Recall: Equivalent definitions for characteristic sets of a differential ideal I:
Let $\mathcal{A}$ be an autoreduced set contained in I w.r.t a fixed ranking. Then
(1) $\mathcal{A}$ is a characteristic set of I
$\Leftrightarrow$ (2) $\mathcal{A}$ is of lowest rank among all autoreduced sets containing in I
$\Leftrightarrow$ (3) For each $f \in \mathcal{A}$, $\delta\text{-rem}(f, \mathcal{A}) = 0$
$\Leftrightarrow$ (4) I doesn't contain a nonzero δ-poly reduced w.r.t. $\mathcal{A}$.

Note. $r = \delta\text{-rem}(f, \mathcal{A})$, the diff remainder of f w.r.t. $\mathcal{A}$, satisfies the relation:

$$\prod_{A \in \mathcal{A}} S_A^{i_A} I_A^{t_A} \cdot f \equiv r \mod [\mathcal{A}] \text{ for some } i_A, t_A \in \mathbb{N}.$$

By (3), $I \subseteq \mathrm{sat}(\mathcal{A}) \triangleq [\mathcal{A}] : H_{\mathcal{A}}^{\infty}$. $\left(H_{\mathcal{A}}^{\infty} = \{ \prod_{A \in \mathcal{A}} S_A^{i_A} I_A^{t_A} \mid i_A, t_A \in \mathbb{N} \} \right)$.
By (4), $S_A, I_A \notin I$ for $\forall A \in \mathcal{A}$.

A differential ring is called Ritt-Noetherian if the set of radical differential ideals satisfies ACC condition (Equivalently, every radical differential ideal is finitely generated as a radical δ-ideal.)

Theorem 2.2.3 (Ritt-Raudenbush basis Theorem)
Let (K, δ) be a differential field of characteristic 0.
Then $K\{y_1, \ldots, y_n\}$ is Ritt-Noetherian.

Proof. Sps the contrary. Then $\exists$ a radical δ-ideal $\subseteq K\{y_1, \ldots, y_n\}$ not finitely generated as a radical δ-ideal.
By Zorn's lemma, $\exists$ a maximal radical δ-ideal J not finitely generated as a radical δ-ideal.
Claim: J is prime.
If not, $\exists\, a, b \in K\{y_1, \ldots, y_n\}$, $a, b \notin J$ but $ab \in J$.

So $J \subsetneq \{J, a\}$ and $J \subsetneq \{J, b\}$. By the maximality of J,
$\exists\ f_1, \cdots, f_s, g_1, \cdots, g_t \in J$ s.t.

$$\{J, a\} = \{f_1, \cdots, f_s, a\} \text{ and } \{J, b\} = \{g_1, \cdots, g_t, b\}.$$

(Here, we use $\mathrm{char}(K) = 0$. Think why.)

$$\begin{aligned} \text{So } J^2 &\subseteq \{J, a\} \cdot \{J, b\} = \{f_1, \cdots, f_s, a\} \cdot \{g_1, \cdots, g_t, b\} \\ &\subseteq \{f_i g_j, f_i b, a g_j : 1 \le i \le s, 1 \le j \le t\} \\ &\subseteq J. \end{aligned}$$

$\Rightarrow J = \{f_i g_j, f_i b, a g_j : 1 \le i \le s, 1 \le j \le t\}$ is finitely generated $\rightarrow\leftarrow$.

Thus, J is prime and the claim is proved.

Select a ranking $\mathcal{R}$ on $\Theta(Y)$ and take a characteristic set $\mathcal{A} = A_1, \cdots, A_p$ of J. Denote $IS = \prod_{i=1}^{p} (S_{A_i} I_{A_i})$.

J is prime $\Rightarrow J = \mathrm{sat}(\mathcal{A}) = [\mathcal{A}] : H_{\mathcal{A}}^{\infty} \subseteq \{\mathcal{A}\} : IS$. (Indeed, $\forall f \in \mathrm{sat}(\mathcal{A})$ $\exists\ k_i, t_i \in \mathbb{N}$ s.t. $\prod_{i=1}^{p} I_{A_i}^{k_i} S_{A_i}^{t_i} f \in [\mathcal{A}] \subseteq \{\mathcal{A}\}$. Take $t = \max\{k_i + 1, t_i + 1\}$. Then $(IS)^t f^t \in \{\mathcal{A}\}$. So $IS \cdot f \in \{\mathcal{A}\}$.) Since $IS \notin J$, $\exists\ f_1, \cdots, f_l \in J$ s.t.

$$\{J, IS\} = \{f_1, \cdots, f_l, IS\}.$$

$$\begin{aligned} \text{Thus, } J^2 &\subseteq J \cdot \{J, IS\} = J \cdot \{f_1, \cdots, f_l, IS\} \\ &\subseteq \{f_1, \cdots, f_l, \mathcal{A}\} \\ &\subseteq J. \end{aligned}$$

Hence, $J = \{f_1, \cdots, f_l, A_1, \cdots, A_p\}$, a contradiction to the hypothesis that J is not finitely generated radically.

Whenhence, $K\{Y_1, \cdots, Y_n\}$ is Ritt-Noetherian. $\boxtimes$.

Theorem 2.2.4. Let R be a δ-ring which is Ritt-Noetherian and $\mathbb{Q} \subseteq R$. Then for every radical δ-ideal $I \subsetneqq R$, there exist a finite number of prime δ-ideals $P_1, \cdots, P_\ell$ s.t.

$$I = \bigcap_{i=1}^{\ell} P_i . \quad (*)$$

Moreover, if $(*)$ is irredundant ($\forall i, \bigcap_{j \neq i} P_j \not\subseteq P_i$), then this set of prime ideals is unique. In this case, $P_1, \cdots, P_\ell$ are called prime components of I.

Proof. Suppose the statement is false, i.e., the set

$$A = \{ I \mid I \subsetneqq K\{Y_1, \cdots, Y_n\} \text{ is a radical } \delta\text{-ideal and } I \text{ is not a finite intersection of prime } \delta\text{-ideals} \}$$ is not empty.

Since R is Ritt-Noetherian, every ascending chain of radical δ-ideal has an upper bound in A. By Zorn's lemma, A has a maximal element $J \in A$. Clearly, J is not prime. So $\exists\, a, b \notin J$, but $ab \in J$. Thus, $\{J, a\} \supsetneqq J$ and $\{J, b\} \supsetneqq J$.

Also, $\{J, a\} \neq R$. Indeed, if not, then $1 \in \{J, a\}$. Since $\mathbb{Q} \subseteq R$, $1 \in [J, a]$ and $1 = f + \sum * \delta^k(a)$. $(f \in J)$ By $ab \in J$ and J is radical, $b\delta^k(a) \in J$ $\forall k \in \mathbb{N}$. So $b = fb + \sum * b\delta^k(a) \in J$, contradicting to $b \notin J$.

Similarly, $\{J, b\} \neq R$ could be shown.

By the maximality of J, $\exists\, P_1^a, \cdots, P_\ell^a, P_{\ell+1}^b, \cdots, P_{\ell+t}^b$ prime δ-ideals in R s.t.

$$\{J, a\} = P_1^a \cap \cdots \cap P_\ell^a \text{ and}$$
$$\{J, b\} = P_{\ell+1}^b \cap \cdots \cap P_{\ell+t}^b .$$

Now show $J = \{J, a\} \cap \{J, b\}$. Indeed, let $f \in \{J, a\} \cap \{J, b\}$, then

$$f^2 \in \{J, a\} \cdot \{J, b\} \subseteq \{J, ab\} \subseteq J \Longrightarrow f \in J.$$

Thus, $J = \{J, a\} \cap \{J, b\} = P_1^a \cap \cdots \cap P_\ell^a \cap P_{\ell+1}^b \cap \cdots \cap P_{\ell+t}^b$, contradicting to the hypothesis $J \in A$. So the first statement is valid.

Uniqueness. Suppose $I = \bigcap_{i=1}^{l} P_i = \bigcap_{j=1}^{t} Q_j$ be irredundant intersections.
For each $j = 1, \cdots, t$,
$$\bigcap_{i=1}^{l} P_i \subseteq Q_j.$$
Then $\exists\, i_0 \in \{1, \cdots, l\}$ s.t. $P_{i_0} \subseteq Q_j$. Indeed, sps the contrary, then $\exists$ $f_i \in P_i \setminus Q_j$ for each $i = 1, \cdots, l$. Thus, $f_1 f_2 \cdots f_l \in \bigcap_{i=1}^{l} P_i \subseteq Q_j \rightarrow\leftarrow$.
Similarly, $\exists\, j_0 \in \{1, \cdots, t\}$ s.t. $Q_{j_0} \subseteq P_{i_0} \subseteq Q_j$. Since $I = \bigcap_{j=1}^{t} Q_j$ is irredundant, $j_0 = j$ and $P_{i_0} = Q_j$.
Thus, $l = t$ and $\exists$ a permutation $\sigma \in S_l$ s.t. $P_i = Q_{\sigma(i)}$. $\boxtimes$.

Cor. 2.2.5 Every proper radical δ-ideal $I \subsetneqq K\{y_1, \cdots, y_n\}$ $(\mathrm{char}(K) = 0)$ can be written as a finite intersection of prime δ-ideals.
If $I = \bigcap_{i=1}^{l} P_i$ is irredudant, P_i are called prime components of I.

Example. $I = \{y'^2 - 4y\} \subseteq \mathbb{Q}\{y\}$. Then
$$I = \{y'^2 - 4y,\ y'' - 2\} \cap \{y\} \text{ (chapter 3).}$$

We end this chapter by giving an example illustrating a δ-ideal is not finitely generated as a δ-ideal.
Example 2. The radical δ-ideal $\{xy\} \subseteq K\{x, y\}$ is not finitely generated as a δ-ideal. In other words, there doesn't exist finitely many δ-poly $f_1, \cdots, f_s \in K\{x, y\}$ s.t. $\{xy\} \neq [f_1, \cdots, f_s]$.
Proof. Let $I = \{xy\} \subseteq K\{x, y\}$ and $J = (x^{(i)} y^{(j)} : i, j \in \mathbb{N}) \subseteq K\{x, y\}$.
Claim A $I = J$.
Indeed, $J \subseteq I$, for $xy \in I \xrightarrow{\text{Lemma 1.10}} \forall\, i, j \in \mathbb{N},\ x^{(i)} y^{(j)} \in I$.

It is easy to show that J is a δ-ideal and the following fact:

$$f \notin J \iff f \text{ has a term not involving any } y^{(j)} \text{ (or } x^{(i)}\text{)}.$$

The fact implies that $J \subseteq \{xy\}$ is a radical δ-ideal and $I = J$ follows.

Now, suppose the contrary, i.e., $\exists\, f_1, \cdots, f_s \in K\{x, y\}$ s.t. $I = [f_1, \cdots, f_s]$.

Since $I = J$, then $\exists\, q_e \in \mathbb{N}$ s.t. $f_i \in [\, x^{(i)} y^{(j)} : 0 \le i, j \le q_e\,]$.

Hence, $I = [\, x^{(i)} y^{(j)} : 0 \le i, j \le q_e\,]$.

In particular, $x^{(q_e+1)} y^{(q_e+1)} \in [\, x^{(i)} y^{(j)} : 0 \le i, j \le q_e\,]$

$\Downarrow$ substitute $x = y$ in the expression of $x^{(q_e+1)} y^{(q_e+1)}$ in terms of $x^{(i)} y^{(j)}$, $(i, j \le q_e)$

$$\left(y^{(q_e+1)}\right)^2 \in [\, y^{(i)} y^{(j)} : \ i, j \le q_e\,]. \quad (*)$$

$\left(y^{(q_e+1)}\right)^2 \notin \delta^2(V_{2q_e})$

Use the notation in Example 1, $\left(y^{(q_e+1)}\right)^2 \in V_{2q_e+2}$ and $V_{2q_e+2} = \mathrm{Span}_K\{\delta^2(V_{2q_e}), \left(y^{(q_e+1)}\right)^2\}$.

But, $[\, y^{(i)} y^{(j)} : i, j \le q_e\,] \cap V_{2q_e+2} = \mathrm{Span}_K(\delta^2(V_{2q_e}))$.

(Indeed, $V_{2q_e} \subseteq [\, (y^{(i)})^2 : i \le q_e\,] \subseteq [\, y^{(i)} y^{(j)} : i, j \le q_e\,]$

$$\Rightarrow \delta^2(V_{2q_e}) \subseteq [\, y^{(i)} y^{(j)} : i, j \le q_e\,] \cap V_{2q_e+2}.$$

On the other hand, $\forall\, f \in [\, y^{(i)} y^{(j)} : i, j \le q_e\,] \cap V_{2q_e+2}$,

$$f = \sum_{0 \le i, j \le q_e} c_{ij}\, \delta^{2q_e+2-i-j}\left(y^{(i)} y^{(j)}\right)$$

$$= \sum_{0 \le i, j \le q_e} c_{ij}\, \delta^2\Big(\underbrace{\delta^{2q_e-i-j}\left(y^{(i)} y^{(j)}\right)}_{\in V_{2q_e}}\Big) \in \mathrm{Span}_K(\delta^2(V_{2q_e})).\ )$$

Since $\left(y^{(q_e+1)}\right)^2 \notin \mathrm{span}_K(\delta^2(V_{2q_e}))$,

$\left(y^{(q_e+1)}\right)^2 \notin [\, y^{(i)} y^{(j)} : i, j \le q_e\,]$, contradicts to $(*)$. $\boxtimes$.

Chapter 3 The Differential Algebra-Geometry Dictionary

Let (K, δ) be a differential field of characteristic 0.

Let $K\{Y\} = K\{y_1, \cdots, y_n\}$ be the differential polynomial ring in the differential variables $y_1, \cdots, y_n$ over K. Any $\Sigma \subseteq K\{Y\}$ defines a system of algebraic diff equations. The main objective of differential algebra is to study the solutions of any such system (i.e., differential varieties, our main protagonists).

§3.1 Ideal-Variety correspondence in differential algebra

Recall the definitions of differential closed fields and differential varieties:

For $f \in K\{Y\} = K\{y_1, \cdots, y_n\}$ and $\eta = (\eta_1, \cdots, \eta_n) \subseteq L^n$ w/ $(L, \delta) \supseteq (K, \delta)$,

η is a differential zero of f if $f(\eta) = 0$. Here, $f(\eta)$ means replacing $\delta^k y_i$ by $\delta^k(\eta_i)$ in $f(y_1, \cdots, y_n)$.

(E, δ) is <u>differentially closed</u> if for all $F \subseteq E\{y_1, \cdots, y_n\}$, whenever $\exists (L, \delta) \supseteq (E, \delta)$ and $\eta \in L^n$ s.t. $F(\eta) = 0$, there exists $\xi \in E^n$ s.t. $F(\xi) = 0$.

Let $(K, \delta) \subseteq (E, \delta)$. (E, δ) is called a differential closure of (K, δ) if

1) (L, δ) is differentially closed and 2) for every differential closed field $(M, \delta) \supseteq (K, \delta)$, there is a differential embedding $\varphi: L \hookrightarrow M$ w/ $\varphi|_K = id_K$.

Throughout this chapter, $(E, \delta) \supseteq (K, \delta)$ is a fixed differential closed field.

By a differential affine space, we mean any E^n for $n \in \mathbb{N}$. An element $(\eta_1, \cdots, \eta_n) \in E^n$ is called a point.

A set $V \subseteq E^n$ is called a <u>δ-variety</u> over K if $\exists \Sigma \subseteq K\{Y\}$ s.t.

$$V = \mathbb{V}(\Sigma) \triangleq \{ \eta \in E^n \mid f(\eta) = 0,\ \forall f \in \Sigma \}.$$

Let $\mathcal{T} = \{\delta\text{-varieties in } E^n \text{ over } K\}$.

Then $\mathcal{T}$ satisfies: 1) $\phi, E^n \in \mathcal{T}$;

2) if $V_1, V_2 \in \mathcal{T}$, $V_1 \cup V_2 \in \mathcal{T}$;

3) Any intersection of elements of $\mathcal{T}$ is an element of $\mathcal{T}$.

So $\mathcal{T}$ is a topology on E^n, called the Kolchin topology, as compared to the Zariski topology in algebraic geometry. A δ-variety is called a Kolchin-closed set. For a set $S \subseteq E^n$, the smallest δ-variety (with respect to inclusion) containing S is called the Kolchin closure of S, denoted by S^{Kol}.

For a subset $S \subseteq E^n$, define

$$\mathbb{I}(S) = \{ f \in K\{y_1, \dots, y_n\} \mid \forall\, \eta \in S,\ f(\eta) = 0 \}.$$

It is easy to show that $\mathbb{I}(S)$ is a radical δ-ideal in $K\{Y\}$, called the vanishing δ-ideal of S.

Prop 3.1.1 1) If $S_1 \subseteq S_2 \subseteq E^n$, then $\mathbb{I}(S_2) \subseteq \mathbb{I}(S_1)$.

2) If $P_1 \subseteq P_2 \subseteq K\{Y\}$, then $\mathbb{V}(P_2) \subseteq \mathbb{V}(P_1)$.

3) If $S \subseteq E^n$, then $V = \mathbb{V}(\mathbb{I}(S))$ is the Kolchin closure of S and $\mathbb{I}(V) = \mathbb{I}(S)$.

Proof. 1) and 2) are straightforward.

To show 3): Let $S^{Kol} = \mathbb{V}(\Sigma)$ for $\Sigma \subseteq K\{Y\}$.

For every $f \in \Sigma$, $f|_S \equiv 0 \Rightarrow f \in \mathbb{I}(S)$. So $\Sigma \subseteq \mathbb{I}(S)$.

Thus, $V = \mathbb{V}(\mathbb{I}(S)) \subseteq \mathbb{V}(\Sigma) = S^{Kol}$. Hence, $S^{Kol} = V$.

$S \subseteq V \Rightarrow \mathbb{I}(V) \subseteq \mathbb{I}(S)$. If $\exists\, f \in \mathbb{I}(S) \setminus \mathbb{I}(V)$, then $\exists\, \eta \in V$ s.t. $f(\eta) \neq 0$. Set $\Sigma_1 = \mathbb{I}(V) \cup \{f\}$. Then $\Sigma_1 \subseteq \mathbb{I}(S) \Rightarrow \mathbb{V}(\Sigma_1) \supseteq \mathbb{V}(\mathbb{I}(S)) = V$.

Since $\eta \in V$, $\eta \in \mathbb{V}(\Sigma_1)$. So $f(\eta) = 0$ $\rightarrow\leftarrow$. $\square$

Now, we have two maps between Π and the set of radical δ-ideals in $K\{Y\} = K\{Y_1, \cdots, Y_n\}$:

$$\mathbb{I} : \{\delta\text{-varieties in } E^n \text{ over } K\} \longrightarrow \{\text{radical } \delta\text{-ideals in } K\{Y\}\}$$
$$V \qquad\qquad\qquad\qquad \mathbb{I}(V)$$

and

$$\mathbb{V} : \{\text{radical } \delta\text{-ideals in } K\{Y\}\} \longrightarrow \{\delta\text{-varieties in } E^n \text{ over } K\}$$
$$J \qquad\qquad\qquad\qquad \mathbb{V}(J)$$

Corollary 3.1.2 For every δ-variety V, $\mathbb{V}(\mathbb{I}(V)) = V$.

Hence, $\mathbb{I}$ is injective and $\mathbb{V}$ is surjective.

Proof. By Prop 3.1.1, $V = V^{Kol} = \mathbb{V}(\mathbb{I}(V))$. ☒.

A point $\eta \in L^n$ ($L \supseteq K$ a δ-extension field) is a generic zero of a differential ideal I if $I = \mathbb{I}(\eta)$. Clearly,

A differential ideal I is prime $\Leftrightarrow$ I has a generic zero.

"$\Rightarrow$" If I prime, set $L = \mathrm{Frac}(K\{Y_1, \cdots, Y_n\}/I)$. Then $(\bar{Y}_1, \cdots, \bar{Y}_n) \in L^n$ is a generic zero of I.

Next section, we will give the differential Nullstellensatz theorem (both the weak and strong analogues of the Hilbert's Nullstellensatz thorem). Combining Corollary 3.1.2, we will show $\mathbb{I}$ and $\mathbb{V}$ are inclusion-reversing bijective maps. For the content in this section, all the results are valid even if E is not differentially closed. But for the differential Nullstellensatz theorem to be valid, E is required to be differentially closed.