Recall: Let (R,δ) be a differential ring and I be a differential ideal of (R,δ) (i.e., $I \trianglelefteq R$ and $\delta(I) \subseteq I$).

- I is radical if $f^n \in I$ for some $n \in \mathbb{N}_{>0}$ implies $f \in I$.
- I is prime if $I \neq R$ and $ab \in I \Rightarrow a \in I$ or $b \in I$. $\Leftrightarrow$ I has a generic zero.

Theorem 1.15 Let $I \subsetneq R$ be a radical differential ideal. Then $I = \bigcap_{I \subseteq P \text{ prime}} P$.

Proposition: Let $\mathbb{Q} \subseteq (R,\delta)$. Then we have

(i) $\{S\} = \sqrt{[S]}$ for $S \subseteq R$.

(ii) Each maximal differential ideal is prime.

- Differential polynomial ring $K\{y_1,\cdots,y_n\} \stackrel{\triangle}{=} K[\delta^i y_j : i \in \mathbb{N}, j=1,\cdots,n]$
- Each $f \in K\{y_1,\cdots,y_n\}$ is called a differential polynomial.
- Let $E \supseteq K$ be a differential closed field containing K and $F \subseteq K\{y_1,\cdots,y_n\}$

$\mathbb{V}_E(F) = \{(a_1,\cdots,a_n) \in E^n \mid \underbrace{f(a_1,\cdots,a_n)=0}_{\text{replacing } \delta^i y_j \text{ by } \delta^i a_j \text{ in } f} \ \forall f \in F\}$ is called a differential variety defined by F.

In this chapter, we will prove the differential analog of Hilbert basis theorem for the differential polynomial ring, i.e., the Ritt-Raudenbush basis theorem. Before that, we first introduce characteristic set method, which is the main computational tool in differential algebra and also could provide some theoretical insight. The idea behind characteristic sets is similar to the notion of Gröbner basis.

Section 2.1 Differential characteristic sets

Motivated Example (Ideal membership problem):

① In $\mathbb{Q}[x]$, every ideal is of the form $I=(f)$ for some $f \in \mathbb{Q}[x]$.

By the Euclidean division algorithm, $g = qf + r$ w/ $r = \text{rem}(g,f)$. Then $g \in I \Leftrightarrow r = 0$.

② In $\mathbb{Q}[x_1,\cdots,x_n]$, given an ideal $I=(f_1,\cdots,f_s)\subseteq\mathbb{Q}[x_1,\cdots,x_n]$, we use Gröbner basis to test whether $g\in I$.

③ How about the differential ideal membership problem? (diff characteristic sets)

Let (K,δ) be a differential field of characteristic zero. The differential polynomial ring $K\{Y\}\triangleq K\{Y_1,\cdots,Y_n\}$ in the diff variables $Y=\{Y_1,\cdots,Y_n\}$ can be viewed as a polynomial ring in the algebraic variables

$$\Theta(Y)\triangleq\{\delta^i(Y_j)\mid i\in\mathbb{N},\ j=1,\cdots,n\}.\quad (\text{i.e., } K\{Y\}=K[\Theta(Y)]).$$

A (differential) ranking on $\Theta(Y)$ is a total ordering on $\Theta(Y)$ satisfying

(1) $u<\delta(u)$ for all $u\in\Theta(Y)$ and

(2) if $u,v\in\Theta(Y)$ with $u<v$, then $\delta(u)<\delta(v)$.

Example: • The set $\Theta(Y)=\{\delta^i(Y): i\in\mathbb{N}\}$ has a unique ranking $Y<\delta(Y)<\delta^2(Y)<\delta^3(Y)<\cdots$.

• Two important rankings in $\Theta(Y)$ are the following:

1) Elimination ranking: $Y_i>Y_j \Rightarrow \delta^k(Y_i)>\delta^l(Y_j)$ for any $k,l\in\mathbb{N}$.

2) Orderly ranking: $k>l\Rightarrow \delta^k(Y_i)>\delta^l(Y_j)$ for all $i,j\in\mathbb{N}$.

Lemma 2.1.1 Every ranking is a well-ordering (i.e., every nonempty subset of $\Theta(Y)$ has a least element).

Proof. Let $U\subseteq\Theta(Y)$ and $U\neq\phi$. For each $j\in\{1,\cdots,n\}$, if $\exists\, i\in\mathbb{N}$ s.t. $\delta^i(Y_j)\in U$ then set $k_j=\min\{i\mid \delta^i(Y_j)\in U\}$ and set $u_j=\delta^{k_j}(Y_j)$. Then the least elt of U is the least elt in the finite set of u_j's. ☒.

Until the end of this subsection, we assume a ranking $\mathcal{R}$ is fixed. And by convention, $1 < \delta^i(y_j)$. (Denote $\delta(a), \delta^2(a), \delta^k(a)$ by $a', a'', a^{(k)}$ $(k \geq 3)$ respectively).

Def 2.1.2 Let $f \in K\{y_1, \cdots, y_n\} \setminus K$. The leader of f is the largest elt of $\Theta(Y)$ w.r.t. $\mathcal{R}$ which appears effectively in f, denoted by u_f or $\mathrm{ld}(f)$.
By the two conditions in the definition of ranking, for each $i \in \mathbb{N}$, $\mathrm{ld}(\delta^i(f)) = \delta^i(\mathrm{ld}(f))$. We write f as a univariate poly of u_f, then

$$f = I_d (u_f)^d + I_{d-1} (u_f)^{d-1} + \cdots + I_1 u_f + I_0, \quad \text{where } I_i \text{ is free of } u_f$$

and $d = \deg(f, u_f)$. The leading coefficient I_d is called the initial of f and denoted by I_f. The pair $\mathrm{rk}(f) := (u_f, d)$ is called the rank of f.

Example. Let $f = (y')^2 - 4y \in \mathbb{Q}\{y\}$. Then $u_f = \mathrm{ld}(f) = y'$ and $I_f = 1$.
Apply δ to f, then we have $\delta(f) = 2y'y'' - 4y'$.
So we get $u_{\delta(f)} = y'' = \delta(u_f)$ and $I_{\delta(f)} = 2y' = \frac{\partial f}{\partial y'}$.

Note that in the above example, $\deg(\delta(f), u_{\delta(f)}) = 1$ and $I_{\delta(f)} = \frac{\partial f}{\partial u_f}$.
Def 2.1.3. Let $f \in K\{y_1, \cdots, y_n\} \setminus K$. $\frac{\partial f}{\partial u_f}$ is called the separant of f, denoted by S_f.

Remark: 1) $f = \sum_{i=0}^{d} I_i u_f^i \Rightarrow \delta(f) = \sum_{i=1}^{d} I_i \delta(u_f^i) + \sum_{i=0}^{d} \delta(I_i) u_f^i$
$= \left(\sum_{i=1}^{d} I_i i u_f^{i-1}\right) \delta(u_f) + \sum_{i=0}^{d} \delta(I_i) u_f^i = S_f \cdot \delta(u_f) + \sum_{i=0}^{d} \delta(I_i) u_f^i$.
Note that $u_{\delta(f)} = \delta(u_f)$, $I_{\delta(f)} = S_f$ and $\deg(\delta(f), u_{\delta(f)}) = 1$. (char $K = 0$).
Also, for $k > 0$, $\delta^k(f) = S_f \cdot \delta^k(u_f)$ + tail poly involving derivatives less than $\delta^k(u_f)$.
So $u_{\delta^k(f)} = \delta^k(u_f)$, $I_{\delta^k(f)} = S_f$, $\deg(\delta^k(f), u_{\delta^k(f)}) = 1$.
((K, δ) a δ-field, c alg over $K \Longrightarrow$ there is a unique way to make $(K(c), \delta)$ a δ-field)

2) By convention, for $f \in K \setminus \{0\}$, $u_f = 1$.

Def 2.1.4. Let $f, g \in K\{Y\}$. We say that f is partially reduced w.r.t. g if none of the $\underbrace{\text{proper derivatives of } u_g}_{(\delta^i(u_g) \text{ with } i>0)}$ appears effectively in f.

Example. 1) Let $f = Y^2$, $g = Y+1$. Since $u_g = Y$ and none of the proper derivatives of Y appears in f, f is partially reduced w.r.t. g.
2) Let $f = 2Y\delta(Y)^2 + Y$ and $g = Y+1$. Since $\delta(u_g) = \delta(Y)$ appears in the first term of f, f isn't partially reduced w.r.t. g.

Def 2.1.5. We say f is reduced w.r.t. g if
1) f is partially reduced w.r.t. g, and
2) $\deg(f, u_g) < \deg(g, u_g)$.

Def 2.1.6. A subset $\mathcal{A} \subseteq K\{Y_1, \cdots, Y_n\}$ is called an autoreduced set if any elt of $\mathcal{A}$ is reduced w.r.t. any other elt of $\mathcal{A}$.
Remark. If an autoreduced set $\mathcal{A}$ contains an elt $A \in K \setminus \{0\}$, then $\mathcal{A} = \{A\}$.

Lemma 2.1.7. Every autoreduced set of $K\{Y_1, \cdots, Y_n\}$ is finite.
Proof. Let $\mathcal{A}$ be an autoreduced set. For each $i = 1, \cdots, n$, there exists at most one δ-poly $A \in \mathcal{A}$ such that $ld(A) = \delta^k(Y_i)$ for some $k \in \mathbb{N}$, for two δ-poly A_1, A_2 with $ld(A_j) = \delta^{k_j}(Y_i)$ couldn't be reduced w.r.t. each other. Thus, $|\mathcal{A}| \leq n$. ∎
(For the partial differential case, we need to use Dickson lemma to show every autoreduced set is finite.)

Def 2.1.8 Let $f, g \in K\{Y_1, \cdots, Y_n\} \setminus K$. We say f has lower rank than g ($f < g$) if $rk(f) <_{lex} rk(g)$. ($<_{lex}$ is a well-ordering of $\Theta(Y) \times \mathbb{N}^*$.)
By convention, each elt of $K \setminus \{0\}$ has lower rank than elts of $K\{Y\} \setminus K$.

Notation: We use $f \leq g$ to denote either $f < g$ or f and g have the same rank. ("$\leq$" is a pre-order among $K\{Y_1, \cdots, Y_n\}$.)

In the following, we write an autoreduced set in the order of increasing rank i.e., $\mathcal{A} = A_1, \cdots, A_p$ with $rk(A_1) <_{lex} rk(A_2) <_{lex} \cdots <_{lex} rk(A_p)$.

Let $\mathcal{A} = A_1, \cdots, A_p$ and $\mathcal{B} = B_1, \cdots, B_q$ be two autoreduced set. We say $\mathcal{A} < \mathcal{B}$ if either 1) $\exists\, k (\leq \min\{p, q\})$ s.t. $\forall\, i < k$, $rk(A_i) = rk(B_i)$ and $A_k < B_k$
or 2) $p > q$ and for each $i \leq q$, $rk(A_i) = rk(B_i)$.
If neither $\mathcal{A} < \mathcal{B}$ or $\mathcal{B} < \mathcal{A}$, we say $\mathcal{A}$ and $\mathcal{B}$ are of the same rank.
$\mathcal{A}$ and $\mathcal{B}$ have the same rank $\Leftrightarrow$ $p = q$ and $\forall\, i \leq p$, $rk(A_i) = rk(B_i)$.
Say $\mathcal{A} \leq \mathcal{B}$ iff $\mathcal{A} < \mathcal{B}$ or $\mathcal{A}$ and $\mathcal{B}$ have the same rank. ("$\leq$" is a pre-order)

Example. Consider $K\{Y_1, Y_2\}$ and take the orderly ranking with $Y_1 < Y_2$.
Let $\mathcal{A} = \{A_1 = (Y_2')^2 + 1,\ A_2 = Y_1'' + Y_2\}$, $\mathcal{B} = \{B_1 = Y_2' + 2\}$ and $\mathcal{C} = \{C_1 = (Y_2')^2 + 2\}$.
Since $rk(A_1) > rk(B_1)$, $\mathcal{B} < \mathcal{A}$. Since $rk(A_1) = rk(C_1)$ and $|\mathcal{A}| > |\mathcal{C}|$, $\mathcal{A} < \mathcal{C}$.

Prop 2.1.9 Any nonempty set of autoreduced sets in $K\{Y\} = K\{Y_1, \cdots, Y_n\}$ contains an autoreduced set of lowest rank.

Proof. Let U be any nonempty set of autoreduced sets of $K\{Y\}$. Define by induction a sequence of subsets of U as follows:

$U_0 \triangleq U$.

for $i>0$, define $U_i = \left\{ \mathcal{A} \in U_{i-1} \;\middle|\; \begin{array}{l} \mathrm{card}(\mathcal{A}) \geq i, \\ \text{the } i\text{-th elt of } \mathcal{A} \text{ is of lowest rank} \end{array} \right\}$

Then $U_0 \supseteq U_1 \supseteq \cdots$. By lemma 2.1.7, $\exists\, i \in \mathbb{N}$ (actually $i \leq n$ in the ordinary differential case) s.t. $U_i \neq \phi$ and $U_{i+1} = \phi$. Actually, any elt of U_i is an autoreduced set in U of lowest rank. $\blacksquare$.

Def 2.1.10. Let $I \subseteq K\{Y\}$ be a differential ideal. An autoreduced set of lowest rank contained in I is called a characteristic set of I (with respect to the given ranking)

Next class: ① Differential pseudo-division of a δ-poly f w.r.t. an autoreduced set $\mathcal{A}$;

② Another characterization of characteristic sets;

③ Ritt-Raudenbush basis theorem.