Recall: • Let $(K, \delta) \subset L$. Then δ could be extended to a derivation on L. And the extension is unique iff L is algebraic over K. (If $\alpha \in (L,')$ is algebraic over the constant field of K, then $\alpha' = 0$)

• Let $K \subseteq L \subseteq M$ be differential fields. Then M is differential algebraic over $K \Leftrightarrow M$ is differential algebraic over L and L is differential algebraic over K.

• Nonvanishing theorem of differential polynomials: Let K be a non-constant differential field of characteristic 0. If G is a nonzero differential polynomial in $K\{y_1, \ldots, y_n\}$, there exist elements $\eta_1, \ldots, \eta_n$ in K such that $G(\eta_1, \ldots, \eta_n) \neq 0$. In particular, if $0 \neq G \in K\{y\}$ is of order r and $\xi \in K$ is a nonconstant, there exists

$$\eta = c_0 + c_1\xi + \cdots + c_r\xi^r$$

where all the c_i's are constants in K, satisfying $G(\eta) \neq 0$.

Now, we are ready to show that when (K, δ) is a nonconstant differential field, every finitely generated differential algebraic extension field of K is generated by a single element.

Theorem 4.2.3 (Differential Primitive Element Theorem)**.** *Let (K, δ) be a* ***non-constant*** *differential field of characteristic 0 (i.e., $\exists\, b \in K,\ \delta(b) \neq 0$). Assume $K\langle\alpha_1, \ldots, \alpha_n\rangle$ is differential algebraic over K. Then $\exists\, \xi \in K\langle\alpha_1, \ldots, \alpha_n\rangle$ s.t. $K\langle\alpha_1, \ldots, \alpha_n\rangle = K\langle\xi\rangle$.*

Proof. It suffices to show that if γ, β are differential algebraic over K, then $\exists\, e \in K$ s.t.

$$K\langle\gamma, \beta\rangle = K\langle\gamma + e\beta\rangle.$$

Introduce a new differential indeterminate t over $K\langle\gamma, \beta\rangle$ and consider $\gamma + t\beta \in K\langle t\rangle\langle\gamma, \beta\rangle$. By Lemma 4.1.6, $\gamma + t\beta$ is differential algebraic over $K\langle t\rangle$. Consider the prime differential ideal $\mathbb{I}(\gamma + t\beta) \subseteq K\langle t\rangle\{y\}$ and suppose $A(y) \in K\langle t\rangle\{y\}$ is a characteristic set of $\mathbb{I}(\gamma + t\beta)$. Then $A(\gamma + t\beta) = 0$ but $\mathrm{S}_A(\gamma + t\beta) \neq 0$. Assume $\mathrm{ord}(A) = s$. Clearing denominators when necessary, we can take $A \in K\{t, y\}$ and write $A(t, y)$ for convenience.

Now we have $A(t, \gamma + t\beta) = 0$ but $\frac{\partial A}{\partial y^{(s)}}(t, \gamma + t\beta) \neq 0$. Note that

$$\frac{\partial((\gamma + t\beta)^{(k)})}{\partial t^{(s)}} = \begin{cases} 0, & k < s \\ \beta, & k = s \end{cases} \quad \text{for } k \leq s.$$

Take the partial derivative of $A(t, \gamma + t\beta) = 0$ w.r.t. $t^{(s)}$, we have

$$\tfrac{\partial A}{\partial t^{(s)}}(t, \gamma + t\beta) + \beta \cdot \tfrac{\partial A}{\partial y^{(s)}}(t, \gamma + t\beta) = 0.$$

Since $\frac{\partial A}{\partial y^{(s)}}(t, \gamma + t\beta) \neq 0$ belongs to $K\langle\gamma, \beta\rangle\{t\}$, by the proof of Lemma 4.2.2, $\exists\, e \in K$ s.t. $\frac{\partial A}{\partial y^{(s)}}(e, \gamma + e\beta) \neq 0$. Thus, $\beta = -\frac{\frac{\partial A}{\partial t^{(s)}}(e, \gamma + e\beta)}{\frac{\partial A}{\partial y^{(s)}}(e, \gamma + e\beta)} \in K\langle\gamma + e\beta\rangle$ and $K\langle\gamma, \beta\rangle = K\langle\gamma + e\beta\rangle$ follows. □

Corollary 4.2.4. *Let (K, δ) be a non-constant differential field. Let $K\langle\eta_1, \ldots, \eta_n\rangle$ be a differential algebraic extension field of K. Then $\exists\, e_1, \ldots, e_n \in K$ s.t. $K\langle\eta_1, \ldots, \eta_n\rangle = K\langle e_1\eta_1 + \cdots + e_n\eta_n\rangle$.*

Remark: G. Pogudin proved the differential primitive theorem for the case

$$\begin{cases} ① K' = \{0\}; \\ ② K\langle\eta_1, \ldots, \eta_n\rangle \text{ has a nonconstant} \end{cases}.$$

("The primitive element theorem for differential fields with zero derivation on the ground field. J. Pure Appl. Algebra, 4035-4041, 2015.")

4.3 Differential transcendence bases

Let R be a differential ring. Elements $\alpha_1, \ldots, \alpha_n$ in a differential over-ring S of R are called **differentially algebraically dependent over** R if there exists a nonzero $G \in R\{y_1, \ldots, y_n\}$ s.t. $G(\alpha_1, \ldots, \alpha_n) = 0$. Otherwise, $\alpha_1, \ldots, \alpha_n$ are called differentially (δ-) algebraically independent over R. A subset of S is called **δ-algebraically independent over** R if all its subsets are δ-algebraically independent over R.

Definition 4.3.1. *Let $K \subseteq L$ be differential fields and $A \subseteq L$. An element $b \in L$ is called* **δ-algebraically dependent on** *A (over K) if b is δ-algebraic over $K\langle A\rangle$. A subset B of L is called δ-algebraically dependent on A (over K) if every element of B is δ-algebraically dependent on A.*

Since K is our fixed base differential field, for simplicity, we usually omit "over K".

Lemma 4.3.2. *Let $K \subseteq L$ be an extension of δ-fields, $A \subseteq L$ and $b \in L$. Then b is δ-algebraically dependent on A if and only if $\exists\, f \in K\{y_1, \ldots, y_n, z\}$ and $a_1, \ldots, a_n \in A$ such that $f(a_1, \ldots, a_n, z) \neq 0$ and $f(a_1, \ldots, a_n, b) = 0$.*

Proof. Assume b is δ-algebraically dependent on A. Then by definition, b is δ-algebraic over $K\langle A\rangle$, so $\exists$ a nonzero $g \in K\langle A\rangle\{z\}$ s.t. $g(b) = 0$. Let $\{a_1, \ldots, a_n\} \subseteq A$ be the subset appearing effectively in the coefficients of g. After multiplying g by an appropriate element from $K\{a_1, \ldots, a_n\}$, we can assume $g \in K\{a_1, \ldots, a_n, z\}$. Thus, this $g(y_1, \ldots, y_n, z)$ satisfies the desired property. The converse is obvious. □

Lemma 4.3.3. *Let $K \subseteq L$ be an extension of δ-fields and A be a subset of L which is δ-algebraically independent over K. Let $b \in L$. If A, b are δ-algebraically dependent over K, then b is δ-algebraic over $K\langle A\rangle$.*

Proof. Since A, b are δ-algebraically dependent over K, then there exists a nonzero differential polynomial $f \in K\{y_1, \ldots, y_n, z\}$ s.t. $f(a_1, \ldots, a_n, b) = 0$ for some $a_1, \ldots, a_n \in A$. Since $a_1, \ldots, a_n$ are δ-algebraically independent over K, $f(a_1, \ldots, a_n, z) \neq 0$. Thus, b is δ-algebraic over $K\langle A\rangle$. □

Lemma 4.3.4 (Transitivity of δ-algebraic dependence)**.** *Let $(K, \delta) \subseteq (L, \delta)$ and $A, B, C \subseteq L$. If A is δ-algebraically dependent on B and B is δ-algebraically dependent on C, then A is δ-algebraically dependent on C.*

Proof. By the assumption, $K\langle B\rangle\langle A\rangle$ is δ-algebraic over $K\langle B\rangle$ and $K\langle C\rangle\langle B\rangle$ is δ-algebraic over $K\langle C\rangle$. By Lemma 4.1.7, $K\langle C, B, A\rangle$ is δ-algebraic over $K\langle C\rangle$. Thus, each element of A is δ-algebraic over $K\langle C\rangle$. □

Lemma 4.3.5 (The exchange property)**.** *Let $a_1, \ldots, a_n, b$ be elements from a δ-extension field of K. If b is δ-algebraically dependent on $a_1, \ldots, a_n$ but not on $a_1, \ldots, a_{n-1}$, then a_n is δ-algebraically dependent on $a_1, \ldots, a_{n-1}, b$.*

Proof. Since b is δ-algebraically dependent on $a_1, \ldots, a_n$, by Lemma 4.3.2, there exists a nonzero $g \in K\{y_1, \ldots, y_n, z\}$ s.t. $g(a_1, \ldots, a_n, z) \neq 0$ and $g(a_1, \ldots, a_n, b) = 0$. Regard g as a univariate δ-polynomial in y_n with coefficients from $K\{y_1, \ldots, y_{n-1}, z\}$, i.e., $g = \sum_i g_i(y_1, \ldots, y_{n-1}, z) M_i(y_n)$ where the $M_i(y_n)$ are distinct δ-monomials. Then there exists i_0 s.t. $g_{i_0}(a_1, \ldots, a_{n-1}, z) \neq 0$, for otherwise, we would get $g(a_1, \ldots, a_{n-1}, a_n, z) = 0$. Since b is not δ-algebraically dependent on $a_1, \ldots, a_{n-1}$, $g_{i_0}(a_1, \ldots, a_{n-1}, b) \neq 0$. So $g(a_1, \ldots, a_{n-1}, y_n, b) \neq 0$ and consequently, a_n is δ-algebraically dependent on $a_1, \ldots, a_{n-1}, b$. □

Proposition 4.3.6. *Let $K \subseteq L$ be an extension of δ-fields and $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_m\}$ be two subsets of L. Assume that 1) A is δ-algebraically independent over K and 2) A is δ-algebraically dependent on B. Then $n \leq m$.*

Proof. Let $r = |A \cap B|$. If $r = n$, i.e., $A \subseteq B$, then we are done. Now assume $r < n$ and write $B = a_1, \ldots, a_r, b_{r+1}, \ldots, b_m$. Since a_{r+1} is δ-algebraically dependent on $a_1, \ldots, a_r, b_{r+1}, \ldots, b_m$ but not on $a_1, \ldots, a_r$, there will be a b_j $(r+1 \leq j \leq m)$ s.t. a_{r+1} is δ-algebraically dependent on $a_1, \ldots, a_r, b_{r+1}, \ldots, b_j$ but not δ-algebraically dependent on $a_1, \ldots, a_r, b_{r+1}, \ldots, b_{j-1}$. Bt the exchange property (Lemma 4.3.5), b_j is δ-algebraically dependent on $a_1, \ldots, a_r, b_{r+1}, \ldots, b_{j-1}, a_{r+1}$, and thus δ-algebraically dependent on $B_1 := (B \backslash \{b_j\}) \cup \{a_{r+1}\}$. Therefore, B is δ-algebraically dependent on B_1. Since A is δ-algebraically dependent on B, by Lemma 4.2.4, A is δ-algebraically dependent on B_1. Note that $|B_1| = m$ and $|A \cap B_1| = r + 1$. Continuing in this way, we will eventually get some B_{n-r} with $|A \cap B_{n-r}| = n$, i.e., $A \subseteq B_{n-r}$. So $n \leq m$. □

Definition 4.3.7. *Let $(K, \delta) \subseteq (L, \delta)$. A subset A of L is called a* **δ-transcendence basis** *of L over K if 1) A is δ-algebraically independent over K and 2) L is δ-algebraic over $K\langle A\rangle$.*

By the size of a set, we mean its cardinality if the set is finite, and ∞ otherwise.

Theorem 4.3.8. *Let $(K, \delta) \subseteq (L, \delta)$. Then every δ-generating set of $L \supseteq K$ contains a δ-transcendence basis of L over K. In particular, there exists a δ-transcendence basis of L over K. Moreover, any two δ-transcendence bases of L over K are of the same size.*

Proof. Let M be a δ-generating set of L over K, i.e., $L = K\langle M\rangle$. Let

$$N = \{S \subseteq M \mid S \text{ is } \delta\text{-algebraically independent over } K\}.$$

Then $\emptyset \in N \neq \emptyset$. Clearly, the union of every chain of elements in N is again in N. So by Zorn's lemma, there exists a maximal element A in N.

Claim: A is a δ-transcendence basis of L over K.

We now show the claim. For any $a \in M$, a, A are δ-algebraically dependent over K. By Lemma 4.3.3, a is δ-algebraic over $K\langle A\rangle$, so M is δ-algebraic over $K\langle A\rangle$. And by Lemma 4.1.6, $L = K\langle M\rangle$ is δ-algebraic over $K\langle A\rangle$. Thus, $A \subseteq M$ is a δ-transcendence basis of L over K.

Now suppose A and B are both δ-transcendence bases of L over K. By symmetry, it suffices to show that the size of $A \geq$ the size of B. If A is an infinite set, it is automatically valid. So we may assume A is finite. Let B_1 be any finite subset of B. Since A is a δ-transcendence basis of L over K, each element of B_1 is δ-algebraic over $K\langle A\rangle$, and B_1 is δ-algebraically dependent on A. By Proposition 4.3.6, $|B_1| \leq |A|$. Thus, $|B| \leq |A|$. □

Corollary 4.3.9. *Let $(K, \delta) \subseteq (L, \delta)$ and $L = K\langle M\rangle$. If A is a maximal δ-algebraically independent subset of M, then A is a δ-transcendence basis of L over K.*

Theorem 4.3.8 guarantees we can make the following definition:

Definition 4.3.10. *Let $(K, \delta) \subseteq (L, \delta)$. The size of a δ-transcendence basis of L over K is called the* **δ-transcendence degree** *of L over K. It is denoted by δ-tr.deg(L/K).*

Corollary 4.3.11. *Let $(K, \delta) \subseteq (L, \delta)$ and $L = K\langle a_1, \ldots, a_n\rangle$. Then δ-tr.deg$(L/K) \leq n$, and the δ-transcendence degree of a finitely δ-generated δ-field extension is finite.*

Proof. It is clear from Corollary 4.3.9. □

Corollary 4.3.12. *Let $(K,\delta) \subseteq (L,\delta)$. If L contains n number of δ-independent elements, then $n \leq \delta$-tr.deg(L/K). In fact,*

$$\delta\text{-tr.deg}(L/K) = \sup\{n \in \mathbb{N} \mid \exists\, a_1,\ldots,a_n \in L \ \textit{ differentially algebraically independent over } K\}.$$

Proof. Let $a_1,\ldots,a_n \in L$ be δ-algebraically independent over K. We can enlarge $\{a_1,\ldots,a_n\}$ to a δ-generating set B of L over K. Then $\{a_1,\ldots,a_n\}$ is contained in a maximal δ-algebraically independent subset $A' \subseteq B$. By Corollary 4.3.9, A' is a δ-transcendence basis of L over K. Thus, $n \leq \delta$-tr.deg(L/K) and also

$$\sup\{n \in \mathbb{N} \mid \exists\, a_1,\ldots,a_n \in L \text{ that are } \delta\text{-algebraically independent over } K\} \leq \delta\text{-tr.deg}(L/K).$$

The reverse estimate is clear, for a δ-transcendence basis is δ-algebraically independent over K. □

Theorem 4.3.13. *Let $K \subseteq L \subseteq M$ be δ-fields. Then*

$$\delta\text{-tr.deg}(M/K) = \delta\text{-tr.deg}(M/L) + \delta\text{-tr.deg}(L/K).$$

(Here, $\infty + a(\infty) = \infty$).

Proof. Let A be a transcendence basis of L over K and B a δ-transcendence basis of M over L.

Claim: $A \cup B$ is a δ-transcendence basis of M over K.

First, since B is δ-algebraically independent over $K\langle A\rangle$ $(\subseteq L)$, $A \cup B$ is δ-algebraically independent over K. It remains to show M is δ-algebraic over $K\langle A, B\rangle$. Since each element of M is δ-algebraic over $L\langle B\rangle$ and each element of L is δ-algebraic over $K\langle A\rangle$, M is δ-algebraic over $K\langle A, B\rangle$. Thus, $A \cup B$ is a δ-transcendence basis of M over K and $A \cap B = \emptyset$ implies that δ-tr.deg$(M/K) = \delta$-tr.deg$(M/L) + \delta$-tr.deg(L/K). □

Adjoining the differential primitive element theorem, we have

Proposition 4.3.14. *Let $L = K\langle a_1,\ldots,a_n\rangle$ and suppose K contains a nonconstant element in the case $d = \delta$-tr.deg$(L/K) = 0$. Then L is δ-generated by no more than $d+1$ elements.*

Proof. In the case $d = 0$, this is the differential primitive element theorem. Assume $d > 0$. Then $\exists\,\{\xi_1,\ldots,\xi_d\} \subseteq \{a_1,\ldots,a_n\}$ s.t. $\xi_1,\ldots,\xi_d$ is a δ-transcendence basis of L over K, and denote the others by $\xi_{d+1},\ldots,\xi_n$. Then by the differential primitive element theorem, there exist $a_i \in K\langle \xi_1,\ldots,\xi_d\rangle$ s.t. $L = K\langle \xi_1,\ldots,\xi_d\rangle\langle \xi_{d+1},\ldots,\xi_n\rangle = K\langle \xi_1,\ldots,\xi_d\rangle\langle a_{d+1}\xi_{d+1} + \cdots + a_n\xi_n\rangle$. ($d > 0 \Rightarrow K\langle \xi_1,\ldots,\xi_d\rangle$ is a non-constant δ-field). □