathbb{Z}[[x]] \cap \mathbb{Q}(x)$$

be such that $(n+1) \mid a_n$ for all $n \in \mathbb{N}$. Then $\int f(x)\,dx$ is rational.

Remark. This follows from the fact by Polya: For $f \in \mathbb{Z}[[x]]$, f is rational iff f is globally bounded and df/dx is rational.

Theorem (André, 1989). Let K be a number field and $f \in K[[x]]$. Then f is algebraic iff f is globally bounded and df/dx is algebraic.

Mahler's functions and automatic numbers

Definition. A function $F \in \mathbb{C}[[x]]$ is k-Mahler if there exist $d \in \mathbb{N}$ and $a_0, \ldots, a_d \in \mathbb{C}[x]$ with $a_0 a_d \neq 0$ s.t.

$$a_0(x)F(x) + a_1(x)F(x^k) + \cdots + a_d(x)F(x^{k^d}) = 0.$$

Theorem (Noshioka, 1996). A k-Mahler function is either rational or transcendental.

Definition. A real number $\alpha \in (0,1)$ is automatic if the digits of its b-ary expansion can be generated by a finite automata.

Theorem (Adamczewski, Bugeaud, Luca, 2004). An automatic number is either rational or transcendental.

Hadamard's problem on power series

In 1892, Hadamard in his thesis said that

> *"Indeed, the Taylor expansion does not reveal the properties of the function represented, and even seems to mask them completely. "*

Hadamard then considered the following problem:

> *What relationships are there between the coefficients of a power series and the singularities of the function it represents?*

Two special cases of the problem have been studied:

- Power series with rational or integral coefficients;
- Power series with finitely distinct coefficients.

Power series with rational coefficients

$$f(x) = \sum_{n \geq 0} a_n x^n, \quad \text{where } a_n \in \mathbb{Q}.$$

Gotthold Eisenstein (1823-1852)

G. Eisenstein, Uber eine allgemeine Eigenschaft der Reihenentwicklungen aller algebraischen Funcktionen, Belin, Sitzber, 441-443, 1852

On the general properties of the series expansions of algebraic functions

Theorem (Eisenstein 1852, Heine 1853). If $f(x)$ represents an algebraic function over $\mathbb{Q}(x)$, then $\exists\ T \in \mathbb{Z}$, s.t.

$$\sum_{n \geq 0} a_n T^n x^n \in \mathbb{Z}[[x]].$$

Power series with integral coefficients

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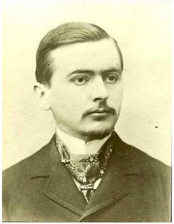

Séries trigonométriques et séries de Taylor

P. Fatou

Acta Mathematica **30**, 335–400(1906)

Fatou's PhD thesis *Séries trigonométriques et séries de Taylor* (Fatou 1906) was the first application of the Lebesgue integral to concrete problems of analysis, mainly to the study of analytic and harmonic functions in the unit disc. In this work, Fatou studied for the first time the Poisson integral of an arbitrary measure on the unit circle. This work of Fatou is influenced by Henri Lebesgue who invented his integral in 1901.

Fatou's Lemma. If $f(x)$ represents a rational function, then

$$f(x) = \frac{P(x)}{Q(x)}, \quad \text{where } P, Q \in \mathbb{Z}[x] \text{ and } Q(0) = 1.$$

Fatou's Theorem. If $f(x)$ converges inside the unit disk, then it is either rational or transcendental over $\mathbb{Q}(x)$.

Power series with integral coefficients

$$f(x) = \sum_{n \geq 0} a_n x^n, \quad \text{where } a_n \in \mathbb{Z}.$$

George Pólya (1887-1985)

George Pólya, *Uber Potenzreihen mit ganzzahligen Koeffizienten*, Math. Ann. **77** (1916), no. 4, 497–513.

Fritz Carlson, *Über Potenzreihen mit ganzzahligen Koeffizienten*, Math. Z. **9** (1921), no. 1-2, 1–13.

Pólya-Carlson Theorem. If $f(x)$ converges inside the unit disk, then either it is rational or has the unit circle as natural boundary.

Corollary (Fatou1906). If $f(x)$ is algebraic, then it is rational.

Power series with finitely distinct coefficients

$$f(x) = \sum_{n\geq 0} a_n x^n, \quad \text{where } a_n \in \Delta \subseteq \mathbb{C} \text{ with } |\Delta| < +\infty.$$

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Fatou's Theorem. A power series with finitely distinct coefficients in $\mathbb{C}$ is either rational or transcendental over $\mathbb{C}(x)$.

Power series with finitely distinct coefficients

$$f(x) = \sum_{n\geq 0} a_n x^n, \quad \text{where } a_n \in \Delta \subseteq \mathbb{C} \text{ with } |\Delta| < +\infty.$$

Gábor Szegő (1895-1985)

From 1917 to 1922, there are four papers with the same title:

Über Potenzreihen mit endlich vielen verschiedenen Koeffizienten.

Power Series with Finitely Distinct Coefficients

1. G. Polya in 1917, Math. Ann.
2. R. Jentzsch in 1918, Math. Ann.
3. F. Carlson in 1919, Math. Ann.
4. G. Szego in 1922, Math Ann.

Szegö's Theorem (1922)

A power series with finitely distinct coefficients in $\mathbb{C}$ is either rational or has the unit circle as its natural boundary.

Arithmetical aspects of power series

Problem. Decide whether a given power series is rational, algebraic, transcendental, or hyper-transcendental?

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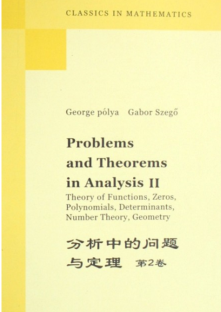

Chapter 3. Arithmetical Aspects of Power Series

Arithmetical aspects of power series

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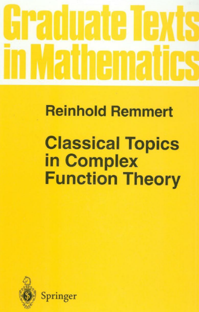

Project. Arithmetic theory of power series in several variables

D-finite power series

Let K be a field of characteristic zero.

Definition. A series $f(x_1, \ldots, x_d) \in K[[x_1, \ldots, x_d]]$ is D-finite if

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Definition. A series $f(x_1,\ldots,x_d) \in K[[x_1,\ldots,x_d]]$ is D-finite if all derivatives $D_{x_1}^{i_1}\cdots D_{x_d}^{i_d}(f)$ form a finite-dimensional vector space over $\mathbb{K}(x_1,\ldots,x_d)$.

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Let K be a field of characteristic zero.
Definition. A series $f(x_1,\ldots,x_d) \in K[[x_1,\ldots,x_d]]$ is D-finite if for each $i \in \{1,\ldots,d\}$, f satisfies a LPDE:

$$p_{i,r_i} D_{x_i}^{r_i}(f) + p_{i,r_i-1} D_{x_i}^{r_i-1}(f) + \cdots + p_{i,0} f = 0.$$

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- R. P. Stanley. Differentiably Finite Power Series. *European Journal of Combinatorics*, 1: 175–188, 1980.
- L. Lipshitz. D-Finite Power Series. *Journal of Algebra*, 122: 353–373, 1989.
- M. Kauers. D-Finite Functions. Springer, 2023, 664 pages.

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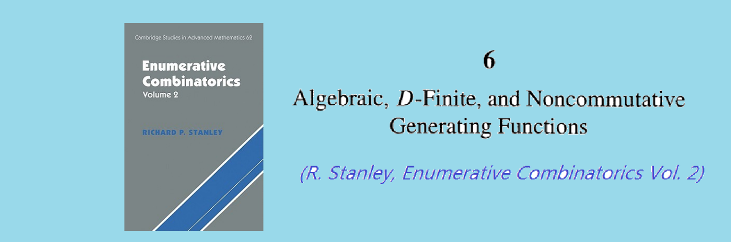

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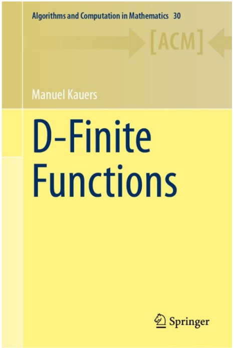

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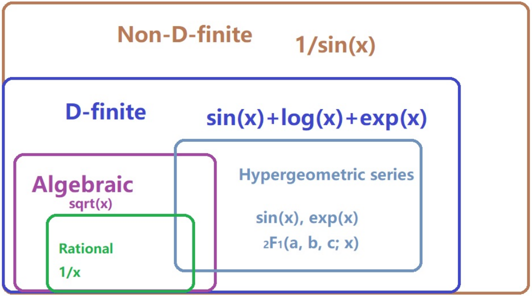

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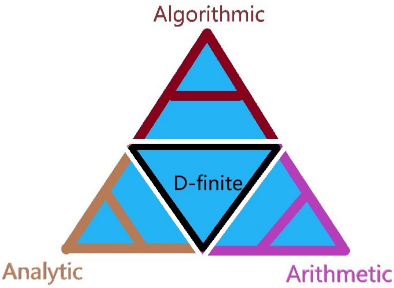

Algebraic Numbers $\rightsquigarrow$ D-finite Functions

Power series with integral coefficients (the multivariate case)

Multivariate extensions of the Pólya-Carlson Theorem:

- André Martineau, *Extension en n-variables d'un théorème de Pólya-Carlson concernant les séries de puissances à coefficients entiers*, C. R. Acad. Sci. Paris Sér. A-B **273** (1971), A1127–A1129. MR 0291495

- V. P. Šeĭnov, *Transfinite diameter and certain theorems of Pólya in the case of several complex variables*, Sibirsk. Mat. Ž. **12** (1971), 1382–1389.

- Emil J. Straube, *Power series with integer coefficients in several variables*, Comment. Math. Helv. **62** (1987), no. 4, 602–615. MR 920060

Theorem (BellChen, 2016) If the multivariate power series

$$F = \sum f(n_1, \ldots, n_d) x_1^{n_1} \cdots x_d^{n_d} \in \mathbb{Z}[[x_1, \ldots, x_d]]$$

is D-finite and converges on the unit polydisc, then it is rational.

Power series with finitely distinct coefficients (the multivariate case)

Theorem (van der Poorten & Shparlinsky, 1994).
Let $a_n : \mathbb{N} \to \Delta$, where Δ is a finite subset of $\mathbb{Q}$. If the generating function $f(x) = \sum_n a_n x^n$ is D-finite, then it is rational.

Remark. This follows from Szegö's theorem.

Theorem (BellChen, JCTA 2017). Let $a_{n_1,\ldots,n_d} : \mathbb{N}^d \to \Delta$, where Δ is a finite subset of K with $\mathrm{char}(K) = 0$. If the generating function

$$f(x_1, \ldots, x_d) = \sum a_{n_1,\ldots,n_d} x_1^{n_1} \cdots x_d^{n_d}$$

is D-finite, then it is rational.

Nonnegative integer points on algebraic varieties

Let V be an algebraic variety over an algebraically closed field K of characteristic zero. We define the listing generating function

$$F_V(x_1, \ldots, x_d) := \sum_{(n_1,\ldots,n_d) \in V \cap \mathbb{N}^d} x_1^{n_1} \cdots x_d^{n_d}$$

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We may ask the following questions:

When F_V is a rational function?

Remark. If F_V is rational, then nonnegative integer points distribute semi-linearly.

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Corollary.

$$F_V \text{ is } D\text{-finite} \quad \Leftrightarrow \quad F_V \text{ is rational.}$$

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We may ask the following questions:

When F_V is a D-finite function?

Theorem.

The problem of testing whether F_V is rational is undecidable!

Nonnegative integer points on algebraic curves

Theorem. Let $p(x,y) \in \mathbb{C}[x,y]$. If the generating function

$$F_p(x,y) := \sum_{(n,m)\in V(p)\cap\mathbb{N}^2} x^n y^m$$

is rational. Then $p = f \cdot g$, where $f, g \in \mathbb{C}[x,y]$ s.t.

$$f = \prod_i (s_i \cdot x + t_i \cdot y + c_i) \quad \text{with } s_i, t_i \in \mathbb{Z} \text{ and } c_i \in \mathbb{C}$$

and g has only finite zeros in $\mathbb{N}^2$.

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Example. Let $p = x^2 - y$. Since p is not a product of integer-linear polynomials, the power series $F_p(x,y)$ is not D-finite.

Beyond D-finite

Definition. $F \in K[[x_1, \ldots, x_d]]$ is differentially algebraic if the transcendence degree of the field generated by the derivatives $D_{x_1}^{i_1} \cdots D_{x_d}^{i_d}(F)$ with $i_j \in \mathbb{N}$ over $K(x_1, \ldots, x_d)$ is finite.

Conjecture. Let V be an algebraic variety over $\mathbb{C}$. Then the power series

$$\sum_{(n_1,\ldots,n_d) \in V \cap \mathbb{N}^d} x_1^{n_1} \cdots x_d^{n_d}$$

is differentially algebraic if and only if it is rational.

Open problems

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Example. Let $p = x^2 - y$. Then the power series

$$F_p(x,y) := \sum_{m\geq 0} x^m y^{m^2}$$

is not differentially algebraic, otherwise, $F_p(x,2) = \sum 2^{m^2} x^m$ is differentially algebraic. By Mahler's lemma, we get a contradiction

$$2^{m^2} \ll (m!)^c \quad \text{for any positive constant } c.$$

Open problems

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is differentially algebraic if and only if it is rational.

Conjecture (Chowla-Chowla-Lipshitz-Rubel). The power series

$$f := \sum_{n\in\mathbb{N}} x^{n^3} \in \mathbb{C}[[x]]$$

is not differentially algebraical, i.e., satisfies no ADE.

Remark. The power series $\sum x^{n^2}$ is differentially algebraic.

P-recursive sequences

Definition. A sequence $s : \mathbb{N} \longrightarrow K$ is P-recursive over K if

$$p_d \cdot s(n+d) + p_{d-1} \cdot s(n+d-1) + \cdots + p_0 \cdot s(n) = 0,$$

where $p_i \in K[n]$. If all p_i are constants in K, we call $s(n)$ C-finite.

Theorem (Stanley, 1980). Let $f(x) = \sum_{n \geq 0} a_n x^n \in K[[x]]$. Then

$$a_n \text{ is P-recursive} \quad \Leftrightarrow \quad f \text{ is D-finite}$$

Remark. This correspondence is not true for multivariate sequences.

$$\sum_{n_1, n_2 \geq 0} \frac{1}{n_1^2 + n_2^2 + 1} \cdot y_1^{n_1} y_2^{n_2} \text{ is not D-finite!}$$

Skolem-Mahler-Lech theorem: the C-finite case

Let $s : \mathbb{N} \to K$ be C-finite over K with char(K)=0. Define

$$\mathbf{Z}_s := \{i \in \mathbb{N} \mid s(i) = 0\}.$$

Theorem. (Skolem 1934, Mahler 1935, Lech 1953) $\mathbf{Z}_s$ is a union of finitely many arithmetic progressions, i.e.,

$$\mathbf{Z}_s = \left(\bigcup_{j=1}^{t} \{d_j n + c_j \mid d_j, c_j, n \in \mathbb{N}\} \right) \cup \{i_1, \ldots, i_s\}, \quad \text{where } s, t < +\infty \ .$$

Example. Let $s(n)$ be the sequence defined by

$$s(n+6) = 6s(n+4) - 12s(n+2) + 8s(n)$$

with $(s(0), \ldots, s(5))$=$(8, 0, 9, 0, 8, 0)$. Then $\mathbf{Z}_s$=$\{8\} \cup \{2n+1 \mid n \in \mathbb{N}\}$.

Skolem-Mahler-Lech theorem: the P-recursive case

Let $s(n)$ be a P-recursive sequence over K with

$$p_d \cdot s(n+d) + p_{d-1} \cdot s(n+d-1) + \cdots + p_0 \cdot s(n) = 0.$$

Define

$$\mathbf{Z}_s := \{i \in \mathbb{N} \mid s(i) = 0\}.$$

Rubel's Conjecture (1983). $\mathbf{Z}_s$ is a union of finitely many arithmetic progressions.

Remark. This conjecture is the linear case of Dynamical Mordell-Lang Conjecture on algebraic dynamics.

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Theorem. (Bell–Burris–Yeats, 2012) If $p_d = 1$ and p_{d-1} is nonzero constant polynomial, then Rubel's Conjecture is true.

Skolem-Mahler-Lech theorem: the P-recursive case

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Theorem. (Bézivin, 1989; Bell-Chen-Hossian, ANT2021)
Let $G \subseteq K^{\times}$ be a finitely generated abelian group. Then Rubel's Conjecture is true if $s(n)$ is P-recursive and $s(n) \in G \cup \{0\}$.

Height

Height is a measure of average-complexity of algebraic numbers.

Definition. For $\alpha \in \overline{\mathbb{Q}}$ with its minimal polynomial

$$p(x) = a_d(x-\beta_1)\cdots(x-\beta_d) \in \mathbb{Z}[x],$$

we let

$$M(\alpha) := |a_d| \cdot \prod_i \max(1, |\beta_i|).$$

Then the absolute logarithmic Weil height $h(\alpha)$ is $\log(M(\alpha))/d$.

Remark. For $r = a/b \in \mathbb{Q}$ with $a, b \in \mathbb{Z}$, we have

$$h(a/b) = \max(\log|a|, \log|b|).$$

Height pattern of D-finite series I

Theorem (Bell-Nguyen-Zannier, 2019)

Let $f(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{N}_0^m} a_{\mathbf{n}}\mathbf{x}^{\mathbf{n}} \in \overline{\mathbb{Q}}[[\mathbf{x}]]$. If f is D-finite and

$$\lim_{\|\mathbf{n}\|\to\infty} \frac{h(a_{\mathbf{n}})}{\log\|\mathbf{n}\|} = 0. \tag{1}$$

Then f is rational.

Remark. This result generalizes the rationality theorem of Bell and Chen, helped to inspire Dimitrov's recent spectacular solution of the Schinzel-Zassenhauss conjecture from the 1960s.

J. P. Bell, K. Nguyen and U. Zannier. D-finiteness, Rationality, and Height. Trans. Amer. Math. Soc. 373 (2020), 7: 4889–4906.

Height pattern of D-finite series II

Theorem (Bell-Nguyen-Zannier, 2023). Let $f(z) = \sum_n a_n z^n \in \overline{\mathbb{Q}}[[z]]$ be D-finite and r be the radius of convergence of f.

(a) If $r \in \{0, \infty\}$ and $f \notin \overline{\mathbb{Q}}[z]$ then $h(a_n) = O(n \log n)$ and $h(a_n) \gg n \log n$ on a set of positive upper density.

(b) If $r \notin \{0, \infty\}$ then one of the following holds:

  (i) $h(a_n) \gg n$ on a set of positive upper density;

  (ii) $\mathrm{den}(a_n) \gg n$, and hence $h(a_n) > \frac{1}{[K:\mathbb{Q}]} \log n + O(1)$, on a set of positive upper density;

  (iii) $f(z)$ is a rational function .

J. P. Bell, K. Nguyen and U. Zannier. D-finiteness, Rationality, and Height II: Lower bounds over a set of positive density. Adv. Math. 414, 1 February 2023, 108859.

Height pattern of D-finite series II

Problem (Height gaps). Let $f(z) \in \overline{\mathbb{Q}}[[z]]$ be D-finite, is it true that one of the following holds?

(i) $h(a_n) = O(n \log n)$ and $h(a_n) \gg n \log n$ for n in a set of positive density;

(ii) $h(a_n) = O(n)$ for every n and $h(a_n) \gg n$ for n in a set of positive density;

(iii) $h(a_n) = O(\log n)$ for every n and $h(a_n) \gg \log n$ for n in a set of positive density;

(iv) $h(a_n) = O(1)$ for every n.

Height pattern of D-finite series III

Theorem (Bell-Chen-Nguyen-Zannier, 2023)

Let $m \in \mathbb{N}$ and let $F(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{N}_0^m} f(\mathbf{n})\mathbf{x}^{\mathbf{n}} \in \overline{\mathbb{Q}}[[\mathbf{x}]]$ be D-finite. For $N \in \mathbb{N}_0$, let $h_N = \max\{h(f(\mathbf{n})) : |\mathbf{n}| \leq N\}$ and

$$d_N = \mathrm{lcm}\{\mathrm{den}(f(\mathbf{n})) : |\mathbf{n}| \leq N\}$$

If $h_N = o(N)$ and $\log d_N = o(N)$ as $N \to \infty$, then

(a) F is a rational function.

(b) Up to scalar multiplication, every irreducible factor of the denominator of F has the form:

$$1 - \zeta \mathbf{x}^{\mathbf{n}}$$

where ζ is a root of unity and $\mathbf{n} \in \mathbb{N}_0^m \setminus \{0\}$.

Hadamard product

Definition. Let $f = \sum a(\mathbf{i})\mathbf{x}^{\mathbf{i}}$ and $g = \sum b(\mathbf{i})\mathbf{x}^{\mathbf{i}}$ be in $K[[\mathbf{x}]]$, where $\mathbf{x}^{\mathbf{i}} = x_1^{i_1}\cdots x_n^{i_n}$. The Hadamard product of f and g is

$$f \odot g = \sum a(\mathbf{i})b(\mathbf{i})\mathbf{x}^{\mathbf{i}}.$$

n	f	g	$f \odot g$
1	rational	rational	rational
1	rational	alg.	alg.
1	alg.	alg.	maybe trans.
2	rational	rational	alg.
2	rational	alg.	maybe trans.
$n > 2$	rational	rational	maybe trans.
$n \geq 1$	D-finite	D-finite	D-finite

Rationality theorems

In 1980, Stanley conjectured that for all $k \geq 2$, the series

$$\sum_{n=0}^{+\infty} \binom{2n}{n}^k x^n$$

is transcendental.

Remark. This conjecture was proved independently by Flajolet in 1987 and by Woodcock and Sharif in 1989.

Theorem (BenzaghouBézivin1992). If $f(x) \in \mathbb{Q}[[x]]$ is D-finite and $f(x) \odot f(x)$ is rational, then $f(x)$ is rational.

Remark. With Singer and Zannier, we found two more proofs: one is arithmetic and another one is Galois-theoretical.

Conjecture (Zannier 2023 ???). If $f(x) \in \mathbb{Q}[[x]]$ is algebraic and $f(x) \odot f(x)$ is algebraic, then $f(x)$ is rational.

Summary

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Thanks!