Recall: • A differential ring (R,δ): (1) R is a commutative ring with unity 1 and
(2) $\delta : R \longrightarrow R$ is a derivation (i.e., additive and leibniz rule.)

(If R is a domain, δ can be extended uniquely to $\mathrm{Frac}(R)$ and $(\mathrm{Frac}(R),\delta)$) is a differential field.

• A differential ideal $\mathcal{I} \subset R$: an ideal with $\delta(\mathcal{I}) \subset \mathcal{I}$.

• Notation: Given $S \subset R$, (S), $[S]$, $\{S\}$ are respectively the ideal, differential ideal and radical differential ideal generated by S in R.

• In general, $\{S\} \neq \sqrt{[S]}$ and a maximal differential ideal may not be prime.

But if $\mathbb{Q} \subset R$, $\{S\} = \sqrt{[S]}$ always hold:

Theorem 1.2.3 Let (R,δ) be a differential ring, $\mathbb{Q} \subseteq R$ and let $I \subseteq (R,\delta)$ be a differential ideal. Then, $\sqrt{I}$ is a radical differential ideal.

Proof. It suffices to show $\sqrt{I}$ is a differential ideal. For this purpose, for each $a \in \sqrt{I}$ (i.e., $\exists\, a \in \mathbb{N}, a^n \in I$), to show $\delta(a) \in \sqrt{I}$. Claim: For each $k, 1 \leq k \leq n$, $a^{n-k}(\delta(a))^{2k-1} \in I$. We show the claim by induction on k and $\delta(a) \in \sqrt{I}$ will follow by allowing $k = n$ $\left((\delta(a))^{2n-1} \in I \Rightarrow \delta(a) \in \sqrt{I}\right)$. If $k = 1$, $\delta(a^n) = na^{n-1}\delta(a) \in I$. Since $\mathbb{Q} \subseteq R$, $a^{n-1}\delta(a) \in I$. Suppose $a^{n-k}(\delta(a))^{2k-1} \in I$ for some $1 \leq k < n$. Then, $\delta(a^{n-k}(\delta(a))^{2k-1}) = (n-k)a^{n-(k+1)}(\delta(a))^{2k} + a^{n-k}(2k-1)\delta(a)^{2k-2}\delta^2(a) \in I$. Multiply the above by $\delta(a)$, we get $a^{n-(k+1)}(\delta(a))^{2k+1} \in I$ and we are done. □

For simplicity, from Section 1.3 to Chapte 5, we shall focus on the ordinary differential case.

1.3 Decomposition of radical differential ideals

In computational algebraic geometry, we have studied decompositions of radical ideals. In differential algebra, we have analogous arguments (Theorem 1.3.5).

Lemma 1.3.1. *Let (R,δ) be a differential ring and I a radical differential ideal of R. If $ab \in I$, then $a\delta(b) \in I$ and $\delta(a)b \in I$.*

Proof. $ab \in I \Rightarrow \delta(ab) = \delta(a)b + a\delta(b) \in I \Rightarrow a\delta(b)\cdot\delta(ab) = (a\delta(b))^2 + ab\delta(a)\delta(b) \in I \Rightarrow (a\delta(b))^2 \in I$. Since I is radical, $a\delta(b) \in I$ and $\delta(a)b \in I$ follows. □

Lemma 1.3.2. *Let I be a radical differential ideal of the differential ring R and $S \subseteq R$ be any subset. Then $I : S = \{a \in R \mid aS \subseteq I\}$ is a radical differential ideal.*

Proof.
- $\forall a, b \in I : S$, $r \in R$, $aS \subseteq I$ and $bS \subseteq I \Rightarrow (a+b)S \subseteq I$ and $raS \subseteq I$ $\Rightarrow a + b \in I : S, ra \in I : S$. So $I : S$ is an ideal.
- $\forall a \in I : S, aS \subseteq I$. By Lemma 1.3.1, $\delta(a)S \subseteq I \Rightarrow \delta(a) \in I : S$. So $I : S$ is a differential ideal.
- $\forall a \in R$, suppose $\exists\, n \in \mathbb{N}, a^n \in I : S$. Then $a^nS \subseteq I$. So for $\forall s \in S$, $a^ns \in I \overset{\times s^{n-1}}{\Longrightarrow} a^ns^n \in I \Rightarrow as \in I$ for $\forall s \in S \Rightarrow a \in I : S$.

Thus, $I : S$ is a radical differential ideal.

□

Lemma 1.3.3. *Let $S, T \subset R$. Then we have $\{S\}\{T\} \subseteq \{ST\}$ and $\{S\} \cap \{T\} = \{ST\}$.*

Proof. For each $a \in S$, by Lemma 1.3.2, $\{aT\} : a$ is a radical differnetial ideal. Since $T \subseteq \{aT\} : a, \{T\} \subseteq \{aT\} : a$. So $a\{T\} \subseteq \{aT\} \subseteq \{ST\}$. Thus, $S \subset \{ST\} : \{T\}$. Again by Lemma 1.3.2, $\{S\} \subset \{ST\} : \{T\}$, and $\{S\}\{T\} \subseteq \{ST\}$ follows.

The assertion $\{S\} \cap \{T\} = \{ST\}$ follows from i) and ii):

i) $ST \subseteq \{S\}, \{T\} \Rightarrow \{ST\} \subseteq \{S\} \cap \{T\}$;

ii) $\forall a \in \{S\} \cap \{T\}, a^2 \in \{S\} \cdot \{T\} \subseteq \{ST\}$. So $a \in \{ST\}$.

□

We now use the above lemmas to show the following result.

Lemma 1.3.4. *Let $T \subseteq R$ be a subset closed under multiplication and let P be maximal among radical differential ideals that do not intersect T. Then P is prime.*

Proof. Suppose the contrary, i.e., P is not prime. Let $a, b \in R$ be such that $ab \in P$ but $a \notin P$ and $b \notin P$. Hence $P \subsetneqq \{P, a\}, P \subsetneqq \{P, b\}$. Thus, $\exists\, t_1 \in \{P, a\} \cap T$, $\exists\, t_2 \in \{P, b\} \cap T$. So $t_1 t_2 \in T$ but $t_1 t_2 \in \{P, a\} \cdot \{P, b\} \subseteq \{P, ab\} = P$, a contradiction to $P \cap T = \emptyset$. □

In a commutative ring R, the nilradical $\sqrt{(0)}$ of R is the intersection of all the prime ideals of R and every radical ideal of R is the intersection of all prime ideals containing it. In differential algebra, we have a similar result which is our main theorem of this section.

Theorem 1.3.5. *Let $I \subsetneqq R$ be a radical differential ideal. Then I can be represented as an intersection of prime differential ideals.*

Proof. We first construct for each $x \notin I$ a prime differential ideal P_x such that $P_x \supseteq I$ and $x \notin P_x$. Let $T = \{x^n \mid n \in \mathbb{N}\}$. The set $U = \{P \subseteq R \mid P$ is a radical differential ideal of $R, I \subseteq P, P \cap T = \emptyset\}$ is nonempty since $I \in U$. By Zorn's Lemma, $\exists$ a maximal element P_x in U. P_x is prime by Lemma 1.3.5, and since $P_x \cap T = \emptyset$, $x \notin P_x$. Clearly, $I = \bigcap_{x \notin I} P_x$ is an intersection of prime differential ideals.

□

In Section 1.2, we gave an example showing a maximal differential ideal might not be prime. But if $\mathbb{Q} \subseteq R$, then a maximal differential ideal in R is always prime.

Corollary 1.3.6. *Let $\mathbb{Q} \subseteq (R, \delta)$ and M be maximal among proper differential ideals. Then M is prime.*

Proof. Consider $\{M\} = \sqrt{[M]} = \sqrt{M}$. If $\sqrt{M} = R$, then $1 \in \sqrt{M} \Rightarrow 1 \in M$, which contradicts M being proper. Therefore, $\sqrt{M} = M$, M is a radical differential ideal. By Theorem 1.3.6, $M = \bigcap_{\alpha \notin M} P_\alpha$ where P_α is a prime differential ideal. Therefore, for all $\alpha \notin M, M = P_\alpha$ and thus, M is prime.

□

Remark: A differential ring R with $\mathbb{Q} \subseteq R$ is called a *Ritt Algebra*. We have shown in Section 1.2 and Section 1.3, in a Ritt Algebra:

1) The radical differential ideal $\{S\} = \sqrt{[S]}$;

2) A maximal differential ideal is a prime differential ideal;

3) Even in a Ritt Algebra R, the quotient R/M (M is a maximal differential ideal) might not be a differential field.
   Example: Let $R = \mathbb{Q}[x]$ with $\delta(x) = 1$. Then $[0]$ is the unique maximal differential ideal. $R/[0] = R$ is not a differential field.

Chapter 2

Differential polynomial rings and the basis theorem

2.1 The ring of differential polynomials

Let (K, δ) be a fixed differential field of characteristic 0. We hope to develop an algebraic structure and algebraic theory for ordinary differential equations.

Definition 2.1.1. *Let (L, δ) be a differential field extension of (K, δ). A subset S of L is said to be differentially dependent over K if the set $(\delta^k(s))_{k \in \mathbb{N}, s \in S}$ is algebraically dependent over K. In the contrary case, S is said to be δ-independent over K, or a family of differential indeterminates over K. In the case $S = \{\alpha\}$, we say that α is differentially algebraic over K or differentially transcendental over K respectively.*

Example: Let $(K, \delta) = (\mathbb{Q}(x), \frac{\mathrm{d}}{\mathrm{d}x})$ and $(L, \delta) = (\mathbb{C}(x, \mathrm{e}^x), \frac{\mathrm{d}}{\mathrm{d}x})$. Clearly, each $c \in \mathbb{C}$ and $\alpha = \mathrm{e}^x$ are differentially algebraic over K.

Now suppose $\mathbb{Y} = \{y_1, \ldots, y_n\}$ is a set of differential indeterminates over K.

Definition 2.1.2. *The ring of differential polynomials in $y_1, \ldots, y_n$ over K is the ring of polynomials*

$$K[\delta^k y_j \mid k \in \mathbb{N}, j = 1, \ldots, n]^1, \text{ denoted by } K\{y_1, \ldots, y_n\}.$$

Its elements are called differential polynomials. Note that $K\{y_1, \ldots, y_n\}$ is a differential ring with the derivation operator δ extending $\delta \mid_K$ and $\delta(\delta^k y_j) = \delta^{k+1}(y_j)$.

Example:

1) $u_{xx} = v_x \longleftrightarrow \delta^2 y_1 - \delta y_2 = 0.$

2) $(\frac{\mathrm{d}u}{\mathrm{d}t})^2 = 4u\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} \longleftrightarrow (\delta y_1)^2 - 4y_1\delta^2(y_1) = 0.$

Definition 2.1.3. *Let (R_1, δ_1) and (R_2, δ_2) be two differential rings. A* **differential homomorphism** *from (R_1, δ_1) to (R_2, δ_2) is a ring homomorphism $\varphi : R_1 \to R_2$ such that $\varphi \circ \delta_1 = \delta_2 \circ \varphi$. If R_0 is a common differential subring of R_1 and R_2, and $\varphi \mid_{R_0} = \mathrm{id}_{R_0}$, φ is called a differential homomorphism over R_0.*

[1]When there is no confusion, we also write $y_j', y_j'', y_j''', y_j^{(n)}$ for $\delta(y_j), \delta^2(y_j), \delta^3(y_j), \delta^n y_j \ (n > 3)$.

$$\begin{array}{ccc} a & \xrightarrow{\varphi} & \varphi(a) \\ \Big\downarrow{\scriptstyle \delta_1} & & \Big\downarrow{\scriptstyle \delta_2} \\ \delta_1(a) & \xrightarrow{\varphi} & \varphi(\delta_1(a)) \end{array}$$

We give two examples of differential homomorphisms:

1) Let $(K, \delta) \subseteq (L, \delta)$ be two differential fields. Then $\mathrm{id}_K : (K, \delta) \to (L, \delta)$ is a differential homomorphism.

2) (Evaluation homomorphism) Take an element $\eta = (\eta_1, \ldots, \eta_n) \in L^n$, then the map

$$\begin{array}{rcl} \varphi_\eta : K\{y_1, \ldots, y_n\} & \longrightarrow & L \\ \delta^k(y_i) & \longmapsto & \delta^k(\eta_i) \end{array}$$

with $\varphi_\eta = f(\eta_1, \ldots, \eta_n) := f|_{\delta^k(y_i)=\delta^k(\eta_i), k\geq 0}$ is a differential homomorphism over K. Note that the evaluation homomorphism is uniquely determined by the value $\varphi(y_i)$.

Proposition 2.1.4. *Let (R_1, δ) and (R_2, δ) be two differential rings and $\varphi : R_1 \to R_2$ be a differential homomorphism. Then $\mathrm{Ker}(\varphi)$ is a differential ideal.*

Proof. $\mathrm{Ker}(\varphi)$ is an ideal of R, since φ is a homomorphism of rings. For each $r \in \mathrm{Ker}(\varphi), \varphi(r) = 0$, so $\delta(\varphi(r)) = 0 = \varphi(\delta(r)) \Rightarrow \delta(r) \in \mathrm{Ker}(\varphi)$. □

Corollary 2.1.5. *Let (R, δ) be a differential ring and I be an ideal of R. Then I is a differential ideal of $R \iff (R/I, \delta)$ is a differential ring).*

Proof. "⇒" Let $r + I \in R/I$. Define

$$\delta(r + I) = \delta(r) + I. \quad (*)$$

To show $(*)$ is well-defined, let $r_1 + I = r_2 + I$, we need to show $\delta(r_1) + I = \delta(r_2) + I$. Since $r_1 - r_2 \in I$ and I is a differential ideal, $\delta(r_1 - r_2) = \delta(r_1) - \delta(r_2) \in I$. So $\delta(r_1) + I = \delta(r_2) + I$.
To show $(*)$ is a derivation on R/I. Let $r_1 + I, r_2 + I \in R/I$, then $\delta(r_1 + I + r_2 + I) = \delta(r_1 + r_2 + I) = \delta(r_1) + \delta(r_2) + I = \delta(r_1 + I) + \delta(r_2 + I)$ and $\delta((r_1 + I)(r_2 + I)) = \delta(r_1r_2 + I) = \delta(r_1)r_2 + r_1\delta(r_2) + I = \delta(r_1 + I) \cdot (r_2 + I) + (r_1 + I) \cdot \delta(r_2 + I)$.

"⇐" Let $\varphi : R \to R/I$ be defined by $\varphi(r) = r + I$ for each $r \in R$. Then $\forall\, r \in R, \varphi(\delta(r)) = \delta(r) + I = \delta(r + I) = \delta(\varphi(r))$, so φ is a differential homomorphism. By Proposition 2.0.4, $I = \mathrm{Ker}(\varphi)$ is a differential ideal of R. □

Definition 2.1.6. *Let $\Sigma \subseteq K\{y_1, \ldots, y_n\}$ and $\eta = (\eta_1, \ldots, \eta_n)$ be a point from a differential extension field (L, δ) of (K, δ). We call η a* **differential zero** *of Σ if for each $f \in \Sigma, f(\eta) = 0$, (that is, $\Sigma \subseteq \mathrm{Ker}(\varphi_\eta : K\{y_1, \ldots, y_n\} \to L^n)$).*

The point η is called a **generic zero** of a differential ideal $I \subseteq K\{y_1, \ldots, y_n\}$ if for each $f \in K\{y_1, \ldots, y_n\}$, $f(\eta_1, \ldots, \eta_n) = 0 \Leftrightarrow f \in I$.

Example: In the algebraic case, $I = (x^2 + y^2 - 1) \subseteq \mathbb{Q}[x, y]$ has a generic point $(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2})$. Also, $(\cos(\theta), \sin(\theta))$ is another generic point. So generic points are not unique.

Lemma 2.1.7. *Let $P \subseteq K\{y_1, \ldots, y_n\}$ be a differential ideal. Then P has a generic zero if and only if P is prime.*

Proof. "$\Rightarrow$" Suppose η is a generic zero of P. For any $f, g \in K\{y_1, \dots, y_n\}$, if $fg \in P$, then $f(\eta)g(\eta) = 0$ which implies $f \in P$ or $g \in P$. So P is a prime differential ideal.
"$\Leftarrow$" Suppose P is a prime differential ideal. Then $K\{y_1, \dots, y_n\}/P$ is a differential domain. Let $L = \mathrm{Frac}(K\{y_1, \dots, y_n\}/P)$ and $\bar{y}_i = y_i + P$. Then $(\bar{y}_1, \dots, \bar{y}_n) \in L^n$ is a generic zero of P. Indeed, $\forall f \in P, f(\bar{y}_1, \dots, \bar{y}_n) = f(y_1, \dots, y_n) + P = \bar{0} \in L$ and $\forall f \in K\{y_1, \dots, y_n\}$, if $f(\bar{y}_1, \dots, \bar{y}_n) = 0$, then $f(y_1, \dots, y_n) \in P$.

□

Definition 2.1.8. *Let (R, δ) be a differential ring. An element $c \in R$ is said to be a* **constant** *if $\delta(c) = 0$. The set of all constants of R is a differential subring of R, called the* **ring of constants of** *R, denoted by C_R. If R is a differential field, C_R is a field, called the field of constants of R.*

Examples:

1) $R = \mathbb{Q}[x]$, $\delta(x) = 1. C_R = \mathbb{Q}$.

2) $R = \mathbb{Z}_p(x^p)$, $\delta(x) = 1$. Then $C_R = R$.

Lemma 2.1.9. *Let $(\mathcal{F}, \delta)$ be a differential field of characteristic 0 and $C_{\mathcal{F}} = \mathcal{F}$. Let $L \supseteq \mathcal{F}$ be a differential field extension and L be algebraic over $\mathcal{F}$. Then $C_L = L$.*

Proof. Let $a \in L$. Suppose $p(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathcal{F}[x]$ is the minimal polynomial of a. Then $\delta(p(a)) = \frac{\partial p}{\partial x}(a) \cdot \delta(a) + \sum_{i=0}^{n} \delta(a_i) a^i = \frac{\partial p}{\partial x}(a) \cdot \delta(a) = 0$. Since $\mathrm{char}(\mathcal{F}) = 0$ and $\frac{\partial p}{\partial x}(a) \neq 0$. Thus $\delta(a) = 0$. □

Remark: Let $L \supseteq \mathcal{F} \supseteq \mathbb{Q}$ and $a \in L$. If a is algebraic over $C_{\mathcal{F}}$, then $\delta(a) = 0$.