Recall: Comparative ranks on different sets

- Differential ranking on $\Theta(Y)=\{\delta^i y_j \mid i\in\mathbb{N}, j=1,\cdots,n\}$

Fix $\mathcal{R}$, given $f\in K\{Y\}\setminus\{0\}$, $\mathrm{ld}(f)=u_f$, $\mathrm{rk}(f)=(u_f, \deg(f,u_f))$, $\mathrm{ini}(f)=I_f$, $\mathrm{sep}(f)=S_f$

- Pre-ordering on $K\{Y\}\setminus\{0\}$:

$$f\leq g \Longleftrightarrow \mathrm{rk}(f)\leq_{lex} \mathrm{rk}(g).$$

- Pre-ordering on autoreduced sets

Def. Let $I\subseteq K\{Y\}$ be a differential ideal. An autoreduced set of lowest rank contained in I is called a characteristic set of I (with respect to a given ranking).

Remark: By convention. ϕ and $\{a\}$ with $a\in K^*$ are autoreduced sets. (Here, $\mathrm{rk}(a)=(1,1)$).

We start to introduce pseudo-division of differential polynomials:

Lemma 2.1.11 Let $\mathcal{A}=A_1,\cdots,A_p$ be an autoreduced set in $K\{Y\}$ and $F\in K\{Y\}$. Then there exist $\tilde{F}\in K\{Y\}$ and $t_i\in\mathbb{N}$ satisfying

1) $\tilde{F}$ is partially reduced w.r.t. $\mathcal{A}$,
2) the rank of $\tilde{F}$ is not higher than that of F,
3) $\prod_{i=1}^{p} S_{A_i}^{t_i} F\equiv \tilde{F} \mod [\mathcal{A}]$.

More precisely, $\prod_{i=1}^{p} S_{A_i}^{t_i} F-\tilde{F}$ can be expressed as a linear combination of derivatives $\theta(A_i)$ with coefficients in $K\{Y\}$ s.t. $\theta(u_{A_i})\leq u_F$.

<u>Proof.</u> If F is partially reduced w.r.t. $\mathcal{A}$, then set $\tilde{F}=F$ and $t_i=0\,(i\le p)$. Otherwise, F contains a proper derivative $\delta^k(u_{A_i})$ of the leader of some A_i. Let v_F be such derivatives of the maximal rank. We shall prove the lemma by induction on v_F. Suppose for all $G\in K\{Y\}$ that doesn't involve a proper derivative of any u_{A_i} of rank $\geq v_F$, the coeresponding $\tilde{G}$ and natrual numbers are defined satisfying the desired properties. There exists a unique $A\in\mathcal{A}$ s.t. $v_F=\delta^k(u_A)$ for some $k>0$. If $A=\sum_{i=0}^{d} I_i\, u_A^i$, then

$$\delta^k(A)=S_A\,\delta^k(u_A)+T \quad \text{with } T \text{ having lower rank than } \delta^k(u_A)=v_F.$$

Denoting $l=\deg(F,v_F)$ and write F as $F=\sum_{i=0}^{l} J_i\, v_F^i$ where $J_0,\cdots,J_l$ doesn't involve proper derivatives of any u_{A_i} of rank $\geq v_F$.

Hence $S_A^l F=\sum_{i=0}^{l} J_i S_A^{l-i}(S_A v_F)^i \equiv \sum_{i=0}^{l} J_i S_A^{l-i}(-T)^i \mod (\delta^k(A))$.

Clearly, $G=\sum_{i=0}^{l} J_i S_A^{l-i}(-T)^i$ doesn't involve proper derivatives of any u_{A_i} of rank $\geq v_F$. By the induction hypothesis, $\exists\ \tilde{G}$ partially reduced w.r.t. $\mathcal{A}$ and $k_i\in\mathbb{N}$ s.t. $\prod S_{A_i}^{k_i}\, G\equiv \tilde{G} \mod [\mathcal{A}]$.

Now it suffices to set $\tilde{F}=\tilde{G}$, $t_i=\begin{cases} k_i, & A_i\neq A \\ k_i+l, & A_i=A\end{cases}$. $\square$

Remark: $\tilde{F}$ constructed by the process in the proof is called the partial remainder of F w.r.t. A.

Recall the pseudo reduction algorithm in commutative algebra:

Let D be an integral domain and v an indeterminate over D. Let $F, A\in D[v]$ be of respective degrees d_F, d_A. Suppose $A=I_{d_A}v^{d_A}+\cdots+I_1 v+I_0\neq 0$ w/ $I_i\in D$. Let $e=\max\{d_F-d_A+1,0\}$. Then we can compute unique $Q,R\in D[v]$ s.t.

$$I_{d_A}^{e}F=QA+R \text{ and } \deg(R)<\deg(A).$$

Theorem 2.1.12. Let $\mathcal{A} = A_1, \cdots, A_p$ be an autoreduced set in $K\{y_1, \cdots, y_n\}$. If $F \in K\{y_1, \cdots, y_n\}$, then $\exists$ a δ-poly F_0 (δ-remainder of F) and $r_i, t_i \in \mathbb{N}$ such that 1) F_0 is reduced w.r.t. $\mathcal{A}$,

2) The rank of F_0 is no higher than the rank of F.

3) $\prod_{i=1}^{p} S_{A_i}^{t_i} I_{A_i}^{r_i} F \equiv F_0 \mod [\mathcal{A}]$.

Proof. Let $\tilde{F}$ be the partial remainder of F w.r.t. $\mathcal{A}$ and $\prod_{i=1}^{p} S_{A_i}^{t_i} F \equiv \tilde{F} \mod [\mathcal{A}]$

Let $r_p = \max\{0, \deg(\tilde{F}, v_{A_p}) - \deg(A_p, v_{A_p}) + 1\}$. Then $\exists F_{p-1} \in K\{Y\}$ partially reduced w.r.t. $\mathcal{A}$ and reduced w.r.t. A_p s.t. $I_{A_p}^{r_p} \tilde{F} \equiv F_{p-1} \mod (A_p)$.

If $p=1$, then we are done. Otherwise, we can find r_{p-1} and $F_{p-2} \in K\{Y\}$ partially reduced w.r.t. $\mathcal{A}$ and reduced w.r.t. A_{p-1}, A_p s.t. $I_{A_{p-1}}^{r_{p-1}} I_{A_p}^{r_p} \tilde{F} \equiv F_{p-2} \mod (A_{p-1}, A_p)$ and is not higher than $\tilde{F}$. Continuing in this way, we get F_0 satisfing the desired properties. $\square$.

Remark: The reduction procedures above could be summarized in an algorithm, called the Ritt-Kolchin algorithm to compute the δ-remainder of a δ-poly F w.r.t. an autoreduced set $\mathcal{A}$. Denote F_0 above by $\delta\text{-rem}(F, \mathcal{A})$, or $F \xrightarrow[\mathcal{A}]{} F_0$.

Example: Consider $K\{y_1, y_2\}$ and fix the orderly ranking with $y_1 > y_2$.

(1) Let $f = y_1$ and $\mathcal{A} = A_1 = y_2 y_1$. Here $f \xrightarrow[\mathcal{A}]{} 0$, and $I_A f - 0 \in [\mathcal{A}]$.

(2) Let $f = y_1' + 1$ and $\mathcal{A} = A_1 = y_2 y_1^2$. $u_{A_1} = y_1$ and $S_{A_1} = 2y_2 y_1$.

Clearly, f is not partially reduced w.r.t. $\mathcal{A}$. $\delta(A_1) = 2y_2 y_1 y_1' + y_2' y_1^2$

The partial remainder of f w.r.t. $\mathcal{A}$ is $2y_2 y_1 - y_2' y_1^2 = \tilde{f}$ and

$$S_{A_1} f - A_1' = \tilde{f}.$$

$I_{A_1}\tilde{f} - I_{\tilde{f}} A_1 = Y_2(2Y_2Y_1 - Y_2'Y_1^2) - (-Y_2')Y_2Y_1^2 = 2Y_2^2Y_1$, reduced w.r.t $\mathcal{A}$.

So $f \xrightarrow[\mathcal{A}]{} 2Y_2^2Y_1$ and $I_{A_1}S_{A_1}f - 2Y_2^2Y_1 = -Y_2'A_1 + I_{A_1}A_1' \in [\mathcal{A}]$.

Thm 2.1.13 Let $\mathcal{A}$ be an autoreduced set of a proper differential ideal $I \subseteq K\{Y_1, \cdots, Y_n\}$. Then (1) $\mathcal{A}$ is a char set of I

$\Leftrightarrow$ (2) $\forall f \in I$, δ-rem$(f, \mathcal{A}) = 0$.

$\Leftrightarrow$ (3) I doesn't contain a nonzero δ-poly reduced w.r.t. $\mathcal{A}$

Proof. (2) $\Leftrightarrow$ (3) is clear.

"(1) $\Rightarrow$ (3)" Suppose $f \in I \setminus \{0\}$ is reduced w.r.t. $\mathcal{A} = A_1, \cdots, A_p$.

Let $k \in \mathbb{N}$ be maximal such that $rk(A_k) < rk(f)$. Then

$A_1, \cdots, A_k, f$ is an autoreduced set lower than $\mathcal{A}$.

(Here, in the case $rk(f) < rk(A_1)$, take $k=0$ and $\{f\}$ is an autoreded set $< \mathcal{A}$.)

Thus, we get a contradiction and (3) ✓.

"(3) $\Rightarrow$ (1)". Assume (3) is valid. Suppose $\mathcal{A} = A_1, \cdots, A_p$ is not a characteristic set of I. Then $\exists\ \mathcal{B} = B_1, \cdots, B_q$, an autoreduced set of I of lower rank than $\mathcal{A}$. Thus, by definition

either (1) $\exists\ k \le \min\{p, q\}$ s.t. for $i < k$, $rk(A_i) = rk(B_i)$ and $A_k > B_k$

or (2) $q > p$ and for $i \le p$, $rk(A_i) = rk(B_i)$.

Then eith B_k or B_{p+1} is nonzero and reduced w.r.t. $\mathcal{A}$. $\rightarrow\leftarrow$.

Remark: By Theorem 2.1.13, if $\mathcal{A} = A_1, \cdots, A_p$ be a characteristic set of $I \subseteq K\{Y\}$, then $I_{A_i}, S_{A_i} \notin I$ $(\forall i = 1, \cdots, p)$.

A characteristic set of I can be obtained by the following procedure (non-constructive): choose $A_1 \in I$ of minimal rank.

Choose A_2 of minimal rank in the set $\{f\in I \mid f \text{ is reduced w.r.t. } A_1\}$.
Then A_1, A_2 is autoreduced.
Choose A_3 of minimal rank in the set

$$\{f\in I \mid f \text{ is reduced w.r.t. } A_1, A_2\}.$$

Then A_1, A_2, A_3 is autoreduced.
Continue like this. The process must terminate for an autoreduced set is finite.
In the end, we will obtain an autoreduced set $\mathcal{A} := A_1, \dots, A_p$ of I s.t. no poly in I is reduced w.r.t. $\mathcal{A}$. Clearly, $\mathcal{A}$ is a characteristic set of I.

Lemma 2.1.14: Let $\mathcal{A}$ be a characteristic set of a proper δ-ideal $I \subseteq K\{Y\}$.
Denote $H_{\mathcal{A}}^{\infty}$ to be the multiplicative set generated by initials and separants of elts in $\mathcal{A}$ and set $\operatorname{Sat}(\mathcal{A}) := [\mathcal{A}] : H_{\mathcal{A}}^{\infty} = \{f\in K\{Y\} \mid \exists M\in H_{\mathcal{A}}^{\infty},\ Mf\in[\mathcal{A}]\}$.
Then $I \subseteq \operatorname{Sat}(\mathcal{A})$. Furthermore, if I is prime, $I = \operatorname{Sat}(\mathcal{A})$.

Proof. By Theorem 2.1.13, $\forall f\in I$, $\delta\text{-rem}(f, \mathcal{A}) = 0$. Thus,
$\exists i_A, t_A \in \mathbb{N}\ (A\in\mathcal{A})$ s.t. $\prod_{A\in\mathcal{A}} I_A^{i_A} S_A^{t_A} f \in [\mathcal{A}]$, i.e., $f\in \operatorname{Sat}(\mathcal{A})$.
If I is prime, for each $f\in\operatorname{Sat}(\mathcal{A})$, $\exists i_A, t_A$ s.t. $\prod_{A\in\mathcal{A}} I_A^{i_A} S_A^{t_A} f \in [\mathcal{A}] \subseteq I$. Since I_A, S_A are not in I, $f\in I$. $\boxed{\xi}$.
(Since I_A, S_A are reduced w.r.t. $\mathcal{A}$)

Exercise: Develop a division algorithm as follows:
Input: $f\in K\{Y\}$ and an autoreduced set $\mathcal{A} = A_1, \dots, A_p$ w.r.t. a fixed ranking.
Output: $g\in K\{Y\}$, the δ-remainder of f w.r.t. $\mathcal{A}$.
(i.e. 1) g is reduced w.r.t. $\mathcal{A}$
2) $\exists i_k, j_k \in \mathbb{N}$ s.t. $I_{A_1}^{i_1} \cdots I_{A_p}^{i_p} S_{A_1}^{j_1} \cdots S_{A_p}^{j_p} f - g \in [\mathcal{A}]$).

Section 2.2 The Ritt-Raudenbush basis theorem

Hilbert basis theorem: Every ideal of $K[y_1, \cdots, y_n]$ is finitely generated.

(Every ascending chain of ideals in $K[y_1, \cdots, y_n]$ is finite).

One might hope ACC condition holds for differential ideals in $K\{y_1, \cdots, y_n\}$.

However, this is not true.

Non-Example. Consider $K\{y\}$ with $\mathbb{Q} \subseteq (K, \delta)$. The sequence of differential ideals $[y^2] \subseteq [y^2, (y')^2] \subseteq [y^2, (y')^2, (y'')^2] \subseteq \cdots$ doesn't stabilize in $K\{y\}$. (Think about why and we give a proof later.)

Def 2.2.1 A differential ring is called Ritt-Noetherian of the set of radical differential ideals satisfies the ascending chain condition (ACC).

Lemma 2.2.2 Let (R, δ) be a differential ring. Then

R is Ritt-Noethrian $\Longleftrightarrow$ every radical differential ideal I of R is finitely generated as a radical differential ideal (i.e. $\exists f_1, \cdots, f_s \in I$ s.t. $I = \{f_1, \cdots, f_s\}$).

proof. "$\Rightarrow$" Let I be an arbitrary radical differential ideal of R.

Suppose I is not finitely generated as a radical differential ideal. Then we can construct a strict increasing sequence of radical differential ideals, i.e., $\{a_1\} \subsetneq \{a_1, a_2\} \subsetneq \cdots \subsetneq \{a_1, a_2, \cdots, a_p\} \subsetneq \cdots$, $\rightarrow\leftarrow$.

"$\Leftarrow$" Let $I_1 \subseteq I_2 \subseteq \cdots$ be sequence of radical differential ideals. Take $I = \bigcup_{i=1}^{\infty} I_i$. Then I is a radical δ-ideal. Thus, $\exists f_1, \cdots, f_s \in I$ s.t. $I = \{f_1, \cdots, f_s\}$. Since each $f_i \in I$, $\exists m \in \mathbb{N}$ s.t. $f_i \in I_m$ ($\forall i = 1, \cdots, s$). So $\{f_1, \cdots, f_s\} \subseteq I_m \subseteq I \Rightarrow I_m = I_{m+j} = \{f_1, \cdots, f_s\}$ for $j \in \mathbb{N}$. 證.

Theorem 2.2.3 Let (K,δ) be a differential field with $\mathbb{Q}\subseteq K$. The differential polynomial ring $K\{y_1,\cdots,y_n\}$ is Ritt-Noetherian.

Proof. By Lemma 2.2.2, it suffices to prove that every radical differential ideal of $K\{y_1,\cdots,y_n\}$ is finitely generated as radical δ-ideals. Suppose the contrary and $\exists$ a radical δ-ideal of $K\{y_1,\cdots,y_n\}$ that is not finitely generated. By Zorn's lemma, $\exists$ a maximal radical δ-ideal $J\subseteq K\{y_1,\cdots,y_n\}$ that is not finitely generated.

Claim J is a prime δ-ideal.

If not, then $\exists\ a,b\in K\{y_1,\cdots,y_n\}$ s.t. $a,b\notin J$ but $ab\in J$.

Since $\{a,J\}\supsetneqq J$ and $\{b,J\}\supsetneqq J$, $\{a,J\}$ and $\{b,J\}$ are finitely generated as radical δ-ideals. Then $\exists\ f_1,\cdots,f_s,g_1,\cdots,g_t\in J$ s.t.

$\{a,J\}=\{a,f_1,\cdots,f_s\}$ and $\{b,J\}=\{b,g_1,\cdots,g_t\}$.

(Indeed, as $\{a,J\}$ is finitely generated, $\exists\ h_1,\cdots,h_l$ s.t. $\{a,J\}=\{h_1,\cdots,h_l\}$. For each i, $h_i\in\{a,J\}\Rightarrow\exists\, m_i\ h_i^{m_i}\in[a,J]$. So $\exists\ f_1,\cdots,f_s\in J$ s.t. $h_i^{m_i}\in[a,f_1,\cdots,f_s]$. Thus $h_i\in\{a,f_1,\cdots,f_s\}$. $\Rightarrow\{a,J\}=\{h_1,\cdots,h_l\}\subseteq\{a,f_1,\cdots,f_s\}\subseteq\{a,J\}$)

$$\begin{aligned}\text{Hence, } J^2 &\subseteq \{a,J\}\cdot\{b,J\}=\{a,f_1,\cdots,f_s\}\cdot\{b,g_1,\cdots,g_t\}\\ &\subseteq\{ab,ag_j,bf_i,f_ig_j: 1\le i\le s, 1\le j\le t\}\triangleq P\\ &\subseteq J\end{aligned}$$

For each $f\in J$, $f^2\in J^2\subseteq P\Rightarrow f\in P\Rightarrow J=P=\{ab,ag_j,bf_i,f_ig_j\}$, which contradicts to the hypothesis that J is not finitely generated.

Fix a ranking on $\Theta(Y)$ and take a characteristic set $\mathcal{A}$ of J under this ranking. Let $\mathcal{A}=A_1,\cdots,A_p$ and denote $IS\triangleq\prod_{i=1}^{p}(I_{A_i}S_{A_i})\in K\{Y\}$.

Since J is prime, $J = \mathrm{sat}(\mathcal{A}) = [\mathcal{A}] : H_{\mathcal{A}}^{\infty} \subseteq \{\mathcal{A}\} : (IS)$.

Since $I_{A_i}, S_{A_i} \notin J$ for each i, $IS \notin J$. Thus $\{J, IS\}$ is finitely generated as a radical δ-ideal. That is, $\exists\, h_1, \cdots, h_l \in J$ s.t.

$$\{J, IS\} = \{h_1, \cdots, h_l, IS\}.$$

Thus, $J^2 \subseteq J \cdot \{J, IS\} = J \cdot \{h_1, \cdots, h_l, IS\}$

$\subseteq \{h_1, \cdots, h_l, \mathcal{A}\}$ (for $IS \cdot J \subseteq \{\mathcal{A}\}$).

$\subseteq J$.

Hence $J = \{h_1, \cdots, h_l, A_1, \cdots, A_p\}$ $\rightarrow\leftarrow$.

So every radical δ-ideal of $K\{Y_1, \cdots, Y_n\}$ is finitely generated as a radical δ-ideal. $\boxed{}$.

Example 1. $[Y^2] \subsetneqq [Y^2, (Y')^2] \subsetneqq [Y^2, (Y')^2, (Y'')^2] \subsetneqq \cdots$ is an infinite increasing sequence of differential ideals.

Proof. Let $I_n = [Y^2, (Y')^2, \cdots, (Y^{(n)})^2]$ with $n \geq 0$.

Define weight for each $Y^{(i)}Y^{(j)}$ to be $wt(Y^{(i)}Y^{(j)}) = i + j$.

Let V_n be a subspace of $K\{Y\}$ generated by all $Y^{(i)}Y^{(j)}$ of degree 2 and weight n. Then we get

$$V_0 = \mathrm{span}_K(Y^2)$$

$$V_1 = \mathrm{span}_K(YY')$$

$$V_2 = \mathrm{span}_K(YY'', (Y')^2)$$

$$V_3 = \mathrm{span}_K(YY^{(3)}, Y'Y'')$$

$$\vdots$$

$$V_{2n} = \mathrm{span}_K(YY^{(2n)}, Y'Y^{(2n-1)}, \cdots, (Y^{(n)})^2)$$

$$V_{2n+1} = \mathrm{span}_K(YY^{(2n+1)}, Y'Y^{(2n)}, \cdots, Y^{(n)}Y^{(n+1)}).$$

Clearly, $\dim V_{2n} = \dim V_{2n+1} = n+1$ for $n \in \mathbb{N}$.

Claim (1) $V_{2n+2} = \mathrm{span}_K(\delta^2(V_{2n}), (y^{(n+1)})^2)$

(2) $I_n \cap V_{2n+2} = \mathrm{span}_K(\delta^2(V_{2n})) \subsetneq V_{2n+2}$.

To show (1), note that $\delta^2(y^{(k)}y^{(2n-k)}) \in V_{2n+2}$ and

$$\begin{pmatrix} \delta^2(yy^{(2n)}) \\ \delta^2(y'y^{(2n-1)}) \\ \vdots \\ \delta^2((y^{(n)})^2) \\ (y^{(n+1)})^2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 2 & \cdots & 0 & 0 \\ & & \cdots & & & \\ 0 & 0 & 0 & \cdots & 2 & 2 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}}_{A} \begin{pmatrix} yy^{(2n+2)} \\ y'y^{(2n+1)} \\ \vdots \\ y^{(n)}y^{(n+2)} \\ (y^{(n+1)})^2 \end{pmatrix}.$$

$\mathrm{Det}(A) = 1 \Rightarrow \{\delta^2(V_{2n}), (y^{(n+1)})^2\}$ is a basis of $V_{2n+2} \Rightarrow$ (1) ✓.

To show (2), since $V_{2n} \subseteq I_n$ ($(y^{(n)})^2 \in I_n$, $(y^{(n+1)})^2 \in I_n \Rightarrow y^{(n-1)}y^{(n+1)} \in I_n$, $(y^{(n-2)})^2 \in I_n \Rightarrow 2y^{(n+2)}y^{(n-2)} + 8y^{(n+1)}y^{(n-1)} + 6(y^{(n)})^2 \in I_n \Rightarrow y^{(n-2)}y^{(n+2)} \in I_n \cdots yy^{(2n)} \in I_n$),

$\mathrm{span}_K[\delta^2(V_{2n})] \subseteq I_n \cap V_{2n+2}$. And $I_n \cap V_{2n+2} \subseteq \mathrm{span}_K(\delta^{2n+2-2k}(y^{(k)})^2,\ k=0,\cdots,n)$

$= \mathrm{span}_K(\delta^2[\delta^{2n-2k}(y^{(k)})^2] : k=0,\cdots,n) \subseteq \mathrm{span}_K(\delta^2(V_{2n}))$.

Thus, $I_n \cap V_{2n+2} = \mathrm{span}_K(\delta^2(V_{2n})) \subsetneq V_{2n+2}$.

Hence, $V_{2n} \subseteq I_n$ & $V_{2n+2} \not\subseteq I_n$ for all $n \in \mathbb{N}$. Thus, $I_n \subsetneq I_{n+1}$ $\forall n \in \mathbb{N}$. ☒.