抽象代数 II (第三次作业题)
授课老师: 陈绍示(中国科学院数学与系统科学研究院)

1. Let R be a ring without identity and set $S = \mathbb{Z} \times R$. We define the addition and multiplication in S as follows

$$(m, a) + (n, b) = (m + n, a + b) \quad \text{and} \quad (m, a) \cdot (n, b) = (mn, mb + na + ab)$$

for all $m, n \in \mathbb{Z}$ and $a, b \in R$. Show that

(1) $(S, +, \cdot)$ forms a ring with identity $(1, 0)$;

(2) The map $\phi : R \to S$ defined by $\phi(r) = (0, r)$ for $r \in R$ is a injective homomorphism (单同态);

(3) Let $\bar{R}$ be a ring with identity $\bar{e}$ and $\eta : R \to \bar{R}$ be a ring homomorphism. Then $\bar{\eta} : S \to \bar{R}$ defined by $\bar{\eta}((m, a)) = m\bar{e} + \eta(a)$ is also a ring homomorphism.

2. Let M be an R-module and let $\mathrm{Hom}_R(R, M)$ be the abelian group of all R-module homomorphisms from R to M. Define the action of R on $\mathrm{Hom}_R(R, M)$ by

$$(a \cdot f)(r) = f(ra) \quad \text{for } f \in \mathrm{Hom}_R(R, M) \text{ and } a, r \in R.$$

Define the map $\eta : \mathrm{Hom}_R(R, M) \to M$ by $\eta(f) = f(1)$ for $f \in \mathrm{Hom}_R(R, M)$. Show that $\mathrm{Hom}_R(R, M)$ is an R-module and η is an R-module isomorphism (R-模同构).

3. Let V be the linear space of all smooth functions (光滑函数，即无穷次可微函数) over $(0, +\infty)$ and $D : V \to V$ be the derivation (导数) on V. Then V is a $\mathbb{R}[z]$-module via the action $p(z) \cdot f(x) = p(D)(f(x))$ for $p \in \mathbb{R}[z]$ and $f(x) \in V$.

(1) Describe the submodule generated by $\sin(x)$ and the annihilator of $\sin(x)$;

(2) Describe the submodule generated by $1/x$ and show that the annihilator of $1/x$ is (0).

4. Let M and M' be two $\mathbb{Z}$-modules. Show that M and M' are isomorphic (同构) as $\mathbb{Z}$-modules if they are isomorphic as abelian groups.

5. Let M be a finite abelian group and $M \neq 0$. Can M be a left $\mathbb{Q}$-module ?