Symbolic Integration

A Brief Introduction

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Integration Problems

Indefinite Integration. Given a function $f(x)$ in certain class $\mathfrak{C}$, decide whether there exists $g(x) \in \mathfrak{C}$ such that

$$f = \frac{dg}{dx} \triangleq g'.$$

Example. For $f = \log(x)$, we have $g = x\log(x) - x$.

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Definite Integration. Given a function $f(x)$ that is continuous in the interval $I \subseteq \mathbb{R}$, compute the integral

$$\int_I f(x)dx.$$

Example. For $f = \log(x)$ and $I = [1,2]$, we have

$$\int_I f(x)dx = 2\log(2) - 1.$$

Fundamental Theorem of Calculus

Newton–Leibniz Theorem. Let $f(x)$ be a continuous function on $[a,b]$ and let $F(x)$ be defined by

$$F(x) = \int_a^x f(t)dt \quad \text{for all } x \in [a,b].$$

Then $F(x)' = f(x)$ for all $x \in [a,b]$ and

$$\int_a^b f(x)\,dx = F(b) - F(a). \qquad \text{(Newton–Leibniz formula)}$$

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Definite Integration $\rightsquigarrow$ Indefinite Integration

$$\int_1^2 \log(x)\,dx = F(2) - F(1) = 2\log(2) - 1, \quad \text{where } F(x) = x\log(x) - x.$$

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Definite Integration $\rightsquigarrow$ Indefinite Integration

$$\int_0^{+\infty} \exp(-x^2)\,dx = ?$$

What is Elementary Functions?

- Polynomials: $P(x) \in \mathbb{C}[x]$

$$P(x) = p_0 + p_1 x + \cdots + p_n x^n, \quad \text{where } p_i \in \mathbb{C}.$$

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- Rational functions: $f(x) \in \mathbb{C}(x)$

$$f(x) = \frac{P(x)}{Q(x)}, \quad \text{where } P, Q \in \mathbb{C}[x] \text{ and } Q \neq 0.$$

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- Algebraic functions: $\alpha(x) \in \overline{\mathbb{C}(x)}$

$$r_d \alpha^d + r_{d-1} \alpha^{d-1} + \cdots + r_0 = 0, \quad \text{where } r_i \in \mathbb{C}(x).$$

What is Elementary Functions?

- Exponential functions: $f(x) = \exp(g(x))$ with $g \in \overline{\mathbb{C}(x)}$

$$f'(x) = \exp(g(x)) \cdot g'(x) = f(x) \cdot g'(x).$$

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$$f'(x) = \frac{g'(x)}{g(x)}.$$

- Trigonometric functions: $\sin(x), \cos(x), \tan(x), \dots$

$$\sin(x) = \frac{\exp(ix) - \exp(-ix)}{2i}, \quad \cos(x) = \frac{\exp(ix) + \exp(-ix)}{2}.$$

What is Elementary Functions?

$$\mathfrak{E} := (\{\mathbb{C}, x\}, \quad \{+, -, \times, \div\}, \quad \{\exp(\cdot), \log(\cdot), \mathsf{RootOf}(\cdot)\}).$$

Definition. An elementary function is a function of x which is the composition of a finite number of

- binary operations: $+, -, \times, \div$;
- unitary operations: exponential, logarithms, constants, solutions of polynomial equations.

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Example.

$$3x^2 + 3x + 1$$

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Example.

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Example.

$$\exp\left(\sqrt{\frac{1}{3x^2+3x+1}}\right)^2 + x^2 + 1$$

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$$\log\left(\exp\left(\sqrt{\frac{1}{3x^2+3x+1}}\right)^2 + x^2 + 1\right)$$

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Example.

$$\frac{\pi}{\sqrt{\log\left(\exp\left(\sqrt{\frac{1}{3x^2+3x+1}}\right)^2 + x^2 + 1\right)}}$$

Differential Algebra

Differential Ring and Differential Field. Let R be an integral domain. An additive map $D : R \to R$ is called a derivation on R if

$$D(f \cdot g) = f \cdot D(g) + g \cdot D(f). \qquad \text{(Leibniz's rule)}$$

The pair (R, D) is called a differential ring. If R is a field, it is then called a differential field.

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Example.
Polynomial ring: $(\mathbb{C}[x], ')$

$$P = \sum_{i=0}^{n} p_i x^i \quad \rightsquigarrow \quad P' = \sum_{i=0}^{n} i p_i x^{i-1}.$$

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Example.
Rational-function field: $(\mathbb{C}(x), ')$

$$f = \frac{P}{Q} \quad \rightsquigarrow \quad f' = \frac{P'Q - PQ'}{Q^2}.$$

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Example.
Elementary-function field: algebraic case

$$(\mathbb{C}(x)(\alpha), ') \text{ with } \alpha \text{ algebraic over } \mathbb{C}(x)$$

$$r_d\alpha^d + r_{d-1}\alpha^{d-1} + \cdots + r_0 = 0 \quad \rightsquigarrow \quad \alpha'(x) = -\frac{r_d'\alpha^d + \cdots + r_0'}{dr_d\alpha^{d-1} + \cdots + r_1}$$

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Example.
Elementary-function field: exponential case

$$(\mathbb{C}(x)(\exp(x)), ')$$

$$f = \frac{1 + x + \exp(x)}{x^2 + \exp(x)} \quad \rightsquigarrow \quad f' = \frac{x(x\exp(x) - 3\exp(x) - x - 2)}{(x^2 + \exp(x))^2}.$$

Differential Algebra

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Example.
Elementary-function field: logarithmic case

$$(\mathbb{C}(x)(\log(x)), ')$$

$$f = \frac{1 + x + \log(x)}{x^2 + \log(x)} \quad \rightsquigarrow \quad f' = -\frac{2\log(x)x^2 + x^3 - \log(x)x + x^2 + x + 1}{(x^2 + \log(x))^2 x}.$$

Differential Algebra

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$$D(f \cdot g) = f \cdot D(g) + g \cdot D(f). \qquad \text{(Leibniz's rule)}$$

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Example.
Elementary-function field: general case

$$(\mathbb{C}(x)(t_1, t_2, t_3, \ldots, t_n), ')$$

$$t_1 = \sqrt{x^2+1}, \quad t_2 = \log(1+t_1^2), \quad t_3 = \exp\left(\frac{1+t_1}{t_1+t_2^2}\right), \ldots$$

Elementary Extensions

Differential Extension. (R^*, D^*) is called a differential extension of (R, D) if $R \subseteq R^*$ and $D^* |_R = D$.

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Elementary Extension. Let (F, D) be a differential extension of (E, D). An element $t \in F$ is elementary over E if one of the following conditions holds:

- t is algebraic over E;
- $D(t)/t = D(u)$ for some $u \in E$, i.e., $t = \exp(u)$;
- $D(t) = D(u)/u$ for some $u \in E$, i.e., $t = \log(u)$.

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Example. $(E, D) = (\mathbb{C}(x), ')$ and $(F, D) = (\mathbb{C}(x, \log(x)), ')$.

Elementary Functions

Definition. An function $f(x)$ is elementary if $\exists$ a differential extension $(F,')$ of $(\mathbb{C}(x),')$ s.t. $F = \mathbb{C}(x)(t_1,\ldots,t_n)$ and t_i is elementary over $\mathbb{C}(x)(t_1,\ldots,t_{i-1})$ for all $i = 2,\ldots,n$.

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Example.

$$f(x) = \frac{\pi}{\sqrt{\log\left(\exp\left(\sqrt{\frac{1}{3x^2+3x+1}}\right)^2 + x^2 + 1\right)}}$$

Then $f(x)$ is elementary since $\exists$ a differential extension

$$F = \mathbb{C}(x)(t_1,t_2,t_3,t_4),$$

where

$$t_1 = \sqrt{\frac{1}{3x^2+3x+1}}, \quad t_2 = \exp(t_1), \quad t_3 = \log(t_2^2 + x^2 + 1), \quad t_4 = \sqrt{t_3}.$$

Symbolic Integration

Let (E, D) and (F, D) be two differential field such that $E \subseteq F$.

Problem. Given $f \in E$, decide whether there exists $g \in F$ s.t. $f = D(g)$. If such g exists, we say f is integrable in F.

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Elementary Integration Problem. Given an elementary function $f(x)$ over $\mathbb{C}(x)$, decide whether $\int f(x)dx$ is elementary or not.

Example. The following integrals are not elementary over $\mathbb{C}(x)$:

$$\int \exp(x^2)dx, \quad \int \frac{1}{\log(x)}dx, \quad \int \frac{\sin(x)}{x}dx, \quad \int \frac{dx}{\sqrt{x(x-1)(x-2)}}, \cdots$$

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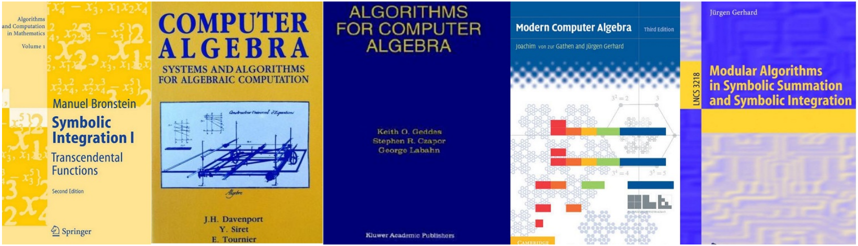
Selected books on Symbolic Integration:



Symbolic Integration: Theoretical Developments

Timeline: from 1827 to 1948

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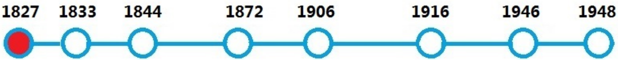
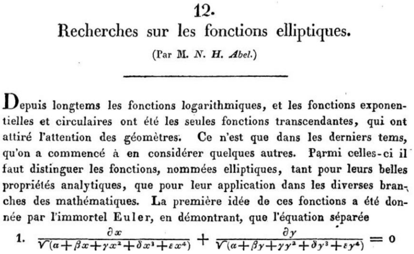


1827: Abel studied the elliptic integrals.



12.

Recherches sur les fonctions elliptiques.

(Par M. *N. H. Abel.*)

Depuis longtems les fonctions logarithmiques, et les fonctions exponentielles et circulaires ont été les seules fonctions transcendantes, qui ont attiré l'attention des géomètres. Ce n'est que dans les derniers tems, qu'on a commencé à en considérer quelques autres. Parmi celles-ci il faut distinguer les fonctions, nommées elliptiques, tant pour leurs belles propriétés analytiques, que pour leur application dans les diverses branches des mathématiques. La première idée de ces fonctions a été donnée par l'immortel Euler, en démontrant, que l'équation séparée

$$1. \quad \frac{\partial x}{\sqrt{(\alpha+\beta x+\gamma x^2+\delta x^3+\varepsilon x^4)}} + \frac{\partial y}{\sqrt{(\alpha+\beta y+\gamma y^2+\delta y^3+\varepsilon y^4)}} = 0$$

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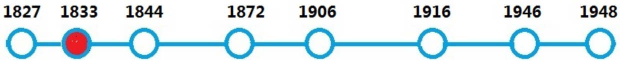

1833-1841: Liouville's theory of elementary integration.

PREMIER MÉMOIRE

Sur la détermination des Intégrales dont la valeur est algébrique;

PAR JOSEPH LIOUVILLE *).

SECOND MÉMOIRE

Sur la détermination des Intégrales dont la valeur est algébrique;

PAR JOSEPH LIOUVILLE.

Liouville's Theorem: Let y be an arbitrary algebraic function of x. If the integral $\int y\,dx$ is expressible in finite explicit form, it is always possible to write

$$\int y\,dx = t + A\log u + B\log v + \cdots + C\log w, \qquad ([2])$$

where $A, B, \ldots, C$ are constants and $t, u, v, \ldots, w$ are algebraic functions of x.

[Liouville 1834c, p. 42]

(1809--1882)

Symbolic Integration: Theoretical Developments

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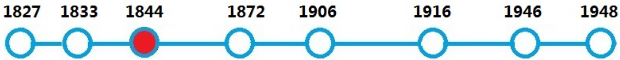
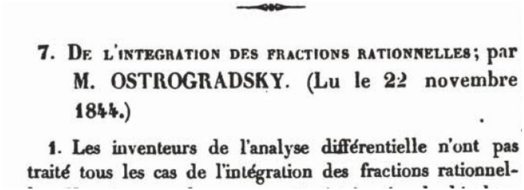


1844: Ostrogradsky presented a method for rational integration.

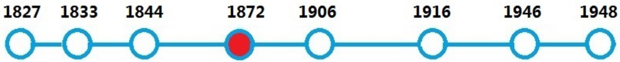


7. De l'intégration des fractions rationnelles; par M. OSTROGRADSKY. (Lu le 22 novembre 1844.)

1. Les inventeurs de l'analyse différentielle n'ont pas traité tous les cas de l'intégration des fractions rationnel-

Symbolic Integration: Theoretical Developments

Timeline: from 1827 to 1948



1872: Hermite gave a reduction method for rational integration.

SUR L'INTÉGRATION

DES

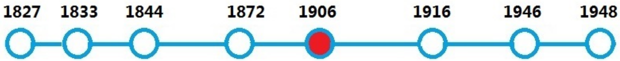
FRACTIONS RATIONNELLES,

PAR M. HERMITE,

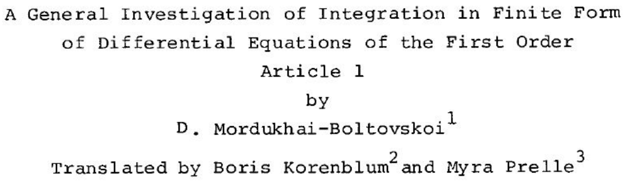
MEMBRE DE L'INSTITUT DE FRANCE.

Symbolic Integration: Theoretical Developments

Timeline: from 1827 to 1948

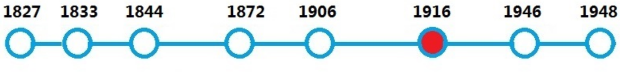


1906: Mordukhai-Boltovskoi studied the problem of solving the differential equations in finite terms.

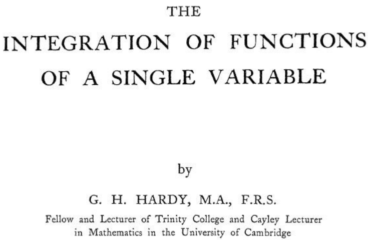

A General Investigation of Integration in Finite Form
of Differential Equations of the First Order
Article 1
by
D. Mordukhai-Boltovskoi[1]
Translated by Boris Korenblum[2] and Myra Prelle[3]

Symbolic Integration: Theoretical Developments

Timeline: from 1827 to 1948



1916: Hardy wrote a book on elementary integration.



THE

INTEGRATION OF FUNCTIONS OF A SINGLE VARIABLE

by

G. H. HARDY, M.A., F.R.S.

Fellow and Lecturer of Trinity College and Cayley Lecturer in Mathematics in the University of Cambridge

(1877--1947)

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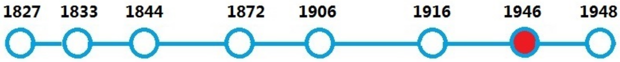

1946: Ostrowski initialized an algebraic approach for elementary integration.

Sur l'intégrabilité élémentaire de quelques classes d'expressions

Par M. A. Ostrowski, Bâle

rationnelle en lg z et z. Nous avons complètement résolu cette question de sorte que l'on peut maintenant à l'aide de calculs purement algébriques reconnaître si l'intégrale de (1) est une fonction élémentaire ou non.

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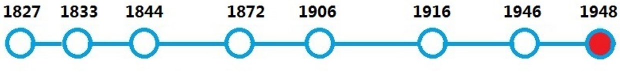
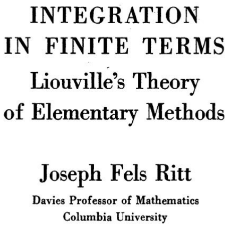


1948: Ritt summarized the works on integration in finite terms.

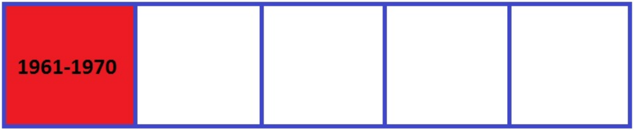

INTEGRATION
IN FINITE TERMS
Liouville's Theory
of Elementary Methods

Joseph Fels Ritt
Davies Professor of Mathematics
Columbia University

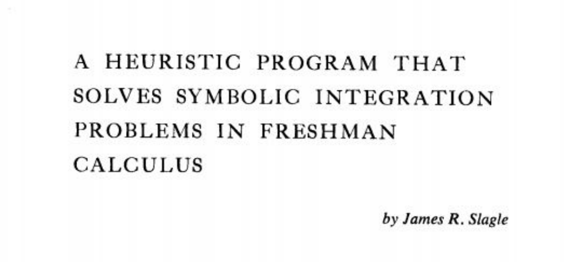

(1893--1951)

Symbolic Integration: Algorithmic Developments

Symbolic Integration: Algorithmic Developments



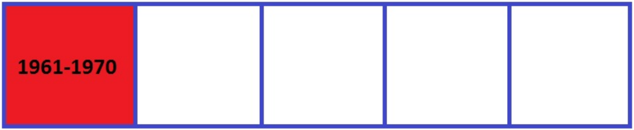
1961: Slagle wrote the program SAINT for symbolic integration.



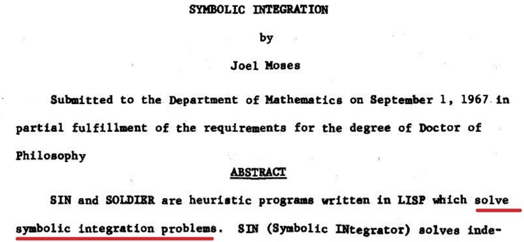
A HEURISTIC PROGRAM THAT SOLVES SYMBOLIC INTEGRATION PROBLEMS IN FRESHMAN CALCULUS

by James R. Slagle

Symbolic Integration: Algorithmic Developments



1967: Moses wrote the programs SIN and SOLDIER for symbolic integration.



SYMBOLIC INTEGRATION

by

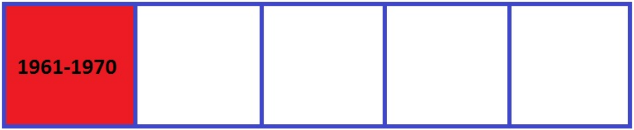
Joel Moses

Submitted to the Department of Mathematics on September 1, 1967 in partial fulfillment of the requirements for the degree of Doctor of Philosophy

ABSTRACT

SIN and SOLDIER are heuristic programs written in LISP which solve symbolic integration problems. SIN (Symbolic INtegrator) solves inde-

Symbolic Integration: Algorithmic Developments



1968: Rosenlicht's differential-algebraic proof of Liouville's theorem.

PACIFIC JOURNAL OF MATHEMATICS
Vol. 24, No. 1, 1968

LIOUVILLE'S THEOREM ON FUNCTIONS WITH ELEMENTARY INTEGRALS

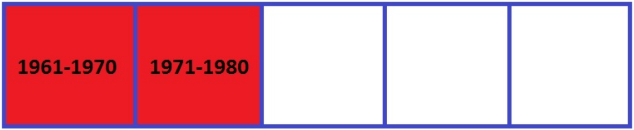
MAXWELL ROSENLICHT

Integration in Finite Terms

Maxwell Rosenlicht

The American Mathematical Monthly, Vol. 79, No. 9 (Nov., 1972), 963-972.

Symbolic Integration: Algorithmic Developments



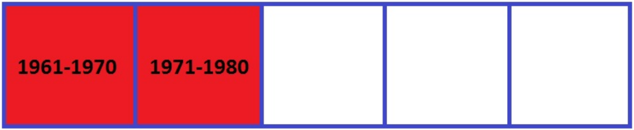
1971: Moses's survey on symbolic integration.

Symbolic Integration: The Stormy Decade

Joel Moses*
Project MAC, MIT, Cambridge, Massachusetts

Three approaches to symbolic integration in the 1960's are described. The first, from artificial intelligence, led to Slagle's SAINT and to a large degree to Moses' SIN. The second, from algebraic manipulation, led to Manove's implementation and to Horowitz' and Tobey's reexamination of the Hermite algorithm for integrating rational functions. The third, from mathematics, led to Richardson's proof of the unsolvability of the problem for a class of functions and for Risch's decision procedure for the elementary functions. Generalizations of Risch's algorithm to a class of special functions and programs for solving differential equations and for finding the definite integral are also described.

Symbolic Integration: Algorithmic Developments



1976: Rothstein's algorithm for integration of transcendental elementary functions

ASPECTS OF SYMBOLIC INTEGRATION AND SIMPLIFICATION OF

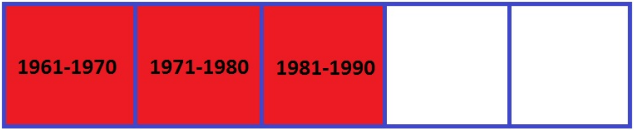
EXPONENTIAL AND PRIMITIVE FUNCTIONS

by

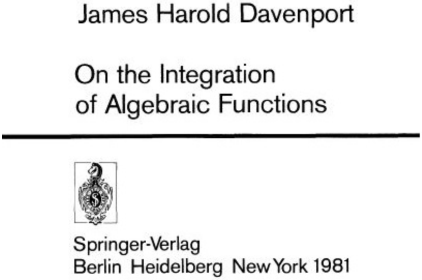
MICHAEL ROTHSTEIN

1976

Symbolic Integration: Algorithmic Developments



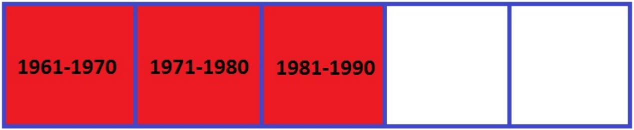
1981: Davenport's algorithm for integration of algebraic functions


James Harold Davenport

On the Integration
of Algebraic Functions

Springer-Verlag
Berlin Heidelberg New York 1981

Symbolic Integration: Algorithmic Developments



1984: Trager's algorithm for integration of algebraic functions

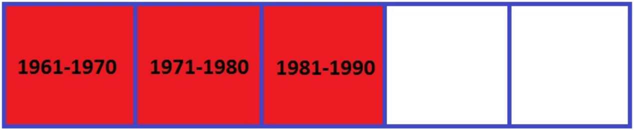
INTEGRATION OF ALGEBRAIC FUNCTIONS

by

BARRY MARSHALL TRAGER

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

September 1984



1985: Singer, Saunders, and Caviness presented an extension of Liouville's theorem

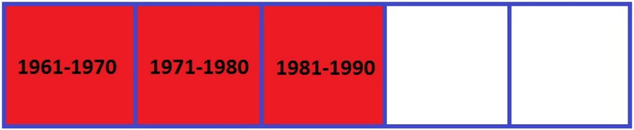
SIAM J. COMPUT.
Vol. 14, No. 4, November 1985

015

AN EXTENSION OF LIOUVILLE'S THEOREM ON INTEGRATION IN FINITE TERMS*

M. F. SINGER†, B. D. SAUNDERS‡ AND B. F. CAVINESS§

Symbolic Integration: Algorithmic Developments



1985: Cherry's algorithm for integration with the error function

J. Symbolic Computation (1985) **1**, 283–302

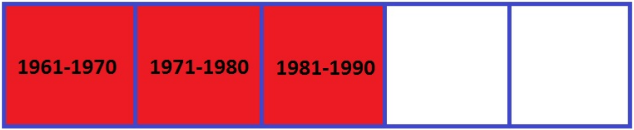
Integration in Finite Terms with Special Functions: the Error Function†

G. W. CHERRY

Tektronix, Inc., Beaverton, Oregon

(*Received* 20 *December* 1984)

Symbolic Integration: Algorithmic Developments



1990: Bronstein's algorithm for integration of elementary functions

J. Symbolic Computation (1990) **9**, 117–173

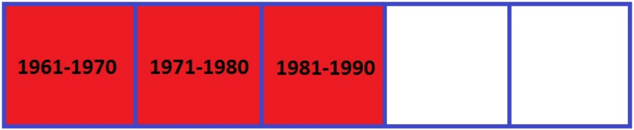
Integration of Elementary Functions[1]

Manuel Bronstein

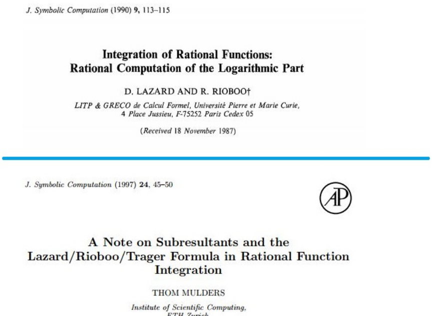
Mathematical Sciences Department, IBM Research Division,
T.J. Watson Research Center, Yorktown Heights, NY 10598, *USA*

(*Received* 1 *September* 1988)

Symbolic Integration: Algorithmic Developments



1990: Computation of the logarithmic part via subresultants


J. Symbolic Computation (1990) 9, 113–115

Integration of Rational Functions:
Rational Computation of the Logarithmic Part

D. LAZARD AND R. RIOBOO†
LITP & GRECO de Calcul Formel, Université Pierre et Marie Curie,
4 Place Jussieu, F-75252 Paris Cedex 05

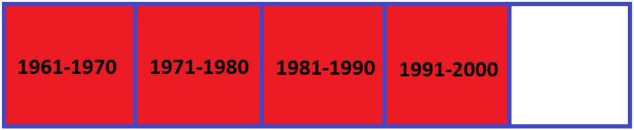
(*Received* 18 *November* 1987)

J. Symbolic Computation (1997) 24, 45–50

A Note on Subresultants and the
Lazard/Rioboo/Trager Formula in Rational Function
Integration

THOM MULDERS
Institute of Scientific Computing,

Symbolic Integration: Algorithmic Developments



1992: Knowles' algorithm for integration with the error function

J. Symbolic Computation (1992) 13, 525–543

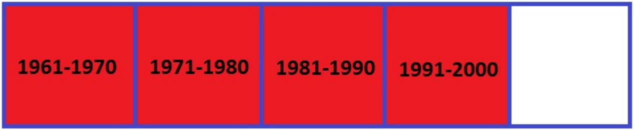
Integration of a Class of Transcendental Liouvillian Functions with Error-Functions, Part I

PAUL H. KNOWLES

D'Youville College, 320 Porter Avenue, Buffalo, NY 14201, U.S.A.

(*Received* 16 *May* 1988)

Symbolic Integration: Algorithmic Developments



1994: Baddoura's algorithm for integration with the dilogarithms

Integration in Finite Terms with Elementary Functions and Dilogarithms

by

Mohamed Jamil Baddoura

Diplôme d'Ingénieur, École Polytechnique (1980)
S.M., Massachusetts Institute of Technology (1982)

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

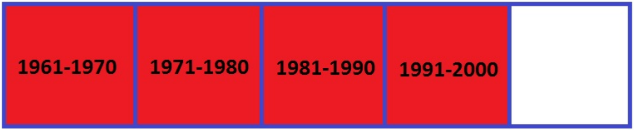
Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

January 1994

Symbolic Integration: Algorithmic Developments



1995: Computation of the logarithmic part via Groebner bases

J. Symbolic Computation (1995) **20**, 163–167

A Note on Gröbner Bases and Integration of Rational Functions

GÜNTER CZICHOWSKI

Institut für Mathematik und Informatik, Universität Greifswald, F.L. Jahnstr.15a, D-17487 Greifswald, Germany

(Received 20 October 1994)

Symbolic Integration: Algorithmic Developments

1961-1970	1971-1980	1981-1990	1991-2000	2001-now

2008: Kauers's algorithm for computing the logarithmic part of algebraic integration

Integration of Algebraic Functions:
A Simple Heuristic for Finding the Logarithmic Part

Manuel Kauers*
RISC-Linz
Johannes Kepler Universität
A-4040 Linz, Austria
mkauers@risc.uni-linz.ac.at

Symbolic Integration: Algorithmic Developments

1961-1970	1971-1980	1981-1990	1991-2000	2001-now

2012: Raab's algorithm for the logarithmic part of the integrals of transcendental functions

Journal of Symbolic Computation 47 (2012) 1290–1296

Contents lists available at SciVerse ScienceDirect

Journal of Symbolic Computation

journal homepage: www.elsevier.com/locate/jsc

Using Gröbner bases for finding the logarithmic part of the integral of transcendental functions

Clemens G. Raab

Research Institute for Symbolic Computation, Johannes Kepler University, 4040 Linz, Austria

Symbolic Integration: Algorithmic Developments

1961-1970	1971-1980	1981-1990	1991-2000	2001-now

2014: Zannier found some unlikely intersections between elementary integration and number theory

Elementary integration of differentials in families and conjectures of Pink

Umberto Zannier

Abstract. In this short survey paper we shall consider, in particular, indefinite integrals of differentials on algebraic curves, trying to express them in *elementary terms*. This is an old-fashioned issue, for which Liouville gave an explicit criterion that may be considered a primordial example of differential

. . . . . .

Symbolic Integration: Algorithmic Developments

1961-1970	1971-1980	1981-1990	1991-2000	2001-now

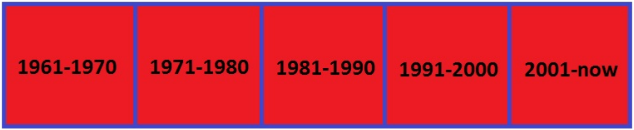
2010: Creative telescoping for rational functions via Hermite reduction

Complexity of Creative Telescoping for Bivariate Rational Functions*

Alin Bostan, Shaoshi Chen, Frédéric Chyzak
Algorithms Project-Team, INRIA
Paris-Rocquencourt
78153 Le Chesnay (France)
{alin.bostan,shaoshi.chen,frederic.chyzak}@inria.fr

Ziming Li
Key Laboratory of Mathematics Mechanization,
Academy of Mathematics and System Sciences
100190 Beijing (China)
zmli@mmrc.iss.ac.cn

Symbolic Integration: Algorithmic Developments



2013: Creative telescoping for hyperexponential functions via Hermite reduction

Hermite Reduction and Creative Telescoping for Hyperexponential Functions*

Alin Bostan[1], Shaoshi Chen[2], Frédéric Chyzak[1], Ziming Li[3], Guoce Xin[4]
[1]INRIA, Palaiseau, 91120, (France)
[2]Department of Mathematics, NCSU, Raleigh, 27695-8025, (USA)
[3]KLMM, AMSS, Chinese Academy of Sciences, Beijing 100190, (China)
[4]Department of Mathematics, Capital Normal University, Beijing 100048, (China)
schen@amss.ac.cn, {alin.bostan, frederic.chyzak}@inria.fr
zmli@mmrc.iss.ac.cn, guoce_xin@163.com

Symbolic Integration: Algorithmic Developments

1961-1970	1971-1980	1981-1990	1991-2000	2001-now

2016: Creative telescoping for algebraic functions via Hermite reduction

Reduction-Based Creative Telescoping for Algebraic Functions*

Shaoshi Chen[1,2], Manuel Kauers[3], Christoph Koutschan[4]

[1]KLMM, AMSS, Chinese Academy of Sciences, Beijing, 100190, (China)
[2]Symbolic Computation Group, University of Waterloo, Ontario, N2L3G1, (Canada)
[3]Institute for Algebra, Johannes Kepler University, Altenberger Straße 69, A-4040 Linz, (Austria)
[4]RICAM, Austrian Academy of Sciences, Altenberger Straße 69, A-4040 Linz, (Austria)
schen@amss.ac.cn, manuel.kauers@jku.at
christoph.koutschan@ricam.oeaw.ac.at

Liouville's Theorem: the Rational Case

Theorem. Let $f \in \mathbb{C}(x)$. Then $f(x)$ is elementary integrable. Moreover,

$$\int f(x)\,dx = \underbrace{g_0}_{\text{rational part}} + \underbrace{\sum_{i=1}^{n} c_i \log(g_i)}_{\text{transcendental part}},$$

where $g_0, g_1, \ldots, g_n \in \mathbb{C}(x)$ and $c_1, \ldots, c_n \in \mathbb{C}$.

Ostrogradsky–Hermite Reduction. Any $f \in \mathbb{C}(x)$ can be decomposed into

$$f = g' + \frac{p}{q},$$

where $g \in \mathbb{K}(x)$, $\deg(p) < \deg(q)$, and q is squarefree. Moreover,

$$\int f\,dx \text{ is rational} \quad \Leftrightarrow \quad p = 0$$

Liouville's Theorem: the Algebraic Case

Theorem (Liouville1834). Let $f(x)$ be algebraic over $\mathbb{C}(x)$. If $\int f(x)\,dx$ is elementary, then

$$\int f(x)\,dx = \underbrace{g_0}_{\text{algebraic part}} + \underbrace{\sum_{i=1}^{n} c_i \log(g_i)}_{\text{transcendental part}},$$

where $g_0, g_1, \ldots, g_n \in \mathbb{C}(x, f(x))$ and $c_1, \ldots, c_n \in \mathbb{C}$.

Liouville's Theorem: the Algebraic Case

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where $g_0, g_1, \ldots, g_n \in \mathbb{C}(x, f(x))$ and $c_1, \ldots, c_n \in \mathbb{C}$.

Remark. With the above theorem, Liouville proved in 1834 that the elliptic integral

$$\int \frac{1}{\sqrt{x(x-1)(x-2)}}$$

is not elementary.

Liouville's Theorem: the Elementary Case

Theorem (Liouville1835). Let $f(x)$ be elementary over $\mathbb{C}(x)$, i.e.,

$$f \in F = \mathbb{C}(x)(t_1, t_2, \ldots, t_n).$$

If $\int f(x)\,dx$ is elementary, then

$$\int f(x)\,dx = \underbrace{g_0}_{F\text{-part}} + \underbrace{\sum_{i=1}^{n} c_i \log(g_i)}_{\text{transcendental part}} ,$$

where $g_0, g_1, \ldots, g_n \in F$ and $c_1, \ldots, c_n \in \mathbb{C}$.

Liouville's Theorem: the Elementary Case

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$$\int f(x)\,dx = \underbrace{g_0}_{F\text{-part}} + \underbrace{\sum_{i=1}^{n} c_i \log(g_i)}_{\text{transcendental part}},$$

where $g_0, g_1, \ldots, g_n \in F$ and $c_1, \ldots, c_n \in \mathbb{C}$.

Remark. With the above theorem, Liouville proved that the integrals

$$\int \exp(x^2)\,dx, \quad \int \frac{1}{\log(x)}\,dx, \quad \int \frac{\sin(x)}{x}\,dx, \ldots$$

are not elementary.

Why $\exp(x^2)$ is not Elementary Integrable?

Let $t = \exp(x^2)$. We prove by contradiction.

Proof. If $\int t\,dx$ is elementary, Liouville's theorem implies that $\exists g_0, \ldots, g_n \in \mathbb{C}(x,t)$ and $c_0, \ldots, c_n \in \mathbb{C}$ s.t.

$$\int t\,dx = g_0 + \sum_{i=1}^{n} c_i \log(g_i) \quad \Leftrightarrow \quad t = g_0' + \sum_{i=1}^{n} c_i \frac{g_i'}{g_i}$$

$$\Downarrow$$

$$t = (ft)' \quad \text{for some } f \in \mathbb{C}(x) \quad \Leftrightarrow \quad 1 = f' + 2xf$$

Claim. The differential equation

$$y(x)' + 2x \cdot y(x) = 1$$

has no rational-function solution!

Welcome to Symbolic Integration!

Thank You!