Recall (Differential Primitive Element Theorem)

Theorem 4.1.3 Let (K,δ) be a nonconstant differential field of char 0. If $K\langle\beta_1,\cdots,\beta_n\rangle$ is differential algebraic over K, then

$$\exists\ \gamma\in K\langle\beta_1,\cdots,\beta_n\rangle \text{ s.t. } K\langle\beta_1,\cdots,\beta_n\rangle=K\langle\gamma\rangle.$$

Key Lemma:

Lemma 4.1.2 (Non-vanishing of differential polynomials)

Let (K,δ) be a nonconstant differential field of char 0. If $G\in K\{y_1,\cdots,y_n\}\setminus\{0\}$, then $\exists\ (\eta_1,\cdots,\eta_n)\in K^n$ s.t. $G(\eta_1,\cdots,\eta_n)\neq 0$. In particular, if $G\in K\{y\}\setminus\{0\}$ and $\xi\in K$, $\xi'\neq 0$, then $\exists\ C_0,\cdots,C_s$ ($s=\mathrm{ord}(G)$) constants of K s.t.

$$G(C_0+C_1\xi+C_2\xi^2+\cdots+C_s\xi^s)\neq 0.$$

§4.2 Differential transcendence bases

Let R be a differential ring. Elements $\alpha_1,\cdots,\alpha_n$ in a differential over-ring S of R are called differentially algebraically dependent over R if there exists a nonzero $G\in R\{y_1,\cdots,y_n\}$ s.t. $G(\alpha_1,\cdots,\alpha_n)=0$. Otherwise, $\alpha_1,\cdots,\alpha_n$ are called differentially$_{(\delta)}$ algebraic independent over R. A subset of S is called δ-algebraically independent over R if all its subsets are δ-algebraically independent over R.

Def 4.2.1 Let $K\subseteq L$ be an extension of δ-fields and $A\subseteq L$. An element $b\in L$ is called δ-algebraically dependent on A (over K) if b is δ-algebraic over $K\langle A\rangle$. A subset B of L is called δ-algebraically dependent on A (over K) if every elt of B is δ-algebraically dependent on A.

Note: Since K is our base δ-field, for simplicity, we usually omit "over K".

Lemma 4.2.2 Let $K \subseteq L$ be an extension of δ-fields, $A \subseteq L$ and $b \in L$. Then b is δ-algebraically dependent on A if and only if $\exists f \in K\{y_1, \cdots, y_n, z\}$ and $a_1, \cdots, a_n \in A$ such that $f(a_1, \cdots, a_n, z) \neq 0$ and $f(a_1, \cdots, a_n, b) = 0$.

Proof. Assume b is δ-alg. dep. on A. Then by definition, b is δ-alg over $K\langle A \rangle$, so $\exists$ a nonzero $g \in K\langle A \rangle\{z\}$ s.t. $g(b) = 0$.
Let $\{a_1, \cdots, a_n\} \subseteq A$ be the subset appearing effectively in the coefficients of g. After multiplying g by appropriate elt from $K\{a_1, \cdots, a_n\}$, we can assume $g \in K\{a_1, \cdots, a_n, z\}$. Thus, this g satisfies the desired property.
The converse is obvious. $\boxtimes$.

Lemma 4.2.3 Let $K \subseteq L$ be an extension of δ-fields and A be a subset of L which is δ-algebraically independent over K. Let $b \in L$. If A, b are δ-algebraically dependent over K, then b is δ-algebraic over $K\langle A \rangle$.

Proof. Since A, b are δ-algebraically dependent over K, then $\exists\ 0 \neq f \in K\{y_1, \cdots, y_n, z\}$ s.t. $f(a_1, \cdots, a_n, b) = 0$ for some $a_1, \cdots, a_n \in A$.
Since $a_1, \cdots, a_n$ are δ-alg indep. over K, $f(a_1, \cdots, a_n, z) \neq 0$.
Thus, b is δ-alg. over $K\langle A \rangle$. $\boxtimes$.

Lemma 4.2.4 (Transitivity of δ-algebraic dependence)
Let $(K, \delta) \subseteq (L, \delta)$ and $A, B, C \subseteq L$. If A is δ-alg. dependent on B and B is δ-alg. dep. on C, then A is δ-alg dep. on C.

Proof. By the assumption, $K\langle B \rangle\langle A \rangle$ is δ-alg over $K\langle B \rangle$ and $K\langle C \rangle\langle B \rangle$ is δ-alg over $K\langle C \rangle$. By Lemma 4.6, $K\langle C, B, A \rangle$ is δ-alg over $K\langle C \rangle$. Thus, each elt of A is δ-alg over $K\langle C \rangle$. $\boxtimes$.

Lemma 4.2.5 (The exchange property)

Let $a_1, \cdots, a_n, b$ be elts from a δ-extension field of K.
If b is δ-algebraically dependent on $a_1, \cdots, a_n$ but not on $a_1, \cdots, a_{n-1}$, then a_n is δ-algebraically dependent on $a_1, \cdots, a_{n-1}, b$.

Proof. Since b is δ-alg. dep. on $a_1, \cdots, a_n$, by Lemma 4.2.2, there exists $0 \neq g \in K\{Y_1, \cdots, Y_n, Z\}$ s.t. $g(a_1, \cdots, a_n, Z) \neq 0$ and $g(a_1, \cdots, a_n, b) = 0$.
Regard g as a univariate δ-poly in Y_n with coefficients from $K\{Y_1, \cdots, Y_{n-1}, Z\}$, and let $g_1, \cdots, g_m \in K\{Y_1, \cdots, Y_{n-1}, Z\}$ be all the nonzero coefficients.
Then $\exists\, i$ $g_i(a_1, \cdots, a_{n-1}, Z) \neq 0$, for otherwise, $g(a_1, \cdots, a_{n-1}, a_n, Z) = 0$.
Since b is not δ-alg dependent on $a_1, \cdots, a_{n-1}$, $g_i(a_1, \cdots, a_{n-1}, b) \neq 0$.
So $g(a_1, \cdots, a_{n-1}, Y_n, b) \neq 0$ and consequently, a_n is δ-alg dependent on $a_1, \cdots, a_{n-1}, b$. $\boxtimes$.

Prop 4.2.6 Let $K \subseteq L$ be an extension of δ-fields and $A = \{a_1, \cdots, a_n\}$, $B = \{b_1, \cdots, b_m\}$ be two subsets of L. Assume that 1) A is δ-algebraically independent over K and 2) A is δ-alg dependent on B.
Then $n \leq m$.

Proof. Let $r = |A \cap B|$. If $r = n$, then we are done.
Now assume $r < n$ and write $B = a_1, \cdots, a_r, b_{r+1}, \cdots, b_m$.
Since a_{r+1} is δ-alg. dependent on $a_1, \cdots, a_r, b_{r+1}, \cdots, b_m$ but not on $a_1, \cdots, a_r$, there will be a b_j ($r+1 \leq j \leq m$) s.t. a_{r+1} is δ-alg. dep. on $a_1, \cdots, a_r, b_{r+1}, \cdots, b_j$ but not δ-alg. dep. on $a_1, \cdots, a_r, b_{r+1}, \cdots, b_{j-1}$.
By the exchange property (Lemma 4.2.5), b_j is δ-alg dep on $a_1, \cdots, a_r, b_{r+1}, \cdots, b_{j-1}, a_{r+1}$, and thus δ-alg dep on $B_1 := (B \setminus \{b_j\}) \cup \{a_{r+1}\}$.

Therefore, B is δ-alg dep on B_1. Since A is δ-alg. dependent on B, by Lemma 4.2.4, A is δ-alg dep on B_1. Note that $|B_1|=m$ and $|A\cap B_1| = r+1$. Continuing in this way, we will eventually arrive at $r=n$, i.e., $A\subseteq B$. So $n\leq m$. ☒

Def 4.2.7 Let $(K,\delta)\subseteq(L,\delta)$. A subset A of L is called a δ-transcendence basis of L/K if 1) A is δ-algebraically indep. over K and 2) L is δ-algebraic over $K\langle A\rangle$.

By the size of a set, we mean its cardinality if the set is finite, and ∞ otherwise.

Theorem 4.2.8 Let $(K,\delta)\subseteq(L,\delta)$. Then every δ-generating set of $L\supseteq K$ contains a δ-transcendence basis of L/K. In particular, there exists a δ-transcendence basis of L/K. Moreover, any two δ-transcendence bases of L/K are of the same size.

proof. Let M be a δ-generating set of L/K, i.e., $L=K\langle M\rangle$. Let $N=\{S\subseteq M \mid S \text{ is } \delta\text{-alg indep over } K\}$. Then $\phi\in N\neq\phi$. Clearly, the union of every chain of elements in N is again in N. So by Zorn's Lemma, there exists a maximal elt A in N.

claim: A is a δ-transcendence basis of L/K.

We now show the claim. For any $a\in M$, a, A are δ-alg dep over K. By Lemma 4.2.3, a is δ-alg over $K\langle A\rangle$, so M is δ-alg over $K\langle A\rangle$. And by Lemma 4.5, $L=K\langle M\rangle$ is δ-algebraic over $K\langle A\rangle$. Thus, $A\subseteq M$ is a δ-transcendence basis of L/K.

Now, suppose A and B are both δ-transcendence bases of L/K. By symmetry, it suffices to show that the size of $A \geq$ the size of B. If A is an infinite set, it is automatically valid. So we may assume A is finite. Let B_1 be a finite subset of B. Since A is a δ-transcendence basis of L/K, each elts of B_1 is δ-alg over $K\langle A\rangle$, and B_1 is δ-alg. dep on A. By Prop 4.2.6, $|B_1| \leq |A|$. Thus, $|B| \leq |A|$. ∎

Corollary 4.2.9 Let $(K,\delta) \subseteq (L,\delta)$ and $L = K\langle M\rangle$. If A is a maximal δ-alg indep subset of M, then A is a δ-transcendence basis of L/K.

Theorem 4.2.8 guarantees we can make the following definition:

<u>Definition 4.2.10</u> Let $(K,\delta) \subseteq (L,\delta)$. The size of a δ-transcendence basis of L/K is called the δ-transcendence degree of L/K. It is denoted by $\delta\text{-tr.deg}(L/K)$.

Corollary 4.2.11. Let $(K,\delta) \subseteq (L,\delta)$ and $L = K\langle a_1, \cdots, a_n\rangle$. Then $\delta\text{-tr.deg}(L/K) \leq n$, and the δ-transcendence degree of a finitely δ-generated δ-field extension is finite.

Proof. It is clear from Cor. 4.2.9.

Corollary 4.2.12 Let $(K,\delta) \subseteq (L,\delta)$. If L contains n δ-independent elts, then $n \leq \delta\text{-tr.deg}(L/K)$. In fact,

$$\delta\text{-tr.deg}(L/K) = \sup\{n \in \mathbb{N} \mid \exists\, a_1, \cdots, a_n \in L \ \delta\text{-algebraically indep over } K\}.$$

Proof. Let $a_1, \dots, a_n \in L$ be δ-alg. indep over K. We can enlarge $\{a_1, \cdots, a_n\}$ to a δ-generating set B of L/K. Then $\{a_1, \cdots, a_n\}$ is contained in a maximal δ-alg indep subset $A' \subseteq B$. By Corollary 4.2.9, A' is a δ-transcendence basis of L/K. Thus, $n \le \delta\text{-tr.deg}(L/K)$ and also $\sup\{n \mid \cdots\} \le \delta\text{-tr.deg}(L/K)$. The reverse estimate is clear, for a δ-transcendence basis is δ-alg indep over K. $\boxtimes$

<u>Theorem 4.2.13</u> Let $K \subseteq L \subseteq M$ be δ-fields. Then

$$\delta\text{-tr.deg}(M/K) = \delta\text{-tr.deg}(M/L) + \delta\text{-tr.deg}(L/K).$$

(Here, $\infty + a(\infty) = \infty$).

Proof. Let A be a δ-transcendence basis of L/K and B a δ-transcendence basis of M/L.

Claim: $A \cup B$ is a δ-transcendence basis of M/K.

First, since B is δ-alg. indep over $K\langle A\rangle$ $(\subseteq L)$, $A \cup B$ is δ-algebraically indep over K.

It remains to show M is δ-alg over $K\langle A, B\rangle$. Since each elt of M is δ-alg over $L\langle B\rangle$ and each elt of L is δ-alg over $K\langle A\rangle$, M is δ-alg over $K\langle A, B\rangle$.

Thus, $A \cup B$ is a δ-transcendence basis of M/K and $A \cap B = \phi$ implies that $\delta\text{-tr.deg}(M/K) = \delta\text{-tr.deg}(M/L) + \delta\text{.tr.deg}(L/K)$. $\boxtimes$

Adjoining the differential primitive element theorem, we have

Prop 4.2.14. Let $L = K\langle a_1, \cdots, a_n\rangle$ and suppose K contains a nonconstant elt in the case $d = \delta\text{-tr.deg}(L/K) = 0$.

Then L is δ-generated by $\leq d+1$ elts. (See Resolvent later)
Proof. In the case $d=0$, this is the diff primitive element theorem. Assume $d>0$. Then $\exists \{\xi_1,\cdots,\xi_d\}\subseteq\{a_1,\cdots,a_n\}$ s.t. $\xi_1,\cdots,\xi_d$ is a δ-transcendence basis of L/K and denote the others by $\xi_{d+1},\cdots,\xi_n$.
Then $L=K\langle\xi_1,\cdots,\xi_d\rangle\langle\xi_{d+1},\cdots,\xi_n\rangle$

$$=K\langle\xi_1,\cdots,\xi_d\rangle\langle a_{d+1}\xi_{d+1}+\cdots+a_n\xi_n\rangle \quad (d>0\Rightarrow K\langle\xi_1,\cdots,\xi_d\rangle'\neq\{0\}). \boxtimes$$

§4.3 Applications to differential varieties

Let (K,δ) be δ-field of char 0 and $V\subseteq\mathbb{A}^n$ an irreducible δ-variety over K. (Here $\mathbb{A}^n=\bar{K}^n$ w/ $(\bar{K},\delta)$ a δ-closed field containing (K,δ)).
The coordinate δ-ring of V is $K\{V\}\triangleq K\{y_1,\cdots,y_n\}/\mathbb{I}(V)$.
Here $K\{V\}$ is a δ-domain and we consider the δ-quotient field

$$K\langle V\rangle=\mathrm{Frac}(K\{V\}).$$

Naturally, $K\langle V\rangle$ is a δ-field extension of K and called the δ-function field of V. Clearly, $(\bar{y}_1,\cdots,\bar{y}_n)\in K\langle V\rangle^n$ is a generic point of V. Given any other generic point $(a_1,\cdots,a_n)$ of V, we have $K\langle V\rangle=K\langle\bar{y}_1,\cdots,\bar{y}_n\rangle\cong K\langle a_1,\cdots,a_n\rangle$ w/ $\bar{y}_i\leftrightarrow a_i$.
In particular, $\delta\text{-tr.deg}\,K\langle\bar{y}_1,\cdots,\bar{y}_n\rangle/K=\delta\text{-tr.deg}\,K\langle a_1,\cdots,a_n\rangle/K$.

Def 4.3.1. Let $V\subseteq\mathbb{A}^n$ be an irreducible δ-variety over K. The δ-dimension of V is defined as the δ-transcendence degree of the δ-function field of V over K. That is,

$$\delta\text{-dim}(V)=\delta.\text{tr.deg}\,K\langle V\rangle/K.$$

For an arbitrary V with irreducible components $V_1,\cdots,V_m$,

$$\delta\text{-dim}(V)=\max_i\,\delta\text{-dim}(V_i).$$

Another equivalent definition of diff dimension in the language of differential ideals given by Ritt:

Def 4.3.2. Let $P \subseteq K\{y_1, \cdots, y_n\}$ be a prime δ-ideal. A δ-variable subset $U \subseteq \{y_1, \cdots, y_n\}$ is called a δ-independent set modulo P if $P \cap K\{U\} = \{0\}$. A parametric set of P is a maximal independent modulo P. The δ-dimension of P (or $\mathbb{V}(P)$) is defined to be the cardinal number of its parametric set.

EX: Please show different parametric sets of a prime δ-ideal have the same cardinal number. And show Def 4.3.1 and Def 4.3.2 are equivalent for prime δ-ideals or irreducible δ-varieties.

Lemma 4.3.2 Let V be a δ-variety and $W \subseteq V$ a δ-subvariety. Then $\delta\text{-dim}(W) \leq \delta\text{-dim}(V)$.

Proof. First assume W and V are both irreducible. $W \subseteq V$ implies that $\mathbb{I}(W) \supseteq \mathbb{I}(V)$. Suppose $\delta\text{-dim}(W) = d$ and $\{y_1, \cdots, y_d\}$ is a parametric set of $\mathbb{I}(W)$. Clearly $\mathbb{I}(V) \cap \{y_1, \cdots, y_d\} = \{0\}$ and $\{y_1, \cdots, y_d\}$ is a δ-independent set modulo $\mathbb{I}(V)$ which could be extended to a parametric set of $\mathbb{I}(V)$. Thus, $\delta\text{-dim}(V) = \delta\text{-dim}(\mathbb{I}(V)) \geq d$.

Now, let V and and W be arbitrary. Let W_1 be an irreducible component of W with $\delta\text{-dim}(W) = \delta\text{-dim}(W_1)$. Then W_1 is contained in an irreducible component V_1 of V. By the above,

$$\delta\text{-dim}(W) = \delta\text{-dim}(W_1) \leq \delta\text{-dim}(V_1) \leq \delta\text{-dim}(V). \quad \square$$

Exercise: Let $W \subseteq V$ be two irreducible δ-varieties with $\delta\text{-dim}(W) = \delta\text{-dim}(V)$. Is $W = V$?